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HARMONIC GREEN FUNCTIONS FOR A PLANE DOMAIN WITH TWO TOUCHING CIRCLES AS BOUNDARY

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Abstract. The harmonic Green and Neumann functions are explicitly constructed for a particular simply connected circular plane domain with a double point at the boundary via the parqueting-reflection principle.

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1. Introduction

A proper method for constructing harmonic Green and Neumann functions for plane domains with boundaries consisting of arcs from straight lines and circles is given by the parqueting-reflection principle (Begehr & Vaitekhovich, 2011a; b; Begehr & Vaitekhovich, 2013). The continued reflections of the domain at the boundary parts have to achieve a parqueting of the entire complex plane C. Such admitted domains are e.g. strips (Begehr, 2016), half, quarter planes, plane sectors (Begehr *et al.*, 2009; Begehr & Vaitekhovich, 2012; Begehr *et al.*, 2010; Gaertner, 2006), discs, disc sectors and ring sectors (Begehr & Vaitekhovich, 2010; Begehr & Vaitekhovich, 2013; Wang, 2011; Shupeyeva, 2013), rectangles (Begehr, 2016), equilateral triangles (Begehr & Vaitekhovich, 2010; Begehr & Vaitekhovich, 2011; Begehr *et al.*, 2010), circular rings (Vaitsiakhovich, 2008a;b;c; Begehr & Vaitekhovich, 2010), lens and lunes (Begehr & Vaitekhovich, 2014), half-hexagons (Shupeyeva, 2013), certain domains in hyperbolic geometry (Akel & Begehr, 2017; Begehr, 2014). Even if the parqueting is achieved the reflections might not necessarily provide proper fundamental solutions to the Laplace operator (Begehr, 2005; Begehr, 1994) as e.g. for domains with ellipses as boundaries (Begehr *et al.*, 2017a;b).

The elements of the set of straight lines and circles in the plane are simply described in complex form as

$$a|z|^2 + \overline{b}z + b\overline{z} + c = 0,\ 0 < |b|^2 - ac,$$

a, c are real, b is complex. A point z in the complex plane C is reflected at this curve onto the point z_{re} satisfying

$$az_{re}\bar{z} + \bar{b}z_{re} + b\bar{z} + c = 0.$$

Obviously, a point on the curve is reflected onto itself.

The principle works as follows. A point z from the respective domain D is reflected at all parts of the boundary ∂D of D onto several points, each in an image domain of D. These points are taken as simple zeroes of a meromorphic function P_1 for which the initial point $z \in D$ is chosen to be a simple pole. Continuing the procedure zeroes are reflected onto poles and poles onto zeroes. In case the process is infinite the resulting product turns out to converge and then $G_1(z,\varsigma) = \log|P_1(z,\varsigma)|^2$ is the harmonic Green function for D. Here $z \in D$ is the initial point and $\varsigma \in D$ is the variable. Choosing z and ζ in any of the image domains of the reflection process the same expression for G_1 turns out as the Green function for this domain.

The same set of reflection points of $z \in D$ provides the harmonic Neumann function for D. All the points are taken as simple poles for a meromorphic function. For the product, in case it is an infinite product, proper convergence producing factors have to be introduced.

This procedure will be applied to the domain

$$D_0 = D = \left\{\frac{1}{2} < \left|z - \frac{1}{2}\right|, \ |z| < 1\right\} = \{1 < |2z - 1|, \ |z| < 1\}.$$

1. Reflecting $z \in D$ at the unit circle, $\partial\mathbb{D} = \{|z| = 1\}$, i.e. $\bar{z} = \dfrac{1}{z_{re}}$ produces

$$\tilde{D}_0 = \left\{\frac{1}{2} < \left|\frac{1}{z_{re}} - \frac{1}{2}\right|, \ \frac{1}{|z_{re}|} < 1\right\} = \{1 < |z|, \ z + \bar{z} < 2\}.$$

2. Reflecting $z \in D$ at $\left\{\left|z - \dfrac{1}{2}\right| = \dfrac{1}{2}\right\}$, i.e. $\bar{z} = \dfrac{z_{re}}{2z_{re} - 1}$ results in

$$\tilde{D}_1 = \left\{\frac{1}{3} < \left|z - \frac{2}{3}\right|, \ \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}.$$

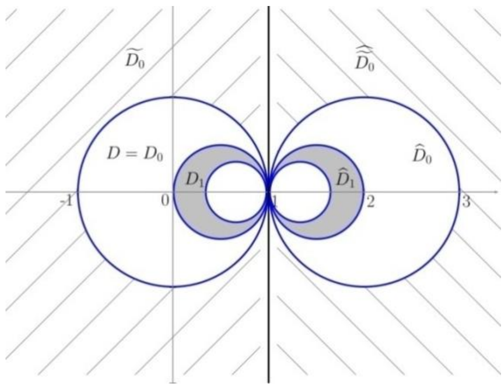

Figure 1. Domain D with the first reflections

3. Inductively, it follows

$$D_k = \left\{ \frac{1}{k+2} < \left| z - \frac{k+1}{k+2} \right|, \left| z - \frac{k}{k+1} \right| < \frac{1}{k+1} \right\}, \ k \in N_0.$$

The $D_k's$ shrink to the point $z = 1$ and provide a parqueting for the unit disc, $\overline{\mathbb{D}} = \bigcup_{k \in N_0} \overline{D}_k$. Moreover, the half plane $\{z + \bar{z} \le 2\} = \overline{\tilde{D}}_0 \cup \overline{\mathbb{D}}$. For a parqueting of the complex plane C all the $D_k's$ have to be reflected at $\{z + \bar{z} = 2\}$. Thus D_k is reflected onto

$$\hat{D}_k = \left\{ \frac{1}{k+2} < \left| z - \frac{k+3}{k+1} \right|, \left| z - \frac{k+2}{k+1} \right| < \frac{1}{k+1} \right\}, \ k \in N_0,$$

while $\tilde{D}_0$ is reflected onto

$$\hat{\tilde{D}}_0 = \{1 < |z - 2|, \ 2 < z + \bar{z}\}.$$

Hence,

$$C = \overline{\tilde{D}}_0 \cup \overline{\hat{\tilde{D}}}_0 \bigcup_{k \in N_0} \left(\overline{D}_k \cup \overline{\hat{D}}_k \right).$$

2. Harmonic Green functions for D.

In order to construct the meromorphic function which will determine the Green function the continued reflection of $z \in D_0$ have to be found.

1. $z \in D_0$ reflected at $\{|z| = 1\}$ gives $\tilde{z}_0 = \frac{1}{\bar{z}} \in \tilde{D}_0$ and reflected at $\left\{ \left| z - \frac{1}{2} \right| = \frac{1}{2} \right\}$ leads to $z_1 = \frac{\bar{z}}{2\bar{z} - 1} \in D_1$.
2. $z_1 \in D_1$ is reflected at $\left| z - \frac{2}{3} \right| = \frac{1}{3}$ onto

$$z_2 = \frac{2\bar{z}_1 - 1}{3\bar{z}_1 - 2} = \frac{1}{z - 2}.$$

Inductively

$$z_{2k} = \frac{(k-1)z - k}{kz - (k+1)} \in D_{2k} = \left\{ \frac{1}{2k+2} < \left| z - \frac{2k+1}{2k+2} \right|, \left| z - \frac{2k}{2k+1} \right| < \frac{1}{2k+1} \right\}$$

is reflected at $\left\{ \left| z - \frac{2k+1}{2k+2} \right| = \frac{1}{2k+2} \right\}$ onto

$$z_{2k+1} = \frac{(k+1)\bar{z} - k}{(k+2)\bar{z} - (k+1)} \in D_{2k+1}$$

$$= \left\{ \frac{1}{2k+3} < \left| z - \frac{2k+2}{2k+3} \right|, \left| z - \frac{2k+1}{2k+2} \right| < \frac{1}{2k+2} \right\}.$$

3. Finally, all these points have to be reflected at the line $\{z+\bar z=2\}$. The original point $z\in D_0$ is mapped onto $\hat z=2-\bar z$ and $\tilde{\hat z}=\frac{1}{\bar z}\in\tilde{\hat D}_0$ onto $\hat{\tilde{\hat z}}=\frac{2z-1}{z}\in\hat{\tilde{\hat D}}_0$, similarly, $z_{2k}\in D_{2k}$ and $z_{2k+1}\in D_{2k+1}$, $k\in N_0$, onto

$$\hat z_{2k}=\frac{(k+1)\bar z-(k+2)}{k\bar z-(k+1)}=\frac{1}{\overline{z_{2k+2}}}\in\hat D_{2k},\ \hat z_{2k+1}=\frac{(k+3)z-(k+2)}{(k+2)z-(k+1)}=\frac{1}{\overline{z_{2k+3}}}\in\hat D_{2k+1}.$$

These reflections lead to the meromorphic function

$$P_1(z,\varsigma)=\frac{z}{\bar z}\frac{1-\bar z\varsigma}{\varsigma-z}\frac{\varsigma+\bar z-2}{z\varsigma+1-2z}\prod_{k=0}^{\infty}\frac{\varsigma-z_{2k+1}}{\varsigma-z_{2k+2}}\frac{\varsigma-\hat z_{2k+2}}{\varsigma-\hat z_{2k+1}}. \tag{1}$$

Lemma 1. The infinite product P_1 from (1) converges for $z,\varsigma\in D,\ \varsigma\neq z$.

Proof. From

$$\frac{\varsigma-z_{2k+1}}{\varsigma-z_{2k+2}}-1=\frac{z_{2k+2}-z_{2k+1}}{\varsigma-z_{2k+2}},\quad \frac{\varsigma-\hat z_{2k+2}}{\varsigma-\hat z_{2k+1}}-1=\frac{\hat z_{2k+1}-\hat z_{2k+2}}{\varsigma-\hat z_{2k+1}},$$

and

$$z_{2k+2}-z_{2k+1}=\frac{1-|z|^2}{[(k+1)z-(k+2)][(k+2)\bar z-(k+1)]},$$

$$\hat z_{2k+1}-\hat z_{2k+2}=\frac{4\left(1-|z|^2\right)}{[(k+2)z-(k+1)][(k+1)\bar z-(k+2)]},$$

follow

$$|z_{2k+2}-z_{2k+1}|\le\frac{1}{\sqrt{k(k+1)(k+2)(k+3)}}<\frac{1}{k(k+2)},$$

$$|\hat z_{2k+1}-\hat z_{2k+2}|\le\frac{4}{\sqrt{k(k+1)(k+2)(k+3)}}<\frac{4}{k(k+2)}.$$

As $\lim_{k\to\infty}z_k=1$ for $z_k,\hat z_k\notin D,\varsigma\in D$ the respective denominators are bounded away from 0. Obviously, $P_1(1,\varsigma)=1$ because for $z=1$ the relations $z_{2k+1}=z_{2k+2}=\hat z_{2k+1}=\hat z_{2k+2}=1$ hold.

Lemma 2. The function $|P_1|$ from (1) is symmetric, i.e. $|P_1(z,\varsigma)|=|P_1(\varsigma,z)|$ for $z,\varsigma\in D,\ \varsigma\neq z.$

Proof. From

$$\frac{\varsigma-z_{2k+1}}{\varsigma-z_{2k+2}}=\frac{\bar z-\overline{\varsigma_{2k+1}}}{z-\hat\varsigma_{2k-1}}\frac{(k+2)\varsigma-(k+1)}{(k+1)\varsigma-k}\frac{(k+1)z-(k+2)}{(k+2)\bar z-(k+1)},$$

$$\frac{\varsigma-\hat z_{2k+2}}{\varsigma-\hat z_{2k+1}}=\frac{\bar z-\overline{\hat\varsigma_{2k+2}}}{z-\varsigma_{2k+4}}\frac{(k+1)\varsigma-(k+2)}{(k+2)\varsigma-(k+3)}\frac{(k+2)z-(k+1)}{(k+1)\bar z-(k+2)},$$

follows

$$\prod_{k=0}^{\infty}\frac{\varsigma-z_{2k+1}}{\varsigma-z_{2k+2}}\frac{\varsigma-\hat z_{2k+2}}{\varsigma-\hat z_{2k+1}}=\prod_{k=0}^{\infty}\overline{\frac{z-\varsigma_{2k+1}}{z-\varsigma_{2k+2}}}\ \overline{\frac{z-\hat\varsigma_{2k+2}}{z-\hat\varsigma_{2k+1}}}\ \frac{z-\varsigma_{2k+2}}{z-\varsigma_{2k+4}}\frac{z-\hat\varsigma_{2k+1}}{z-\hat\varsigma_{2k-1}}\alpha_k,$$

with

$$\alpha_k=\frac{(k+1)\varsigma-(k+2)}{(k+2)\varsigma-(k+3)}\frac{(k+2)\varsigma-(k+1)}{(k+1)\varsigma-k}\frac{(k+1)z-(k+2)}{(k+2)\bar z-(k+1)}\frac{(k+2)z-(k+1)}{(k+1)\bar z-(k+2)}.$$

Observing

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left|\left\{\frac{z-\varsigma_{2k+2}}{z-\varsigma_{2k+4}}\ \frac{z-\hat{\varsigma}_{2k+1}}{z-\hat{\varsigma}_{2k-1}}\alpha_k\right|$$

$$=\lim_{n\to\infty}\left|\frac{z-\hat{\varsigma}_{2n+1}}{z-\hat{\varsigma}_{-1}}\ \frac{z-\varsigma_2}{z-\varsigma_{2n+4}}\ \frac{\varsigma-2}{(n+2)\varsigma-(n+3)}\ \frac{(n+2)\varsigma-(n+1)}{\varsigma}\right|=\left|\frac{z\varsigma+1-2z}{z\varsigma+1-2\varsigma}\right|$$

then $|P_1(z,\varsigma)|=|P_1(\varsigma,z)|$ follows.

Theorem 1. The function $G_1(z,\varsigma)=\log|P_1(z,\varsigma)|^2\ z,\varsigma\in D,\ \varsigma\neq z$ is the harmonic Green function for the domain D.

Proof. Because $P_1(z,\cdot)$ is meromorphic in D with a simple pole at the point z, the function $G_1(z,\cdot)$ is harmonic in $D\setminus\{z\}$. Moreover, $G_1(z,\varsigma)+\log|\varsigma-z|^2$ is harmonic in D. To check its boundary behavior one observes for $|z|=1$

$$P_1(z,\varsigma)=\frac{\bar{z}}{z}\ \frac{z-\varsigma}{\bar{z}\varsigma-1}\ \frac{z\varsigma+1-2z}{2-\bar{z}-\varsigma}\prod_{k=0}^{\infty}\frac{\varsigma-z_{2k+2}}{\varsigma-z_{2k+1}}\ \frac{\varsigma-\hat{z}_{2k+1}}{\varsigma-\hat{z}_{2k+2}}=\frac{1}{P_1(z,\varsigma)},$$

i.e. $|P_1(z,\varsigma)|=1$, as for $|z|=1$ the relations $z_{2k+2}=z_{2k+1},\ \hat{z}_{2k+1}=\hat{z}_{2k+2}$ hold.

Inserting $\left|z-\frac{1}{2}\right|=\frac{1}{2}$, i.e. $2|z|^2=z+\bar{z}$ or $z=z_1$ into the expression for $P_1(z,\varsigma)$ and observing $z_{2k+1}=z_{2k},\ \hat{z}_{2k+1}=\hat{z}_{2k}$ shows $P_1(z_1,\varsigma)=\dfrac{1}{P_1(z,\varsigma)}$, so that again $|P_1(z,\varsigma)|=1$.

3. Poisson kernel for *D*.

The Green function $G_1(z,\varsigma)$ provides the Poisson kernel as

$$g_1(z,\varsigma)=-\frac{1}{2}\partial_{\nu_\varsigma}G_1(z,\varsigma),\ z\in D,\ \varsigma\in\partial D,$$

where ∂_{ν_ς} is the outward normal derivative on ∂D. On $|\varsigma|=1$ it is given by $\partial_{\nu_\varsigma}=\varsigma\partial_\varsigma+\bar{\varsigma}\partial_{\bar{\varsigma}}$, so that applied to real functions $\partial_{\nu_\varsigma}=2\operatorname{Re}\varsigma\partial_\varsigma$. On the part $\left|\varsigma-\frac{1}{2}\right|=\frac{1}{2}$ similarly $\partial_{\nu_\varsigma}=-(2\varsigma-1)\partial_\varsigma-(2\bar{\varsigma}-1)\partial_{\bar{\varsigma}}$, so that for real functions $\partial_{\nu_\varsigma}=-2\operatorname{Re}(2\varsigma-1)\partial_\varsigma$.

Besides the above representation the Green function can also be rewritten as

$$G_1(z,\varsigma)=\log\left|\frac{z}{\bar{z}}\ \frac{1-\bar{z}\varsigma}{\varsigma-z}\ \frac{\varsigma+\bar{z}-2}{z\varsigma+1-2z}\ \frac{\varsigma-z_1}{\varsigma-\hat{z}_1}\prod_{k=1}^{\infty}\frac{\varsigma-z_{2k+1}}{\varsigma-z_{2k}}\ \frac{\varsigma-\hat{z}_{2k}}{\varsigma-\hat{z}_{2k+1}}\right|^2.$$

Thus either

$$\begin{aligned}\partial_\nu G_1(z,\varsigma)=&-\frac{1}{\varsigma-z}-\frac{\bar{z}}{1-\bar{z}\varsigma}-\frac{z}{z\varsigma+1-2z}+\frac{1}{\varsigma-\bar{z}-2}\\&+\sum_{k=1}^{\infty}\left[\frac{1}{\varsigma-z_{2k+1}}-\frac{1}{\varsigma-z_{2k+2}}+\frac{1}{\varsigma-\hat{z}_{2k+2}}-\frac{1}{\varsigma-\hat{z}_{2k+1}}\right]\end{aligned}\tag{2}$$

or

$$\partial_{\varsigma} G_1(z,\varsigma) = -\frac{1}{\varsigma - z} + \frac{1}{\varsigma - z_1} - \frac{\bar{z}}{1-\bar{z}\varsigma} + \frac{z}{z\varsigma + 1 - 2z} + \frac{1}{\varsigma + \bar{z} - 2} - \frac{1}{\varsigma - \hat{z}_1}$$
$$+ \sum_{k=1}^{\infty} \left[\frac{1}{\varsigma - z_{2k+1}} - \frac{1}{\varsigma - z_{2k}} + \frac{1}{\varsigma - \hat{z}_{2k}} - \frac{1}{\varsigma - \hat{z}_{2k+1}} \right]. \quad (3)$$

Both representations are used in the next proof.

Theorem 2. The Poisson kernel satisfies on $|\varsigma| = 1$

$$g_1(z,\varsigma) = \frac{\varsigma}{\varsigma - z} + \frac{\bar{\varsigma}}{\overline{\varsigma - z}} - 1 + O(1-|z|^2) \text{ for } |z| \to 1,$$

$$g_1(z,\varsigma) = O(2|z|^2 - z - \bar{z}) \text{ for } \left|z - \frac{1}{2}\right| \to \frac{1}{2}.$$

For $\left|\varsigma - \frac{1}{2}\right| = \frac{1}{2}$

$$g_1(z,\varsigma) = O(1-|z|^2) \text{ for } |z| \to 1,$$

$$g_1(z,\varsigma) = -\frac{2\varsigma - 1}{\varsigma - z} - \frac{2\bar{\varsigma} - 1}{\overline{\varsigma - z}} + 2 + O(2|z|^2 - z - \bar{z}) \text{ for } \left|z - \frac{1}{2}\right| \to \frac{1}{2}.$$

For the proof two remarks are stated.

Lemma 3. For $|\varsigma| = 1$, $|z| = 1$ but $z \neq 1$ then

$$\varsigma + \bar{z} - 2 \neq 0, \; z\varsigma + 1 - 2z \neq 0.$$

For $\left|\varsigma - \frac{1}{2}\right| = \frac{1}{2}$, $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ but $z \neq 1$ then

$$\varsigma + \bar{z} - 2 \neq 0, \; z\varsigma + 1 - 2z \neq 0.$$

Proof. 1. If $\varsigma + \bar{z} - 2 = 0$ then $|\varsigma|^2 = 1 = 4 - 2(z + \bar{z}) + |z|^2$ i.e. $z + \bar{z} = 2$, so that $\operatorname{Re} z = 1$. As $|z| = 1$, then $\operatorname{Im} z = 0$, i.e. $z = 1$. For $|z| = 1$ then $z\varsigma + 1 - 2z = z(\varsigma + \bar{z} - 2)$.

2. Assume $\varsigma + \bar{z} - 2 = 0$ for $2|\varsigma|^2 = \varsigma + \bar{\varsigma}$ and $2|z|^2 = z + \bar{z}$. Then $\varsigma - \frac{1}{2} = \frac{3}{2} - \bar{z}$, hence, $4 - 2(z + \bar{z}) = 0$, i.e. $\operatorname{Re} z = 1$. Because $2|z|^2 = z + \bar{z}$ then $|z|^2 = 1$. Thus $\operatorname{Im} z = 0$ and $z = 1$.

Similarly, from $z\varsigma + 1 - 2z = 0$, i.e. $z(2-\varsigma) = 1$ follows $2\bar{z} = 2|z|^2(2-\varsigma) = (2-\varsigma)(z+\bar{z}) = 2(z+\bar{z}) - \varsigma(z+\bar{z})$, i.e. $2z = \varsigma(z+\bar{z}) = 2\varsigma|z|^2$, i.e. $1 = \varsigma\bar{z} = \bar{\varsigma}z = |z\varsigma|$, so that $\operatorname{Im} \bar{z}\varsigma = 0$. From $4 = 4|z|^2|\varsigma|^2 = (z+\bar{z})(\varsigma+\bar{\varsigma})$ follows $2 = z\varsigma + \overline{z\varsigma} = 2\operatorname{Re} z\varsigma$. Hence, $\operatorname{Re} z\varsigma = 1 = |z\varsigma|$, i.e. $\operatorname{Im} z\varsigma = 0$.

Thus $\operatorname{Im} z(\varsigma + \bar{\varsigma}) = 0$, i.e. $\operatorname{Im} z \operatorname{Re}\varsigma = 0$.

Either $\operatorname{Re}\varsigma = 0$ for $\left|\varsigma - \frac{1}{2}\right| = \frac{1}{2}$, so that $\varsigma = 0$ or $\varsigma = 1$. If $\varsigma = 0$ then $1 - 2z = 0$, but $\left|z - \frac{1}{2}\right| = \frac{1}{2}$. Or $\varsigma = 1$, then $z = 1$, or $\operatorname{Im} z = 0$ for $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, so that $z = 0$ or $z = 1$. But $z = 0$ contradicts that $z(2-\varsigma) = 1$ holds.

Proof of Theorem 2. Using (2) and observing

$$\frac{1}{\varsigma+\bar{z}-2}-\frac{z}{z\varsigma+1-2z}=\frac{1-|z|^2}{(\varsigma+\bar{z}-2)(z\varsigma+1-2z)}, \tag{4}$$

$$\frac{1}{\varsigma+\bar{z}-2}-\frac{z}{z\varsigma+1-2z}=0 \quad \text{for } z=1,$$

$$z_{2k+1}-z_{2k+2}=\frac{|z|^2-1}{[(k+1)z-(k+2)][(k+2)\bar{z}-(k+1)]}, \tag{5}$$

$$\hat{z}_{2k+1}-\hat{z}_{2k+2}=\frac{4(|z|^2-1)}{[(k+2)z-(k+1)][(k+1)\bar{z}-(k+2)]}, \tag{6}$$

then for $|\varsigma|=1$

$$\varsigma\partial_\varsigma G_1(z,\varsigma)=-\frac{\varsigma}{\varsigma-z}-\frac{\overline{\varsigma}}{\overline{\varsigma-z}}+1+O(1-|z|^2) \text{ for } |z|\to 1,$$

$$\partial_{\nu_\varsigma} G_1(z,\varsigma)=-2\left(\frac{\varsigma}{\varsigma-z}-\frac{\overline{\varsigma}}{\overline{\varsigma-z}}-1\right)+O(1-|z|^2) \text{ for } |z|\to 1.$$

Starting from (3)

$$\partial_\varsigma G_1(z,\varsigma)=\frac{1}{\varsigma-z_1}-\frac{1}{\varsigma-z}-\frac{\bar{z}}{1-\bar{z}\varsigma}-\frac{z}{z\varsigma+1-2z}+\frac{1}{\varsigma+\bar{z}-2}-\frac{1}{\varsigma-\hat{z}_1}$$
$$+\sum_{k=1}^{\infty}\left[\frac{z_{2k+1}-z_{2k}}{(\varsigma-z_{2k})(\varsigma-z_{2k+1})}+\frac{\hat{z}_{2k}-\hat{z}_{2k+1}}{(\varsigma-\hat{z}_{2k})(\varsigma-\hat{z}_{2k+1})}\right]$$

and observing

$$\frac{1}{\varsigma-z_1}-\frac{1}{\varsigma-z}=\frac{z+\bar{z}-2|z|^2}{(2z\varsigma-\varsigma-\bar{z})(\varsigma-z)}, \tag{7}$$

$$\frac{\bar{z}}{1-\bar{z}\varsigma}+\frac{z}{z\varsigma+1-2z}=\frac{z+\bar{z}-2|z|^2}{(1-\bar{z}\varsigma)(2z\varsigma+1-2z)}, \tag{8}$$

$$\frac{1}{\varsigma+\bar{z}-2}+\frac{1}{z\varsigma+1-2z}=\frac{z+\bar{z}-2|z|^2}{(\varsigma+\bar{z}-2)(2\bar{z}\varsigma+2-3z-\varsigma)}, \tag{9}$$

$$z_{2k+1}-z_{2k}=\frac{2|z|^2-z-\bar{z}}{[(k+2)\bar{z}-(k+1)][kz-(k+1)]}, \tag{10}$$

$$\hat{z}_{2k}-\hat{z}_{2k+2}=\frac{2|z|^2-z-\bar{z}}{[k\bar{z}-(k+1)][(k+2)z-(k+1)]}, \tag{11}$$

thus for $|\varsigma|=1$

$$\partial_{\nu_\varsigma} G_1(z,\varsigma)=O(2|z|^2-z-\bar{z}) \text{ for } \left|z-\frac{1}{2}\right|\to\frac{1}{2}.$$

Again starting from (3) and using for $\left|\varsigma-\frac{1}{2}\right|=\frac{1}{2}$ i.e. for $|\varsigma|^2=\varsigma+\bar{\varsigma}$

$$\frac{2\varsigma-1}{\varsigma-z_1}=2+\frac{1}{2\bar{z}\varsigma-\bar{z}-\varsigma}-\frac{2\bar{\varsigma}-1}{\varsigma-z}$$

and the formulas (7) to (11) show for $\left|\varsigma - \frac{1}{2}\right| = \frac{1}{2}$ the relation

$$(2\varsigma - 1)\partial_\varsigma G_1(z,\varsigma) = -\frac{2\varsigma - 1}{\varsigma - z} - \frac{2\overline{\varsigma} - 1}{\overline{\varsigma - z}} + 2 + O(2|z|^2 - z - \overline{z}) \text{ for } \left|z - \frac{1}{2}\right| \to \frac{1}{2}.$$

Finally from (2), using

$$\frac{1}{\varsigma - z} + \frac{\overline{z}}{1 - \overline{z}\varsigma} = \frac{1 - |z|^2}{(\varsigma - z)(1 - \overline{z}\varsigma)}$$

together with (4) to (6), for $\left|\varsigma - \frac{1}{2}\right| = \frac{1}{2}$ the asymptotic behavior

$$(2\varsigma - 1)\partial_{\nu_\varsigma} G_1(z,\varsigma) = O(1 - |z|^2) \quad \textit{for} \quad |z| \to 1$$

is seen.

4. Harmonic Neumann function for *D*.

The parqueting-reflection principle suggests to introduce the infinite product

$$\begin{aligned} Q_1(z,\varsigma) &= z\overline{z}\,\frac{\varsigma - z}{\varsigma - 1}\,\frac{1 - \overline{z}\varsigma}{1 - \varsigma}\,\frac{z\varsigma + 1 - 2z}{\varsigma - 1}\,\frac{\varsigma + \overline{z} - 2}{\varsigma - 1} \\ &\times \prod_{k=0}^{\infty}\left[\frac{\varsigma - z_{2k+2}}{\varsigma - 1}\,\frac{\varsigma - z_{2k+2}}{\varsigma - 1}\,\frac{\varsigma - \overline{z}_{2k+1}}{\varsigma - 1}\,\frac{\varsigma - \overline{z}_{2k+1}}{\varsigma - 1}\right],\ z,\varsigma \in D. \end{aligned} \tag{12}$$

Lemma 4. The infinite product Q_1 from (12) converges for $z,\varsigma \in D$.

Proof.

$$\frac{(\varsigma - z_{2k+1})(\varsigma - \hat{z}_{2k+1})}{(\varsigma - 1)^2} - 1 = \frac{z_{2k+1}\hat{z}_{2k+1} - 1 - \varsigma(z_{2k+1} + \hat{z}_{2k+1} - 2)}{(\varsigma - 1)^2},$$

$$z_{2k+1}\hat{z}_{2k+1} - 1 = -\frac{|z|^2 - 2z + 1}{|(k+2)z - (k+1)|^2},\ 2 - z_{2k+1} - \hat{z}_{2k+1} = \frac{\overline{z} - z}{|(k+2)z - (k+1)|^2},$$

$$\frac{(\varsigma - z_{2k+2})(\varsigma - \hat{z}_{2k+2})}{(\varsigma - 1)^2} - 1 = \frac{z_{2k+2}\hat{z}_{2k+2} - 1 - \varsigma(z_{2k+2} + \hat{z}_{2k+2} - 2)}{(\varsigma - 1)^2},$$

$$z_{2k+2}\hat{z}_{2k+2} - 1 = -\frac{|z|^2 - 2z + 1}{|(k+1)z - (k+2)|^2},\ 2 - z_{2k+2} - \hat{z}_{2k+2} = \frac{\overline{z} - z}{|(k+1)z - (k+2)|^2}.$$

Remark. Obviously, $Q_1(z,\varsigma)$ fails to be symmetric. The reason is the necessity to introduce convergence creating factors. Nevertheless, from the relations of the zeroes

$$\varsigma = z_{2k+1} = \frac{(k+1)\overline{z} - k}{(k+1)\overline{z} - (k+2)} \text{ or } \overline{z} = \frac{(k+1)\varsigma - k}{(k+2)\varsigma - (k+1)} = \overline{\varsigma_{2k+1}},$$

$$\varsigma = z_{2k+2} = \frac{kz - (k+1)}{(k+1)z - (k+2)} \text{ or } z = \frac{(k+2)\varsigma - (k+1)}{(k+1)\varsigma - k} = \hat{\varsigma}_{2k-1},$$

$$\varsigma = \hat{z}_{2k+1} = \frac{(k+3)z - (k+2)}{(k+2)z - (k+1)} \text{ or } z = \frac{(k+1)\varsigma - (k+2)}{(k+2)\varsigma - (k+3)} = \varsigma_{2k+4},$$

$$\varsigma = \hat{z}_{2k+2} = \frac{(k+2)\overline{z} - (k+3)}{(k+1)\overline{z} - (k+2)} \text{ or } \overline{z} = \frac{(k+2)\varsigma - (k+3)}{(k+1)\varsigma - (k+2)} = \overline{\hat{\varsigma}_{2k+2}},$$

it follows that

$$\tilde{Q}_{1,n}(\varsigma,z)=\varsigma\bar{\varsigma}\frac{z-\varsigma}{z-1}\frac{z-\frac{1}{\bar{\varsigma}}}{z-1}\frac{z+\frac{1}{\varsigma-2}}{z-1}\frac{z+\bar{\varsigma}-2}{z-1}$$
$$\times\prod_{k=0}^{n}\left[\frac{z-\varsigma_{2k+1}}{z-1}\frac{z-\hat{\varsigma}_{2k-1}}{z-1}\frac{z-\varsigma_{2k+4}}{z-1}\frac{z-\hat{\varsigma}_{2k+2}}{z-1}\right]$$

converges with n tending to ∞ to the limit

$$\tilde{Q}_1(\varsigma,z)=\varsigma\bar{\varsigma}\frac{z-\varsigma}{z-1}\frac{z\bar{\varsigma}-1}{\bar{\varsigma}(z-1)}\frac{z\varsigma+1-2z}{(\varsigma-2)(z-1)}\frac{z+\bar{\varsigma}-2}{z-1}\frac{z\varsigma+1-2\varsigma}{z\varsigma+1-2z}\frac{\varsigma-2}{\varsigma}$$
$$\times\prod_{k=0}^{\infty}\left[\frac{z-\varsigma_{2k+1}}{z-1}\frac{z-\varsigma_{2k+2}}{z-1}\frac{z-\hat{\varsigma}_{2k+1}}{z-1}\frac{z-\hat{\varsigma}_{2k+2}}{z-1}\right].$$

Hence,

$$\tilde{Q}_1(\varsigma,z)=\frac{\varsigma-z}{1-z}\frac{1-z\bar{\varsigma}}{1-z}\frac{z\varsigma+1-2\varsigma}{z-1}\frac{z+\bar{\varsigma}-2}{z-1},$$
$$\times\prod_{k=0}^{\infty}\left[\frac{z-\varsigma_{2k+1}}{z-1}\frac{z-\varsigma_{2k+2}}{z-1}\frac{z-\hat{\varsigma}_{2k+1}}{z-1}\frac{z-\hat{\varsigma}_{2k+2}}{z-1}\right].$$

Theorem 3. The function $N_1(\varsigma,z)=-\log|Q_1(z,\varsigma)|^2$, $z,\varsigma\in D$, $\varsigma\neq z$, is a harmonic Neumann function for D satisfying, in particular $\partial_{\nu_\varsigma}N_1(z,\varsigma)=0$ for $\varsigma\in\partial D\setminus\{1\}$, $z\in D$.

Proof. As $Q_1(z,\cdot)$ is analytic in D with a simple zero at the point z the function $N_1(z,\varsigma)$ is harmonic in $D\setminus\{z\}$ and continuously differentiable on $\overline{D}$ up to the point $\varsigma=1$. Moreover, $N_1(\varsigma,z)+\log|\varsigma-z|^2$ is harmonic in D.

$$\partial_\varsigma N_1(z,\varsigma)=-\frac{1}{\varsigma-z}+\frac{\bar{z}}{1-\bar{z}\varsigma}-\frac{z}{z\varsigma+1-2z}-\frac{1}{\varsigma+\bar{z}-2}+\frac{4}{\varsigma-1}$$
$$+\sum_{k=0}^{\infty}\left[\frac{1}{\varsigma-z_{2k+1}}+\frac{1}{\varsigma-z_{2k+2}}+\frac{1}{\varsigma-\hat{z}_{2k+1}}+\frac{1}{\varsigma-\hat{z}_{2k+2}}-\frac{4}{\varsigma-1}\right]$$
$$=-\frac{1}{\varsigma-z}+\frac{\bar{z}}{1-\bar{z}\varsigma}+\frac{2}{\varsigma-1}+\frac{1}{\varsigma-1}\left[\frac{1-z}{\varsigma z+1-2z}-\frac{1-\bar{z}}{\varsigma+\bar{z}-2}\right]$$
$$-\frac{1}{\varsigma-1}\sum_{k=0}^{\infty}\left[\frac{1-\bar{z}}{[(k+2)\bar{z}-(k+1)]\varsigma-[(k+1)\bar{z}-k]}\right.$$
$$+\frac{1-z}{[(k+1)z-(k+2)]\varsigma-[kz-(k+1)]}$$
$$-\frac{1-z}{[(k+2)z-(k+1)]\varsigma-[(k+3)z-(k+2)]}$$
$$\left.-\frac{1-\bar{z}}{[(k+1)\bar{z}-(k+2)]\varsigma-[(k+2)\bar{z}-(k+3)]}\right].$$

For $|\varsigma|=1$from

$$\operatorname{Re}\frac{\varsigma}{\varsigma-1}\frac{1-\bar{z}}{[(k+2)\bar{z}-(k+1)]\varsigma-[(k+1)\bar{z}-k]}$$
$$=\operatorname{Re}\frac{\varsigma}{\varsigma-1}\frac{1-z}{[(k+1)z-k)]\varsigma-[(k+2)z-(k+1)]},$$

$$\mathrm{Re}\frac{\varsigma}{\varsigma-1}\frac{1-z}{[(k+1)z-(k+2)]\varsigma-[kz-(k+1)]}$$
$$=\mathrm{Re}\frac{\varsigma}{\varsigma-1}\frac{1-\bar{z}}{[k\bar{z}-(k+1)]\varsigma-[(k+1)\bar{z}-(k+2)]}$$

follows

$$\left\{\mathrm{Re}\frac{\varsigma}{\varsigma-1}\sum_{k=1}^{\infty}\left[\frac{1}{\varsigma-z_{2k+1}}+\frac{1}{\varsigma-z_{2k+2}}+\frac{1}{\varsigma-\hat{z}_{2k+1}}+\frac{1}{\varsigma-\hat{z}_{2k+2}}-\frac{4}{\varsigma-1}\right]\right\}$$
$$=\mathrm{Re}\left\{\frac{\varsigma}{\varsigma-1}\left[\frac{1}{z\varsigma+1-2z}-\frac{1-\bar{z}}{\varsigma+\bar{z}-2}\right]\right\},$$

so that for $|\varsigma|=1,\ \varsigma\neq 1$

$$\mathrm{Re}\,\varsigma\partial_{\varsigma}N_1(z,\varsigma)=\mathrm{Re}\frac{\varsigma+1}{\varsigma-1}=\frac{|\varsigma|^2-1}{|\varsigma-1|^2}.$$

Inserting for $2|\varsigma|^2=\varsigma+\bar{\varsigma}$ the relations

$$|\varsigma|^2-\varsigma=\bar{\varsigma}-|\varsigma|^2,\ \bar{\varsigma}(2\varsigma-1)=\varsigma,\ \frac{\varsigma}{\varsigma-1}=\frac{\bar{\varsigma}}{1-\bar{\varsigma}},$$

into

$$\partial_{\varsigma}N_1(z,\varsigma)=\frac{1-z}{\varsigma-1}\frac{1}{\varsigma-1}+\frac{1-\bar{z}}{\varsigma-1}\frac{1}{1-\bar{z}\varsigma}+\frac{z-1}{\varsigma-1}\frac{1}{2z-z\varsigma-1}+\frac{\bar{z}-1}{\varsigma-1}\frac{1}{\varsigma+\bar{z}-2}$$
$$-\frac{1}{\varsigma-1}\sum_{k=0}^{\infty}\left[\frac{z_{2k+1}-1}{\varsigma-z_{2k+1}}+\frac{z_{2k+2}-1}{\varsigma-z_{2k+2}}+\frac{\hat{z}_{2k+1}-1}{\varsigma-\hat{z}_{2k+1}}+\frac{\hat{z}_{2k+2}-1}{\varsigma-\hat{z}_{2k+2}}\right]$$

shows for $2|\varsigma|^2=\varsigma+\bar{\varsigma}$

$$\mathrm{Re}\,(2\varsigma-1)\partial_{\varsigma}N_1(z,\varsigma)=\mathrm{Re}\left[\frac{\varsigma}{\bar{\varsigma}}\frac{1}{\varsigma-1}\left(\frac{1-z}{\varsigma-z}+\frac{1-\bar{z}}{1-\bar{z}\varsigma}+\frac{1-z}{z\varsigma+1-2z}-\frac{1-\bar{z}}{\varsigma+\bar{z}-2}-\Sigma\right)\right]$$

with

$$\Sigma=\sum_{k=0}^{\infty}\left[\frac{z_{2k+1}-1}{\varsigma-z_{2k+1}}+\frac{z_{2k+2}-1}{\varsigma-z_{2k+2}}+\frac{\hat{z}_{2k+1}-1}{\varsigma-\hat{z}_{2k+1}}+\frac{\hat{z}_{2k+2}-1}{\varsigma-\hat{z}_{2k+2}}\right].$$

As moreover,

$$\mathrm{Re}\frac{1-\bar{z}}{1-\bar{\varsigma}}\frac{1}{[(k+2)\bar{z}-(k+1)]\varsigma-[(k+1)\bar{z}-k]}$$
$$=\mathrm{Re}\frac{1-z}{1-\bar{\varsigma}}\frac{1}{[kz-(k-1)]\varsigma-[(k+1)z-k]},$$
$$\mathrm{Re}\frac{1-z}{1-\bar{\varsigma}}\frac{1}{[(k+1)z-(k+2)]\varsigma-[kz-(k+1)]}$$
$$=\mathrm{Re}\frac{1-\bar{z}}{1-\bar{\varsigma}}\frac{1}{[(k-1)\bar{z}-k]\varsigma-[k\bar{z}-(k+1)]}$$

thus,

$$\mathrm{Re}\frac{\Sigma}{1-\bar{\varsigma}}=\mathrm{Re}\frac{1}{1-\bar{\varsigma}}\left(\frac{1-z}{\varsigma-z}+\frac{1-\bar{z}}{1-\bar{z}\varsigma}+\frac{1-z}{z\varsigma+1-2z}-\frac{1-\bar{z}}{\varsigma+\bar{z}-2}\right).$$

Hence, on $\left|\varsigma-\frac{1}{2}\right|=\frac{1}{2}$

$$\mathrm{Re}\,(2\varsigma-1)\partial_{\varsigma}N_1(z,\varsigma)=0$$

as long as $\varsigma \neq 1$.

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