MATHEMATICAL MODELING OF THE MASS EXCHANGE PROCESSES IN THE GROUND-SOIL MEDIUM

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Abstract. In the present study the mathematical model of the mass exchange process in the porous medium with fractal structure is considered that can be used in the modeling of the filtration problem. Investigation of the mass exchange processes in the fractal structure massives can provide an overview of changes under the external intensity and control the performance characteristics in such environments.

Keywords: fractal structure, elastic environment, variable structure medium, filtration problems.

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Manuscript received: 10 November 2017

1. Introduction

Mathematical modeling of the mass exchange processes in the fractal structure porous media is of particular importance for the solution of the practical problems in oil and gas exploration and many environmental issues [1-3]. For example, in the oil-gas structures under the external influences one may observe unstable phase changes that causes the formation of the fractal structures and Hele-Shaw cells.

Also, as a result of the various factors the groundwater level approaching to the ground surface results a salinization and moisture of the soil. So, it is important to establish a mathematical model of groundwater level variations for keeping the proper level of groundwater, select the options for removing dewatering waters, and control the soil fertility by optimal agrotechnical and meliorative regimes [6].

2. Formulation of the problem

One of the main characteristics of the soil-ground environment is the volume distribution of the pores. This implies a numerical function $f(r)$ with values from the interval $[r, r + dr]$ that is usually proportional to the number $f(r)dr$ of the of the pores. The natural distribution functions of the pores for the different soil solid samples are graphically illustrated in the technical literature for the stationary case. The creation of the mathematical models of such non-linear processes is very important for the investigation of their physical matter and purposeful control.
3. Solution of the problem

Suppose that the distribution in any point of the layer is given in the form of the function \( f = f(r, t) \) at the moment \( t \). Here \( r \) is a radius of the pore channel. At the beginning of the time \( (t = 0) \), i.e. without external influence we have \( f(r, 0) = f^0(r) \) and the condition \( \int_0^\infty f^0(r)dr = 1 \) holds true.

If structural changes that can change the volume and size of porous channels occur, then coordinate and time-dependent evolutionary changes of the distribution function should be studied. Let's assume that the rate of decreasing the radius of the porous channels is denoted by \( v_r (r, t) \), and the velocity of their formation (or disappearance) by \( v_\eta (r, t) \). According to this scheme, the expression that characterizes the variation of the number of channels in the interval \( [r, r+dr] \) at the time interval \( dt \) will be in the form

\[
f(r, t + dt)dr - f(r, t)dr \approx \frac{\partial f}{\partial t} dt dr
\]

or

\[
\frac{\partial f}{\partial t} dr dt = \frac{\partial}{\partial r} [v_r (r, t)f(r, t)] dr dt - v_\eta (r, t) dr dt.
\]

From this we get

\[
\frac{\partial f}{\partial t} - \frac{\partial}{\partial r} (v_r + v_\eta) = 0.
\]

The "individual" character of this process is the participation of the coefficients \( v_r (r, t) \) and \( v_\eta (r, t) \). This equation is sufficiently general and allows one to solve the class of problems of propagation of the liquids in the porous medium. Application of this equation can solve many practical problems of application of smaller particles of nanotubes, irrigation and fertilization of agricultural land in arid zone.

If drainage streams have natural or anthropogenic dispersed particles, we can denote the distribution function over their sizes by \( q = q(v, t) \). Where \( v \) is the particle volume, and in the subsequent considerations, saying particle we mean its volume. If to denote the increasing of the intensity of the size and the number by \( h_v \) and \( h_\xi \) correspondingly, we get

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial v} (h_v q) + h_\xi = 0.
\]

If to denote by \( k(x, y, z, t) \) the current estimate of the value of change of the absolute conductivity of the layer as a result of the structural changes, then we can take it in the form of the product \( k = \tilde{k}(x, y, z, t)k^0 \). Here, \( \tilde{k}(x, y, z, t) \) is a remaind conductivity that can be defined according to the parallel capillary model corresponding to the Hagen-Puazeil approach for the cylindrical channels.

In the beginning of the time the volume of the liquid passing from the channel with cross profile with the length \( \Delta l \) as a result of the pressure lowering \( \Delta P \) is equal to \( Q = k^0 \Delta P / (\mu \Delta l) \). Then following to Hagen-Puazeil formula for the pores bundle we obtain

\[
Q^0 = N \int_0^\infty \frac{\pi r^2 \Delta P}{8 \mu \xi \Delta l} f^0 (r) dr,
\]

\[
Q = N \int_0^\infty \frac{\pi r^2 \Delta P}{8 \mu \xi \Delta l} f (r) dr.
\]
By the expression obtained from this the conductivity can be calculated with high precision.

\[
\tilde{k} = \frac{k}{k_0} = \frac{\int_0^\infty r^4 f \, dr}{\int_0^\infty r^4 f^0 \, dr} \quad \text{or} \quad k = k_0 \frac{\int_0^\infty r^4 f \, dr}{\int_0^\infty r^4 f^0 \, dr}.
\]

The presence of microscopic substances significantly increases the probability of their penetration into the pores and can lead to narrowing of the size of the entrance or exit sections of the porous channels. The mathematical description of the simplest version of this event assumes that the ratio of the radius of the exit section to the radius of the cylindrical portion of the channel for all capillaries is the same for all channels, and if the radius of the particle is equal to or larger than the radius of the channel's exit, then the exit is closed. The conductivity of the adjacent particles is more less than the conductivity of the layer that of layering.

For the average speed of the liquid inside the capillary, we find the following expression

\[
v_{or} = |v| r^2 / (B \xi (\frac{k_0}{\mu_0} + \frac{k_w}{\mu_w}) \mu_w).
\]

By the combination of the Darcy and Puazeil laws for the liquid volume passing from the capillar bundle we get

\[
Q = |V| = (\frac{k_0}{\mu_0} + \frac{k_w}{\mu_w}) \frac{\Delta P}{\Delta t}.
\]

Keeping groundwater levels at the required level is one of the main tools to ensure the availability of reclamation. Control of the groundwater level can be done through the drainage system.

Groundwater level change equation is written as follows [6]

\[
\frac{\partial h}{\partial t} = \frac{k}{2} \left( \frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right) - \frac{k_0}{M_0} (h - H) + W.
\]

Here \(\delta, k, k_0, M_0, H \approx \text{const.}\).

If in the free surface \(y = H_0\) then \(h(x, H_0, t) = f(x, t)\); if \(y = 0\), then \(h(x, 0, t) = 0\) (waterproof condition).

The functions \(h(x, y, t)\) and \(W(x, y, t)\) are taken as a power functions with respect to the parameter \(\varepsilon\)

\[
h(x, y, t) = h_0(t) + \sum_{k=1}^\infty \varepsilon^k h_k(x, y, t),
\]

\[
W(x, y, t) = W_0(t) + \sum_{k=1}^\infty \varepsilon^k W_k(x, y, t).
\]

If we consider these expressions in the equation and deduce the coefficients of the same degrees of \(\varepsilon\) we get the following system of differential equations

\[
\frac{\partial h_0(t)}{\partial t} + \frac{k_0}{M_0} h_0(t) = W_0
\]

\[
\frac{\partial h_0(t)}{\partial t} = k h_0(t) \cdot \Delta h_1 - \frac{k_0}{M_0} \cdot h_1 - W_1
\]

In this case, each equation excepting the first one is the linear equation with variable coefficients. The first equation is an ordinary differential equation of the first order. Its solution is of the form

\[
h_0(t) = C \cdot e^{-at} + \frac{1}{a} \cdot \int_0^t W_0(\tau) \cdot e^{-a(t-\tau)} d\tau,
\]

\[
a = \frac{k_0}{\delta \cdot M_0}, C = h_0 + \Delta + \frac{W_0(0)}{a}
\]
and is an initial value of the infiltration if $W_0(0) = W(x, y, t), t = 0$.

Solving the equation we find

$$h_1(x, y, t) = \ln \frac{t-\tau}{t} + \sum_{k=1}^{\infty} \frac{x^2 + y^2}{k \cdot k! \cdot 4a} \left( \frac{1}{\Delta t} - \frac{1}{t} \right)$$

Taking into account the stochastic displacement of the soil-ground layer and the fractal character of the porous media and using the known expressions for the fractional differentiation one can obtain

$$D_{0t}^a \varphi = \begin{cases} \frac{1}{\Gamma(-a)} \int_0^t \varphi(\tau)d\tau, & a < 0 \\ \varphi(t), & a = 0, \\ \frac{\partial^{[a]+1}}{\partial t^{[a]+1}} D_{0t}^{a-[a]-1} \varphi, & a > 0. \end{cases}$$

Then

$$h_1 = k \Gamma(1-a) \frac{\partial h(x, y, t)}{\partial t},$$

where $D_{0t}^{x-1}$ is an operator of fractional differentiation.

The use of the fractional differentiation operator of M. Caputo sense gives

$$\frac{\partial_{0t} (h(x, y, t))}{\partial t} = D_{0t}^{a-1} \frac{\partial h(x, y, t)}{\partial t}, \quad 0 < a \leq 1.$$ 

Comparing the obtained expressions we get

$$h_t = k \Gamma(1-a) \frac{\partial h(x, y, t)}{\partial t}$$

or

$$k \Gamma(1-a) \frac{\partial_{0t}^a (mh(x, y, t))}{\partial t} = \frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y} \right) - \frac{k_0}{kd_0} K + f_0.$$

This equation is called a generalized Bussinesk equation for time. One can easily check that

$$\lim_{a \rightarrow 1} \frac{\partial_{0t}^a h(x, y, t)}{\partial t} = \frac{\partial h(x, y, t)}{\partial t}.$$

So, for the case

$$k = \frac{1}{\Gamma(1-a)} \text{ and } a \rightarrow 1$$

this equation turns to the classical Bussinesk equation

$$\frac{\partial h}{\partial t} = a \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) + f(x, y, t).$$

More proper form is as follows

$$\frac{\partial (mh)}{\partial t} = \frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y} \right) - \frac{k_0}{kd_0} K + f_0,$$

$$f_0 = - \frac{k_0}{d_0} (h_0 - H_0) + w.$$ 

The linearized solution method of this nonlinear equation considering the physical considerations, is given above.

Taking into account the fractal structure, we get a new expression for the study of the level variations of groundwater

$$h_t = k \int_0^t (t-\tau) \cdot a \frac{\partial h(x, y, \tau)}{\partial \tau} d\tau.$$
From this
\[
\frac{\partial (m \rho)}{\partial t} = \text{div}(\rho \vec{V}), \quad \vec{V} = -\frac{k}{\mu} \text{grad} P \quad (1)
\]

The equations of state \( \rho = \rho(P, T) \) and porosity \( m = m(P, T) = m(\rho) \) also should be added to this equation.

Though solutions to this system have been found in various modifications and applied, many of them do not take into the consideration the "memory" effect of the medium, the spatial correlation of the pores, and generally the non-linearity of the system inadequacy.

Let us generalize system (1) for the implementation of the fractional order derivative
\[
l_0 \partial_{\alpha \tau}^\alpha (m \rho) + t_0 D_0^\beta (\rho V(\xi)) = 0, \quad V(\xi) = -\frac{k}{\rho_0 \mu} D_0^\gamma P(\xi, \tau). \quad (2)
\]

Here is Ritz-Weil derivative.

System (2) may be reduced to the following equation
\[
l_0 \partial_{\alpha \tau}^\alpha (m \rho) - t_0 D_0^\beta \left( \frac{kp}{\mu} \rho D_0^\gamma P(\xi, \tau) \right) = 0. \quad (3)
\]

Equation (3) is a closed loop system together with the state equation \( \rho = \rho(P, T) \).

When the porosity coefficient is constant in the medium, for the non-compressed fluid equation (3) turns
\[
\frac{\partial_{\alpha \tau}^\alpha P(\xi, \tau)}{D_0^\beta P(\xi, \tau)} = B_0 D_0^\beta \left( D_0^\gamma P(\xi, \tau) \right), \quad (4)
\]

where \( B_0 = \frac{b\kappa_0}{m \lg \mu} \).

As an initial condition is taken \( P(\xi, 0) = B(\xi) \). Here \( \alpha \) is fluid inside layer pressure, \( \mu \) is the absolute viscosity of the liquid, \( \kappa \)- conductivity of the medium, \( \beta \)- fluid volume elasticity module.

For the case \( \beta = \gamma = 1 \) equation (4) can be rewritten as
\[
\frac{\partial_{\alpha \tau}^\alpha P(\xi, \tau)}{D_0^\beta P(\xi, \tau)} = B_0 D_0^\beta \frac{d^2}{d\xi^2} P(\xi, \tau). \quad (5)
\]

The general solution of this equation satisfying initial condition \( P(\xi, 0) = P_0 = \text{const} \) indeed is
\[
P(x, t) = P_0 \int_{-\infty}^{\infty} e^{-ikx} \cdot E_{\alpha,1}(-B_0 k^2 t^\alpha) dk. \quad (6)
\]

If to take \( \alpha = 1 \) in expression (6) then we get the solution known from literature.
$$P(\xi, \tau) = \frac{P_0}{\sqrt{4\pi A\tau}} \cdot \int_{-\infty}^{\infty} \exp \left( -\frac{(\xi - \xi')^2}{4A\tau} \right) d\xi'.$$

For $\alpha = 0.5; \alpha = 0.25E_{\alpha,\alpha}(-z^\alpha)$ is Mittag-Leffler function. This function can also be presented in the form

$$E_{\frac{1}{2}, \frac{1}{4}}(-\sqrt{z}) = \frac{1}{\sqrt{\pi}} - \sqrt{z} \cdot e^\frac{z}{1 - \text{erf}(\sqrt{z})},$$

$$E_{\frac{1}{2}, \frac{1}{4}} = \frac{1}{\sqrt{\pi}} \cdot F_1 \left(1; \frac{1}{4}; z\right) + \frac{\sqrt{2}}{\sqrt{\pi}} F_1 \left(1; \frac{3}{4}; z\right) - \frac{3}{\sqrt{2}} \cdot e^{z} \cdot \left(1 + \text{erf}(\sqrt{z})\right) - \frac{z^\frac{3}{2}}{\sqrt{\pi}}.$$ (7)

In the particular case when $B(\xi) = \delta(\xi)$ expression (6) takes the form

$$P(\xi, \tau) = \frac{1}{\pi} \cdot \int_0^\infty E_{\alpha,1}(-B_0^\alpha k^2) \cos(k\xi) d\xi.$$ 

From this solution taking $\alpha = 1$ we obtain the known expression

$$P(\xi, \tau) = \frac{1}{\sqrt{4\pi B_0}} \cdot \exp \left( -\frac{\xi^2}{4B_0}\right).$$

4. Conclusions

Some features of the mathematical modeling of non-local exchange processes in fractal structure porous media have been studied. A mathematical model of the problem of fluid filtration in layered porous media was proposed based on the analysis of the factors that take into account the filtration laws and the fractal characteristics of the porous media, using the formulas of the fractional differentiation and integration. It is more expedient to use fractional dimensional spaces to simulate the geometrical positioning of the pores. The application of the theory of fractional differential equations makes it possible to cover many factors that can not be studied in traditional approaches, to explore a broader classes of problems, better understand the known results, and to get more precise results.

Application fractional integral differential equations for the study of the problem of filtration in the porous media from certain specific aspects is analysed in [5, p.27-54]. We explore a direct and more general approach to the problem. Apparently, the constructed model allows for quantitative research of the filtration flows in the fractal structures. Application of the fractional integro-differential formalism allows to obtain already known results and generalize them. It is provided the deriving and generalization of the known solutions using the fractional derivative technique.

Fractional differential is a new approach to the study of non-local processes in space and time and allows one to determine the most important quantitative characteristics of these processes.

References


