**ΠGβ NORMAL SPACE IN INTUITIONISTIC FUZZY TOPOLOGY**

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**Abstract.** In this paper new notions of normality are introduced in intuitionistic fuzzy topological spaces using πgβ closed sets. In addition, it is established that this property is preserved by the continuous maps. The links between πgβ normality, almost πgβ normality and mildly πgβ normality are further investigated. In particular it is shown that that the addition of simple condition to the definition of β-normal space yields a property called πgβ normal space which is the weaker of β-normality.

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1. **Introduction**

Atanasov [4] generalized the idea of fuzzy sets and the concept of intuitionistic fuzzy sets was introduced. On the other hand Coker [6] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. In a topological space, several new classes of subsets can be obtained by repeatedly using interior and closure operators. Some of them are found to show remarkably interesting properties. A good number of research papers in recent years have been devoted to the study of these classes of sets and various notions related to them. Such class, the πgβ-closed sets, was introduced and studied in Intuitionistic fuzzy topological spaces in (Jenitha Premalatha and Jothimani, 2012). In this paper, we extend the notion of normality called πgβ-normality in Intuitionistic fuzzy topological spaces. The property of almost normality was introduced by the authors Singal and Arya [12]. The notion of mildly normal space was introduced by and Singal and Singal [13] independently. Mahmoud et al. [2] introduced the notion of β-normal spaces and obtained their characterizations and preservation theorems. The notion of quasi β-normal and mildly β-normal spaces were introduced by M. C. Sharma and Hamant Kumar [8].

The problem is that we will discuss here is what happens to these results when the normality is replaced with the πgβ normality in intuitionistic fuzzy topological spaces.
2. Preliminaries

**Definition 2.1.** [2] An intuitionistic fuzzy (IF) set A in X is an object having the form $A = \{<x, \mu_A(x), v_A(x)> / x \in X\}$ where $\mu_A(x)$ and $v_A(x)$ denote the degree of membership and non-membership respectively, and $0 \leq \mu_A(x) + v_A(x) \leq 1$.

**Definition 2.2.** [2] Let A and B be IFSs of the form $A = \{<x, \mu_A(x), v_A(x)> / x \in X\}$ and $B = \{<x, \mu_B(x), v_B(x)> / x \in X\}$. Then
i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $v_A(x) \geq v_B(x)$ for all $x \in X$
ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
iii) $A^c = \{<x, \mu_A(x), \mu_A(x)> / x \in X\}$
iv) $A \cap B = \{<x, \mu_A(x) \land \mu_B(x), v_A(x) \lor v_B(x)> / x \in X\}$
v) $A \cup B = \{<x, \mu_A(x) \lor \mu_B(x), v_A(x) \land v_B(x)> / x \in X\}$.

**Definition 2.3.** [4] An intuitionistic fuzzy topology (IFT for short) on X is a family $\tau$ of IFSs in X satisfying the following axioms.
i) $\emptyset, X \in \tau$
ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
iii) $\bigcup_{i \in J} G_i \in \tau$ for any family $\{G_i / i \in J\} \subseteq \tau$.

**Definition 2.4.** [4] Let $(X, \tau)$ be an IFTS and $A = <x, \mu_A, v_A>$ be an IFS in X. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure are defined by

- $\text{IF-Int}(A) = \bigcup \{G / G$ is an IFOS in X and $G \subseteq A\}$
- $\text{IF-CI}(A) = \bigcap \{K / K$ is an IFC in X and $A \subseteq K\}$

An IF subset A is said to be IF regular open [11] if $A = \text{IF-Int}(\text{IF-CI}(A))$. The finite union of IF regular open sets is said to be IF-$\pi$-open [11]. The complement of a IF-$\pi$-open set is said to be IF-$\pi$-closed [11]. A is said to be IF-$\beta$-open [11] if $A = \text{IF-CI}(\text{IF-Int}(\text{IF-CI}(A)))$. The family of all IF-$\beta$-open sets of X is denoted by $\text{IF-$\beta$O}(X)$. The complement of a IF-$\beta$-open set is said to be IF-$\beta$-closed [1]. The intersection of all IF-$\beta$-closed sets containing A is called IF-$\beta$-closure [2] of A, and is denoted by $\text{IF-$\beta$Cl}(A)$. The IF-$\beta$-Interior [2] of A, denoted by $\text{IF-$\beta$Int}(A)$, is defined as union of all IF-$\beta$-open sets contained in A. It is well known $1 \text{IF-$\beta$-Cl}(A) = A \cap \text{IF-Int}(\text{IF-CI}(\text{IF-$\beta$Int}(A)))$ and $\text{IF-$\beta$-Int}(A) = A \cap \text{IF-CI}(\text{IF-$\beta$Cl}(A)))$.

**Definition 2.5.** [13] An Intuitionistic fuzzy (IF) topological space $(X, \tau)$ is called mildly normal if for any two IF disjoint regularly closed subsets $A$ and $B$ of $X$, there exist two IF open disjoint subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. i.e., any two IF disjoint regularly closed subsets are separated.

**Definition 2.6.** [12] An IF topological space $(X, \tau)$ is called almost normal if for any two disjoint IF closed subsets $A$ and $B$ of $X$, one of which is IF regularly closed, there exist two disjoint IF open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 2.7.** [16] An IF topological space $(X, \tau)$ is called quasi-normal if any two disjoint IF-$\pi$-closed subsets $A$ and $B$ of $X$ there exist two IF-open disjoint subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. 
Definition 2.8. An IF space X is said to be IF$\beta$–normal [2] if for every pair of disjoint IF-closed subsets A, B of X, there exist disjoint IF$\beta$–open sets U, V of X such that $A \subset U$ and $B \subset V$.

Definition 2.9. An IF space X is said to be IF$\pi\beta$–normal [8] if for every pair of disjoint IF-closed subsets A, B of X, one of which is IF$\pi$–closed, there exist disjoint IF$\beta$–open sets U, V of X such that $A \subset U$ and $B \subset V$.

Definition 2.10. An IF subset A of a IF space X is said to be a IF$\beta$–neighborhood [2] of x if there exists a IF $\beta$–open set U such that $x \in U \subset A$.

Definition 2.11. An IF function $f : X \rightarrow Y$ is said to be
(a) IF-regular open [10] if $f(U)$ is IF-regular open in Y for every open set U in X.
(b) IF $\pi$-continuous [7] if $f^{-1}(F)$ is IF$\pi$-closed in X for each IF closed set in Y.
(c) IF pre-$\beta$-closed [2] if $f(F)$ is IF$\beta$-closed set in for every IF $\beta$-closed set Fin X.
(d) IF $\pi\beta$ continuous [14] if $f^{-1}(F)$ is IF$\pi\beta$ closed in X for every IF closed set F in Y.
(e) IF$\pi\beta$ -irresolute [14] if $f^{-1}(F)$ is IF$\pi\beta$ -closed in X for every IF -closed set F in Y.
(f) almost IF$\beta$–irresolute [2] if for each $x \in X$ and IF $\beta$–neighborhood V of f(x) in Y, IF$\beta$-Cl($f^{-1}(V)$) is neighborhood of x in X.

3. $\pi\beta$ Normal Spaces

In this section, we introduce the notion of IF$\pi\beta$ -normal space and study some of its properties.

Definition 3.1. An IF space X is said to be IF$\pi\beta$ -normal [15]) if for every pair of disjoint IF$\pi\beta$ –closed subsets H and K of X, there exist disjoint IF$\beta$–open sets U, V of X such that $H \subset U$ and $K \subset V$.

Theorem 3.1. For an IF topological space X, the following are equivalent:
(a) X is IF$\pi\beta$ -normal.
(b) For every pair of disjoint IF $\pi\beta$ -open subsets U and V of X whose union is X, there exist IF $\beta$-closed subsets G and H of X such that $G \subset U$, $H \subset V$ and $G \cup H = X$.
(c) For every IF $\pi\beta$ -closed set A and every IF $\pi\beta$ -open set B in X such that $A \subset B$, there exists a IF$\beta$-open subset V of X such that $A \subset V \subset IF\beta$-Cl(V) $\subset B$.
(d) For every pair of disjoint IF$\pi\beta$ –closed subsets A and B of X, there exists IF $\beta$-open subset V of X such that $A \subset V$ and IF $\beta$-Cl(V) $\cap B = 0$.
(e) For every pair of disjoint IF$\pi\beta$ –closed subsets A and B of X, there exist IF$\beta$-open subsets U and V of X such that $A \subset U$, $B \subset V$ and IF$\beta$-Cl(U) $\cap$ IF$\beta$-Cl(V) $= 0$.

Proof. (a) $\Rightarrow$ (b) Let U and V be any IF$\pi\beta$ –open subsets of a IF$\pi\beta$ -normal space X such that $U \cup V = X$. Then, $X \setminus U$ and $X \setminus V$ are disjoint IF$\pi\beta$ –closed subsets of X. By IF$\pi\beta$ –normality of X, there exist disjoint IF$\beta$–open subsets $U_1$ and $V_1$ of X such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $G = X \setminus U_1$ and $H = X \setminus V_1$. Then, G and H are IF$\beta$–closed subsets in X such that $G \cup H = X$. 

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(b) ⇒ (c). Let A be a IF-$\pi\beta$-closed and B is IF-$\pi\beta$-open subsets of X such that $A \subseteq B$. Then, $A \cap (X \backslash B) = \emptyset$. Thus, $X \setminus A$ and $B$ are IF-$\pi\beta$-open subsets of $X$ such that $(X \setminus A) \cup B = X$. By the Part (b), there exist IF $\beta$-closed subsets $G$ and $H$ of $X$ such that $G \subseteq (X \setminus A)$, $H \subseteq B$ and $G \cup H = X$. Thus, we obtain that $A \subseteq (X \setminus G) \subseteq H \subseteq B$. Let $V = X \setminus G$. Then $V$ is IF-$\beta$-open subset of $X$ and IF-$\beta$-Cl($V$) $\subseteq H$ as $H$ is IF-$\beta$-closed set in $X$. Therefore, $A \subseteq V \subseteq$ IF-$\beta$-Cl($V$) $\subseteq B$.

(c) ⇒ (d). Let A and B be disjoint IF-$\pi\beta$-closed subset of X. Then $A \subseteq B$, where $X \setminus B$ is IF-$\pi\beta$-open. By the part (c), there exists a IF-$\beta$-open subset $U$ of $X$ such that $A \subseteq U \subseteq$ IF-$\beta$-Cl($U$)$\subseteq X \setminus B$. Thus, IF-$\beta$-Cl($U$)$\cap B = \emptyset$. (d) ⇒ (e). Let A and B be any disjoint IF-$\pi\beta$-closed subset of X. Then by the part (d), there exists a IF $\beta$-open set $U$ containing $A$ such that IF-$\beta$-Cl($U$)$\cap B = \emptyset$. Since IF-$\beta$-Cl($U$) is IF-$\pi\beta$-closed, then it is IF $\pi\beta$-closed. Thus IF-$\beta$-Cl($U$) and B are disjoint IF-$\pi\beta$-closed subsets of $X$. Again by the part (d), there exists a IF-$\beta$-open set $V$ in $X$ such that $B \subseteq V$ and IF-$\beta$-Cl($U$)$\cap$IF-$\beta$-Cl($V$) = $\emptyset$. (e) ⇒ (a) Let A and B be any disjoint IF-$\pi\beta$-closed subsets of X. Then by the part (e), there exist IF-$\beta$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and IF-$\beta$-Cl($U$)$\cap$IF-$\beta$-Cl($V$) = $\emptyset$. Therefore, we obtain that $U \cap V = \emptyset$. Hence $X$ is IF-$\pi\beta$-normal.

**Lemma 3.1.** (a) The image of IF $\beta$-open subset under an IF-$\beta$-continuous function is IF-$\beta$-open.

(b) The inverse image of IF $\beta$-open subset under an open continuous function is IF-$\beta$-open subset.

**Lemma 3.2.** [15] The image of IF regular open subset under open and closed continuous function is IF regular open subset.

**Lemma 3.3.** [15] The image of IF-$\beta$-open subset under IF-$\beta$-open and IF-$\beta$-closed function is IF-$\beta$-open.

**Theorem 3.2.** If $f : X \rightarrow Y$ be an IF-$\beta$-closed and IF-$\beta$-continuous bijection function and A be a IF $\pi\beta$-closed set in $Y$, then $f^{-1}(A)$ is IF $\pi\beta$-closed set in $X$.

**Proof.** Let $A$ be an IF-$\pi\beta$-closed subset of $Y$ and $U$ be any IF $\pi$-open subset of $X$ such that $f^{-1}(A) \subseteq U$. Then by the Lemma 3.3, we have $f^{-1}(U)$ is IF-$\pi$-open subset of $Y$ such that $A \subseteq f^{-1}(U)$. Since $A$ is IF-$\pi\beta$-closed subset of $Y$ and $f(U)$ is IF-$\pi$-open set in $Y$. Thus IF-$\beta$-Cl($A$) $\subseteq U$. By the Lemma 3.1 we obtain that $f^{-1}(A) \subseteq f^{-1}(U)$. Hence, $f^{-1}(A)$ is IF-$\pi\beta$-closed subset in $X$.

**Theorem 3.3.** If $f : X \rightarrow Y$ be an IF-$\beta$-open IF-$\beta$-closed bijective continuous function and $X$ is IF-$\pi\beta$-normal space. Then $Y$ is also IF-$\pi\beta$-normal.

**Proof.** Let $A$ and $B$ be any disjoint IF-$\pi\beta$-closed subsets of $Y$. Then by the Theorem 3.2, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint IF-$\pi\beta$-closed subsets of $X$. By IF-$\pi\beta$-normality of $X$, there exist IF $\beta$-open subsets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. By assumption, we have $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. By the Lemma 3.1, $f(U)$ and $f(V)$ are disjoint IF-$\beta$-open subsets of $Y$ such that $A \subseteq f(U)$ and $B \subseteq f(V)$. Hence, $Y$ is IF-$\pi\beta$-normal.
Definition 3.2. An IF function \( f : X \rightarrow Y \) is called strongly IF\( \pi g\beta \)-open if \( f(U) \subseteq \text{IF-\( \pi g\beta \)-O}(Y) \) for each \( U \in \text{IF-\( \pi g\beta \)-}(X) \).

Definition 3.3. An IF function \( f : X \rightarrow Y \) is called strongly IF\( \pi g\beta \)-closed if \( f(U) \cap \text{IF-\( \pi g\beta \)-}(Y) \) for each \( U \in \text{IF-\( \pi g\beta \)-}(X) \).

Theorem 3.4. An IF function \( f : X \rightarrow Y \) is strongly IF\( \pi g\beta \)-closed if and only if for each IF subset \( B \) in \( X \) and for each IF\( \pi g\beta \)-open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists an IF\( \pi g\beta \)-open set \( V \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

Proof. Suppose that \( f \) is strongly IF\( \pi g\beta \)-closed. Let \( B \) be an IF subset of \( Y \) and \( U \in \text{IF-\( \pi g\beta \)-O}(X) \) containing \( f^{-1}(B) \). Put \( V = Y \setminus (X \setminus U) \), then \( V \) is an IF\( \pi g\beta \)-open set of \( Y \) such that \( B \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Conversely, let \( K \) be any IF\( \pi g\beta \)-closed set of \( X \). Then \( f^{-1}(Y \setminus (K)) \subseteq X \setminus K \) and \( X \setminus K \in \text{IF-\( \pi g\beta \)-O}(X) \). There exists a IF\( \pi g\beta \)-open set \( V \) of \( Y \) such that \( Y \setminus (K) \subseteq V \) and \( f^{-1}(V) \subseteq X \setminus K \). Therefore, we have \( f(K) = Y \setminus V \) and \( K \subseteq f^{-1}(Y \setminus V) \). Hence, we obtain \( f(K) = Y \setminus V \) and \( f(K) \) is IF\( \pi g\beta \)-closed in \( Y \). This shows that \( f \) is strongly IF\( \pi g\beta \)-closed.

Theorem 3.5. If \( f : X \rightarrow Y \) is a strongly IF\( \pi g\beta \)-closed continuous function from an IF\( \pi g\beta \)-normal space \( X \) onto an IF space \( Y \), then \( Y \) is IF\( \pi g\beta \)-normal.

Proof. Let \( S_1 \) and \( S_2 \) be disjoint IF-closed sets in \( Y \). Then \( f^{-1}(S_1) \) and \( f^{-1}(S_2) \) are IF-closed sets. Since \( X \) is IF\( \pi g\beta \)-normal, then there exist disjoint IF\( \pi g\beta \)-open sets \( U \) and \( V \) such that \( f^{-1}(S_1) \subseteq U \) and \( f^{-1}(S_2) \subseteq V \). By the previous theorem, there exist IF\( \pi g\beta \)-open sets \( A \) and \( B \) such that \( S_1 \subseteq A \), \( S_2 \subseteq B \), \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Also, \( A \) and \( B \) are IF disjoint. Thus, \( Y \) is IF\( \pi g\beta \)-normal.

Definition 3.4. An IF function \( f : X \rightarrow Y \) is said to be almost IF\( \pi g\beta \)-irresolute if for each IF point \( x \) in \( X \) and each IF\( \pi g\beta \) neighborhood \( V \) of \( f(x) \), IF\( \pi g\beta \)-Cl\( f^{-1}(V) \) is a IF\( \pi g\beta \) neighborhood of \( x \).

Lemma 3.4. Let \( f : X \rightarrow Y \) be an IF function. Then \( f \) is almost IF\( \pi g\beta \) irresolute if and only if \( f^{-1}(V) \subseteq \text{IF-\( \pi g\beta \)-int(\( \pi g\beta \)-Cl\( f^{-1}(V) \))) \) for every \( V \in \text{IF-\( \pi g\beta \)-O}(Y) \).

Theorem 3.6. An IF function \( f : X \rightarrow Y \) is almost IF\( \pi g\beta \) irresolute if and only if \( f(\text{IF-\( \pi g\beta \)-Cl}(U)) \subseteq \text{IF-\( \pi g\beta \)-Cl}(f(U)) \) for every \( U \in \text{IF-\( \pi g\beta \)-O}(X) \).

Proof. Let \( U \in \text{IF-\( \pi g\beta \)-O}(X) \). Suppose \( y \in \text{IF-\( \pi g\beta \)-Cl}(f(U)) \). Then there exists \( V \in \text{IF-\( \pi g\beta \)-O}(Y, y) \) such that \( V \cap f(U) = 0 \). Hence \( f^{-1}(V) \cap f(U) = 0 \). Since \( U \subseteq \text{IF-\( \pi g\beta \)-O}(X) \) we have \( \text{IF-\( \pi g\beta \)-int(\( \pi g\beta \)-Cl\( f^{-1}(V) \))) \subseteq \text{IF-\( \pi g\beta \)-Cl}(f(U)) \). Therefore \( f^{-1}(V) \subseteq \text{IF-\( \pi g\beta \)-int(\( \pi g\beta \)-Cl\( f^{-1}(V) \))) \). By Lemma 3.4, \( f^{-1}(V) \cap \text{IF-\( \pi g\beta \)-Cl}(U) = 0 \). Hence \( V \cap f(\pi g\beta \>-\text{Cl}(U)) = 0 \). This implies that \( y \in \text{IF-\( \pi g\beta \)-Cl}(U) \).

Conversely, if \( V \subseteq \text{IF-\( \pi g\beta \)-O}(Y) \), then \( M = X \setminus \text{IF-\( \pi g\beta \)-Cl}(f^{-1}(V)) \subseteq \text{IF-\( \pi g\beta \)-O}(X) \). By hypothesis, \( f(\text{IF-\( \pi g\beta \)-Cl}(M)) \subseteq \text{IF-\( \pi g\beta \)-Cl}(f(M)) \) and hence \( X \setminus \text{IF-\( \pi g\beta \)-int(\( \pi g\beta \)-Cl\( f^{-1}(V) \))) = \text{IF-\( \pi g\beta \)-Cl}(M) \subseteq \text{IF-\( \pi g\beta \)-Cl}(f^{-1}(\pi g\beta \)-IFCl\( f(M)) \subseteq f^{-1}(\text{IF-\( \pi g\beta \)-Cl}(f(X \setminus f^{-1}(V)))) \subseteq f^{-1}(\text{IF-\( \pi g\beta \)-Cl}(Y \setminus V)) \subseteq f^{-1}(Y \setminus f^{-1}(V)). \)

Therefore \( f^{-1}(V) \subseteq \text{IF-\( \pi g\beta \)-int(\( \pi g\beta \)-Cl\( f^{-1}(V) \))) \). By Lemma 3.4, \( f \) is almost IF\( \pi g\beta \)-irresolute.

Theorem 3.7. If \( f : X \rightarrow Y \) is a strongly IF\( \pi g\beta \)-open continuous almost IF\( \pi g\beta \)-irresolute function from a IF\( \pi g\beta \)-normal space \( X \) onto a IF space \( Y \), then \( Y \) is IF\( \pi g\beta \)-normal.

Proof. Let \( A \) be an IF-closed subset of \( Y \) and \( B \) be an IF-open set containing \( A \). Then by continuity of \( f \), \( f^{-1}(A) \) is IF-closed and \( f^{-1}(B) \) is an IF open set of \( X \) such that
Almost $\pi\beta$-normal spaces

**Definition 4.1.** An IF space $X$ is said to be almost $\pi\beta$ -normal if for each IF closed set $A$ and each IF regular closed set $B$ such that $A \cap B = 0$, there exist disjoint $\pi\beta$-open sets $A$ and $B$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 4.1.** For a IF space $X$ the following statements are equivalent:

1. $X$ is almost $\pi\beta$-normal,
2. For every pair of sets $U$ and $V$, one of which is IF-open and the other is IF regular open whose union is $X$, there exist IF $\pi\beta$-closed sets $A$ and $B$ such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.
3. For every IF-closed set $A$ and every IF-regular open set $B$ containing $A$, there exists a IF-$\pi\beta$-open set $V$ such that $A \subseteq V \subseteq \pi\beta(\text{Cl}(V)) \subseteq B$.

**Proof.** (i) $\Rightarrow$ (ii): Let $U$ be an IF open set and $V$ be an IF regular open set in an almost $\pi\beta$-normal space $X$ such that $U \cup V = X$. Then $(X \setminus U)$ is a IF-closed set and $(X \setminus V)$ is a IF regular closed set with $(X \setminus U) \cap (X \setminus V) = 0$. By almost $\pi\beta$-normality of $X$, there exist disjoint $\pi\beta$-open sets $U_1$ and $V_1$ such that $X \setminus U \subseteq U_1$ and $X \setminus V \subseteq V_1$. Let $A = X \setminus U_1$ and $B = X \setminus V_1$. Then $A$ and $B$ are IF-$\pi\beta$-closed sets such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(ii) $\Rightarrow$ (iii): Let $A$ be an IF-closed set and $B$ be an IF regular open set containing $A$. Then $X \setminus A$ is IF open and $B$ is IF regular open sets whose union is $X$. Then by (2), there exist IF-$\pi\beta$-closed sets $M_1$ and $M_2$ such that $M_1 \subseteq X \setminus A$ and $M_2 \subseteq B$ and $M_1 \cup M_2 = X$. Then $A \subseteq X \setminus M_1$, $X \setminus B \subseteq X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = 0$. Let $U = X \setminus M_1$ and $V = X \setminus M_2$. Then $U$ and $V$ are disjoint IF-$\pi\beta$-open sets such that $A \subseteq U \subseteq X \setminus V$. As $X \setminus V$ is IF-$\pi\beta$-closed set, we have IF-$\pi\beta$ Cl($U) \subseteq X \setminus V$ and $A \subseteq \pi\beta - \text{Cl}(U) \subseteq B$.

(iii) $\Rightarrow$ (i): Let $A_1$ and $A_2$ be any two disjoint IF closed and IF regular closed sets, respectively. Put $D = X \setminus A_2$, then $A_2 \cap D = 0$. $A_1 \subseteq D$ where $D$ is a IF regular open set. Then by (3), there exists a IF-$\pi\beta$-open set $U$ of $X$ such that $A_1 \subseteq \pi\beta - \text{Cl}(U) \subseteq D$. It follows that $A_2 \cap X \setminus \pi\beta - \text{Cl}(U) = 0$. Hence, $A_1$ and $A_2$ are separated by IF-$\pi\beta$-open sets $U$ and $V$. Therefore $X$ is almost IF-$\pi\beta$-normal.

**Definition 4.1.** An IF function $f : X \to Y$ is called i) IF R-Map, [5] if $f^{-1}(V)$ is an IF regular open in $X$, for every IF regular open set $V$ of $Y$.

ii) Completely Continuous [3] if $f^{-1}(V)$ is a IF regular open in $X$, for every IF open set $V$ of $Y$.

**Theorem 4.2.** If $f : X \to Y$ is a IF continuous, strongly IF-$\pi\beta$-open, IFR-map and almost IF-$\pi\beta$-irresolute surjection from an almost IF-$\pi\beta$-normal space $X$ onto a IF space $Y$, then $Y$ is almost IF-$\pi\beta$-normal.
Proof. Similar to Theorem 3.7

Corollary 4.1: If \( f : X \to Y \) is a completely IF continuous strongly IF \( \pi g \beta \) open and almost IF\( \pi g \beta \) irresolute surjection from an almost IF \( \pi g \beta \) -normal space \( X \) onto a space \( Y \), then \( Y \) is almost \( \pi g \beta \) -normal.

5. Mildly IF\( \pi g \beta \) -normal spaces

Definition 5.1. A IF space \( X \) is said to be mildly IF\( \pi g \beta \) -normal if for every pair of disjoint IF regular closed sets \( A \) and \( B \) of \( X \), there exist disjoint \( \pi g \beta \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

Theorem 5.1. For a IF space \( X \) the following are equivalent:
(i) \( X \) is mildly IF\( \pi g \beta \) -normal, 
(ii) For every pair of IF regular open sets \( U \) and \( V \) whose union is \( X \), there exist IF\( \pi g \beta \) -closed sets \( G \) and \( H \) such that \( G \subset U \) and \( H \subset V \), 
(iii) For any IF regular closed set \( A \) and every IF regular open set \( B \) containing \( A \), there exists an IF\( \pi g \beta \) -open set \( U \) such that \( A \subset U \subset IF\pi g \beta -\text{Cl}(U) \subset B \), 
(iv) For every pair of disjoint IF regular closed sets \( A \) and \( B \), there exist IF\( \pi g \beta \) -open sets \( U \) and \( V \) such that \( A \subset U \subset IF\pi g \beta -\text{Cl}(U) \cap IF\pi g \beta -\text{Cl}(V) = \emptyset \).

Proof. Similar to Theorem 3.1.

Theorem 5.2. If \( f : X \to Y \) is a strongly IF \( \pi g \beta \) -open IFR-map and almost IF\( \pi g \beta \) irresolute function from a mildly IF \( \pi g \beta \) -normal space \( X \) onto a IF space \( Y \) then \( Y \) is mildly IF\( \pi g \beta \) -normal.

Proof. Let \( A \) be a IF regular closed set and \( B \) be a IF regular open set containing \( A \). Then by IFR-map of \( f \), \( f^{-1}(A) \) is an IF regular closed set contained in the IF regular open set \( f^{-1}(B) \). Since \( X \) is mildly IF\( \pi g \beta \) normal, there exists an IF\( \pi g \beta \) open set \( V \) such that \( f^{-1}(A) \subset V \subset IF\pi g \beta -\text{Cl}(V) \subset f^{-1}(B) \) by Theorem 8. As \( f \) is strongly IF\( \pi g \beta \) open and an almost IF\( \pi g \beta \) irresolute surjection, it implies, \( f(V) \subset IF\pi g \beta -\text{Cl}(f(V)) \subset B \). Hence \( Y \) is mildly IF \( \pi g \beta \) -normal.

Theorem 5.3. If \( f : X \to Y \) is IF R-map, strongly IF\( \pi g \beta \) closed function from a mildly IF\( \pi g \beta \) -normal space \( X \) onto a IF space \( Y \), then \( Y \) is mildly IF\( \pi g \beta \) -normal.

Proof. Similar to Theorem 3.5

Theorem 5.4. If \( f : X \to Y \) is a continuous quasi IF- \( \pi g \beta \) closed surjection and \( X \) is IF \( \pi g \beta \) -normal, then \( Y \) is IF normal.

Proof. Let \( V_1 \) and \( V_2 \) be any disjoint IF closed sets of \( Y \). Since \( f \) is IF continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are disjoint IF closed sets of \( X \). Since \( X \) is IF\( \pi g \beta \) -normal, there exist disjoint \( U_i \), \( i = 1, 2 \) such that \( f^{-1}(V_i) \subset U_i \), for \( i = 1, 2 \). Put \( W_i = Y - f(X - U_i) \), then \( V_i \) is IF open in \( Y \) and \( U_i \subset W_i \) for \( i = 1, 2 \). Since \( U_1 \cap U_2 = \emptyset \), and \( f \) is IF surjective, we have \( W_1 \cap W_2 = \emptyset \). This shows that \( Y \) is IF normal.

Theorem 5.5. Let \( f : X \to Y \) be a closed \( \pi g \beta \) -continuous injection. If \( Y \) IF \( \pi g \beta \) -normal, then \( X \) is IF\( \pi g \beta \) -normal.

Proof. Let \( N_1 \) and \( N_2 \) be disjoint IF closed sets of \( X \). Since \( f \) is a IF Closed injection, \( f(N_1) \) and \( f(N_2) \) are disjoint IF closed sets of \( Y \). By the IF\( \pi g \beta \)-normality of \( Y \), there exist disjoint \( V_1, V_2 \) such that \( f(N_i) \subset V_i \), for \( i = 1, 2 \). Since \( f \) is IF\( \pi g \beta \) -continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are disjoint IF\( \pi g \beta \) -open sets of \( X \) and \( N_i \subset f^{-1}(V_i) \) for \( i = 1, 2 \).
Now, put $U_i = \text{IF } \beta - \text{Int}(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \text{IF } \pi \beta \text{O}(X)$, $N_i \subseteq U_i$ and $U_1 \cap U_2 = 0$. This shows that $X$ is IF$\pi \beta$ -normal.

6. Preservation theorems

In this section we investigate preservation theorems concerning IF$\pi \beta$ normal spaces in intuitionistic fuzzy topological spaces.

**Theorem 6.1.** If $f : X \to Y$ is an almost IF$\pi \beta$ -closed injection and $Y$ is mildly IF$\pi \beta$ -normal respectively, then $X$ is mildly IF$\pi \beta$ normal.

**Proof.** Let $A$ and $B$ be any disjoint IF regular sets of $X$. Since $f$ is an almost IF$\pi \beta$ -closed injection, $f(A)$ and $f(B)$ are disjoint IF regular IF$\pi \beta$ -closed sets of $Y$. By the mild IF$\pi \beta$ -normality of $Y$, there exist disjoint IF open sets $U$ and $V$ of $Y$ such that $(A) \subseteq U$ and $(B) \subseteq V$. Now, put $G = \text{IF } \text{Int}((\text{IF } \text{Cl}(U) ))$ and $H = \text{IF } \text{Int}(\text{IF } \text{Cl}(V))$, then $G$ and $H$ are disjoint IF regular -open sets such that $f(A) \subseteq G$ and $(B) \subseteq H$. Since $f$ is almost IF $\pi \beta$ -continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint IF$\pi \beta$ -open sets containing $A$ and $B$, respectively. It follows from Theorem 5.1 that $X$ is mildly IF$\pi \beta$ normal.

**Lemma 6.1.** A surjection $f : X \to Y$ is almost IF $\pi \beta$ closed if and only if for each subset $S$ of $Y$ and each $U \in \text{IF } \pi \beta$ -RO$(X)$ containing $f^{-1}(S)$ there exists respectively an IF$\pi \beta$ -open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Proof.** Necessity. Suppose that $f$ is almost IF$\pi \beta$ -closed. Let $S$ be a subset of $Y$ and let $U \subseteq \text{IF } \pi \beta$ -RO$(X)$ contain $f^{-1}(S)$. Put $V = Y f (X \cup U)$, then $V$ is a IF$\pi \beta$ -open set of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Sufficiency.** Let $F$ be any regular IF$\pi \beta$ -closed set of $X$. Then $f^{-1}(Y \setminus (X \setminus F)) \subseteq (X \setminus F)$ and $(X \setminus F) \subseteq \text{IF } \pi \beta$ -RO$(X)$. There exists a regular IF $\pi \beta$ -open set $V$ of $Y$ such that $(Y \setminus f(F)) \subseteq V$ and $f^{-1}(V) \subseteq (X \setminus F)$. Therefore, we have $f(F) \supseteq Y \setminus V$ and $F \subseteq f^{-1}(Y \setminus V)$. Hence $f(F) = Y \setminus V$, and $f(F)$ is regular IF$\pi \beta$ -closed in $Y$. This shows that $f$ is almost IF $\pi \beta$ -regular -closed.

**Theorem 6.2.** If $f : X \to Y$ is a completely IF$\pi \beta$ -continuous almost IF$\pi \beta$ -closed surjection and $X$ is mildly IF$\pi \beta$ normal, then $Y$ is IF$\pi \beta$ -normal.

**Proof.** Let $A$ and $B$ be any disjoint IF closed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint IF $\pi \beta$ closed sets of $X$. Since $X$ is mildly IF$\pi \beta$ -normal, there exist disjoint IF open sets $U$ and $V$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Let $G = \text{IF } \text{Int}((\text{IF } \text{Cl}(U)))$ and $H = \text{IF } \text{Int}((\text{IF } \text{Cl}(V)))$, then $G$ and $H$ are disjoint IF regular open sets such that $f^{-1}(A) \subseteq G$ and $f^{-1}(B) \subseteq H$. By Lemma 6.1, there exist IF$\pi \beta$ -open sets $K$ and $L$ of $Y$ such that $A \subseteq K$, $B \subseteq L$, $f^{-1}(K) \subseteq G$ and $f^{-1}(L) \subseteq H$. Since $G$ and $H$ are disjoint, so are $K$ and $L$, and $A$ and $B$ are IF $\pi \beta$ -open, we obtain $A \subseteq \text{IF } \text{Int}(K)$, $B \subseteq \text{IF } \text{Int}(L)$ and $[\text{IF } \text{Int}(K) \cap \text{IF } \text{Int}(L)] = 0$. This shows that $Y$ is IF $\pi \beta$ -normal.

**Corollary 6.1.** If $f : X \to Y$ is a completely IF$\pi \beta$ -continuous $\pi \beta$ closed surjection and $X$ is mildly IF$\pi \beta$ -normal, then $Y$ is IF $\pi \beta$ -normal.

**Theorem 6.3:** Let $f : X \to Y$ be an IF$\pi \beta$-IFR-map (almost $\pi \beta$ -continuous) and almost IF$\pi \beta$ -regular-closed surjection. If $X$ is mildly IF$\pi \beta$ -normal, then $Y$ is mildly IF$\pi \beta$ normal.

**Proof.** Let $A$ and $B$ be any disjoint regular IF$\pi \beta$ -closed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular IF$\pi \beta$-closed or IF closed sets of $X$. Since $X$ is
respectively mildly IF$\pi$g$\beta$ normal there exist disjoint IF open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Put $G = \text{IF-Int}(\text{IF-CI}(U))$ and $H = \text{IF-Int}(\text{IF-CI}(V))$, then $G$ and $H$ are disjoint IF regular open sets of $X$ such that $f^{-1}(A) \subseteq G$ and $f^{-1}(B) \subseteq H$. By Theorem 6.2, there exist IF $\pi g\beta$-open sets $K$ and $L$ of $Y$ such that $A \subseteq K$, $B \subseteq L$, $f^{-1}(K) \subseteq G$ and $f^{-1}(L) \subseteq H$. Since $G$ and $H$ are disjoint, so are $K$ and $L$. It follows from Theorem 4.2 that $Y$ is mildly IF $\pi g\beta$ normal.

**Definition 6.1.** A function $f : X \to Y$ is said to be IF $\pi$-irresolute [3] if $f^{-1}(F)$ is IF$\beta$-closed in $X$ for every IF$\beta$-closed set $F$ in $Y$.

**Theorem 6.4.** If $f : X \to Y$ is IF $\pi$-irresolute, IF pre $\beta$-closed and $A$ is a IF$\pi g\beta$-closed subset of $X$, then $f(A)$ is IF$\pi g\beta$-closed subset of $Y$.

**Proof:** Since $f$ is IF$\pi g\beta$-irresolute function, then $f^{-1}(U)$-IF-open in $X$. Since $A$ is IF$\pi g\beta$-closed set in $X$ and $A \subseteq f^{-1}(U)$ then IF$\beta$cl$_{X}(A) \subseteq f^{-1}(U)$. This implies that $f(\text{IF}\beta\text{cl}_{X}(A)) \subseteq U$. Since $f$ is IF pre $\beta$-closed and IF$\beta$cl$_{X}(A)$ is IF$\beta$-closed set in $X$, then $f(\text{IF}\beta\text{cl}_{X}(A))$ is IF$\beta$-closed in $Y$. Thus, we have $f(\text{IF}\beta\text{cl}_{X}(f(A))) \subseteq U$. Hence, $f(A)$ is IF$\pi g\beta$-closed subset of $Y$. 

**Corollary 6.2.** If $f : X \to Y$ is IF$\pi$-continuous, IF pre $\beta$-closed and $A$ is a IF$\pi g\beta$-closed subset of $X$, then $f(A)$ is IF$\pi g\beta$-closed subset of $Y$.

**Theorem 6.5.** If $f : X \to Y$ is IF$\pi$-irresolute, IF pre $\beta$-closed and IF$\beta$-irresolute injection function from an IF space $X$ to a IF $\pi g\beta$-normal $Y$, then $X$ is IF$\pi g\beta$-normal.

**Proof.** Let $A$ and $B$ be any two disjoint IF$\pi g\beta$ closed subsets of $X$. By the Theorem 5.2 $f(A)$ and $f(B)$ are disjoint IF$\pi g\beta$-closed subsets of $Y$. By IF$\pi g\beta$-normality of $Y$, there exist disjoint IF$\beta$-open subsets $U$ and $V$ of $Y$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \emptyset$. Since $f$ is IF$\beta$-irresolute injection function, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint IF$\beta$-open sets in $X$ such that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$. Hence $X$ is IF$\pi g\beta$-normal.

**Corollary 6.3.** If $f : X \to Y$ is IF $\pi$-continuous, IF pre $\beta$-closed and IF$\beta$-irresolute injection function from a IF space $X$ to a IF $\pi g\beta$-normal $Y$, then $X$ is IF $\pi g\beta$-normal.

**Lemma 6.1.** If the IF bijection function $f : X \to Y$ is IF $\pi$-continuous and regular open, then $f$ is $\pi g\beta$ -irresolute.

**Theorem 6.6.** If $f : X \to Y$ is $\pi g\beta$-irresolute ,IF pre $\beta$-closed bijection function from a IF$\pi g\beta$-normal space $X$ to a IF space $Y$, then $Y$ is IF$\pi g\beta$-normal.

**Proof.** Let $A$ and $B$ be any two disjoint IF$\pi g\beta$-closed subsets of $Y$. Since $f$ is IF$\pi g\beta$-irresolute, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint IF $\pi g\beta$-closed subsets of $X$. By IF$\pi g\beta$-normality of $X$, there exist disjoint IF$\beta$-open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since $f$ is IF pre $\beta$-open and IF bijection function, we have $f(U)$ and $f(V)$ are disjoint IF$\beta$-open sets in $Y$ such that $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Therefore, $X$ is IF$\pi g\beta$-normal.

**Corollary 6.4.** If $f : X \to Y$ is IF$\beta$-continuous, IF regular open and IF pre $\beta$-open bijection function from a IF$\pi g\beta$-normal space $X$ to a IF space $Y$, then $Y$ is IF$\pi g\beta$-normal.
Theorem 6.7. If f: X → Y is a IF pre β-open, IF πgβ - irresolute and IF almost β– irresolute surjection function from a IF πgβ - normal space X onto a IF space Y, then Y is IFπgβ - normal.

**Proof.** Let A be a IF πgβ – closed subset of Y and B be a IFπgβ - open subset of Y such that A ⊆ B. Since f is IFπgβ - irresolute, we obtain that f⁻¹(A) is IFπgβ – closed in X and f⁻¹(B) is IFπgβ - open in X such that f⁻¹(A) ⊆ f⁻¹(B). Since X is IFπgβ - normal, then by the Part (c) of the Theorem 3.1, there exists a IFβ–open set U of X such that f⁻¹ (A) ⊆ U ⊆ IFβclX(U) ⊆ f⁻¹(B). Then, f(f⁻¹(A)) ⊆ f(U) ⊆ f(IFβclX(f(U))) ⊆ f(f⁻¹(B)). Since f is IF pre β–open, IF almost β– irresolute surjection, we obtain that A ⊆ f(U) ⊆ IFβclY(f(U)) ⊆ B and f(U) is IFβ–open set in Y. Hence by the Theorem 3.6, we have Y is IFπgβ – normal.

**Definition 6.2.** An IF topological space (X, τ) is said to be Strongly IF πgβ -normal [9] if for each pair A, B ⊆ X of disjoint IF πgβ - closed sets, there exist disjoint IFπgβ-open sets U and V of X such that A ⊆ U and B ⊆ V.

**Theorem 6.8.** For IF topological space (X, τ), the following are equivalent:

(i) (X, τ) is IFπgβ - normal.

(ii) For every IFπgβ - closed set A and every IFπgβ - open set U containing A, there is a IF β-clopen set V such that A ⊆ V ⊆ U.

**Proof.**

(i)⇒(ii). Let A be IFπgβ - closed and U be IFβ-open with A ⊆ U. Now, we have A∩(X\U)= 0imet, hence there exist disjoint IFβ-open sets W₁ and W₂ such that A⊆ W₁ and X\U⊆ W₂. If V=IFβcl(W₁), then V is a IFβ-clopen set satisfying A⊆ V⊆ U.

(ii)⇒(i). Obvious.

**Definition 6.3.** A IF space (X, τ) is called weakly IF πgβ - normal if disjoint IFπgβ - closed set can be separated by disjoint closed sets.

**Theorem 6.9.** If f : (X, τ)→(Y, σ) is an IF injective ,IF- contra β-continuous always πgβ - closed function and (Y, σ) is weakly IF πgβ -normal, then (X,τ) is IF- πgβ - normal.

**Proof.** Suppose that A₁, A₂ ⊆ X are IFπgβ - closed and disjoint. Since f is always IFπgβ - closed and injective, f(A₁), f(A₂)⊆Y are IF πgβ - closed and disjoint. Since (Y, σ) is weakly IF πgβ - normal, f(A₁) and f(A₂) can be separated by disjoint IF closed sets B₁, B₂⊆ Y. More over as f is IF contra β- continuous, A₁ and A₂ can be separated by disjoint IF β-open sets f⁻¹(B₁) and f⁻¹(B₂). Thus (X,τ) is IFπgβ - normal.

7. **Application of Normality**

The Property of normality of a topological space is involved in a plenty of remarkable results. Especially, in proving Urysohn’s Lemma and Tietze extension theorem. Most spaces encountered in mathematical analysis are normal Hausdorff spaces, or at least normal regular spaces. Another applications is a characterization of perfectly normal spaces, which leads to the characterization of hereditarily normal. To prove the given spaces are metrizable, it is enough to prove it as normal. An important example of a non-normal topology is given by
the Zariski topology, which is used in algebraic geometry. The Sorgenfrey plane, is based on the phenomenon that the product of normal spaces is not necessarily normal. Normality in one of the separation axiom and it plays an important role in determining duals of spaces of continuous functions (in functional analysis). The property of normality is applied in the ideal theory of C*-algebras and related topics. Normal space is applied in summability theory and artificial neural networks.

8. Conclusion

The πgβ closed sets are used to introduce the concepts πgβ-normal space. Also, the characterization, the preservation & hereditary nature of πgβ-normal spaces have been framed and analyzed. In general, the entire content will be a successful tool for the researchers for finding the path to obtain the results in the context of normal spaces in bi topology and can be extended to Functional Analysis.

By the definitions stated above and in preliminaries. We have the following diagram:

\[
\begin{array}{ccc}
\text{normal} & \rightarrow & \text{almost normal} & \rightarrow & \text{mildly normal} \\
\downarrow & & \downarrow & & \downarrow \\
\beta\text{-normal} & \rightarrow & \text{almost } \beta\text{-normal} & \rightarrow & \text{mildly } \beta\text{-normal} \\
\downarrow & & \downarrow & & \downarrow \\
\pi g\beta\text{-normal} & \rightarrow & \text{almost } \pi g\beta\text{-normal} & \rightarrow & \text{mildly } \pi g\beta\text{—normal} \\
\end{array}
\]

References