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SOME INTEGRAL INEQUALITIES FOR HARMONICALLY CONVEX STOCHASTIC PROCESSES ON THE CO-ORDINATES

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Abstract. The purpose of this paper is to introduce harmonically convex stochastic processes on the co-ordinates in order to generalize the classical convex stochastic processes and to obtain new estimations. Hermite-Hadamard type inequalities and estimation for harmonically convex stochastic processes with this purpose in mind are obtained. These results are particularly interesting from optimization view point, since it provides a broader setting to study the optimization and mathematical programming problems and to compare the maximum and minimum values of a stochastic process.

Keywords: *Harmonically convex, stochastic process on co-ordinates, mean-square differentiable, mean-square integral, Hermite-Hadamard inequality.*

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1. Introduction

It is well known that, for every real convex function f on the interval $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

These are celebrated Hermite-Hadamard inequalities. In probabilistic words, they say that

$$f(EX) \leq_{cx} Ef(X) \leq_{cx} Ef(X^*), f \in C_{cx},$$

where E denotes mathematical expectation, X (respectively, X^*) is a random variable having the uniform distribution on the interval $[a,b]$ (respectively, on the set $\{a,b\}$), C_{cx} is the set of all real convex functions on $[a,b]$ and $\leq_{cx}$ stands for the so called convex order of random variables (De la Cal *et al.*, 2006).

There are many studies in recent years on some types of convexity for stochastic processes and Hermite-Hadamard inequalities for related convex stochastic processes, and it is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature. Convex stochastic processes were proposed and some properties were given for classical convex stochastic processesby Nikodem (1980). Stochastic convexity and its applications were defined by Shaked et al (1988). Jensen-convex, λ-convex stochastic processes were introduced by Skowronski (1992).The classical Hermite-Hadamard inequality to convex stochastic processes was extended by Kotrys (2012). Convex stochastic processes on the co-ordinates were introduced and Hermite-Hadamard type inequalities for these processes were obtained by Set et al (2015). Harmonically convex stochastic processes were defined and Hermite-Hadamard type inequalities were obtained by Okur et al (2018).

The authors' findings led to our motivation to build our work. The main subject of this paper is to adapt some obtained results concerning harmonically convex functions on the co-ordinates by Noor et al (2015). To harmonically convex stochastic processes on the co-ordinates and to obtain Hermite-Hadamard type inequalities and estimation for these processes. Thus, the harmonically convex stochastic processes which defined by Okur et al (2018) is extentedon two-dimensional interval in this study.

2. Preliminaries

Let us recall some important types of convexity for stochastic processes:

Definition 1. (Kotrys, 2012) Let (Ω, I, P) be an arbitrary probability space. A function $X: \Omega \to \mathbb{R}$ is called a random variable if it is I-measurable. $X: I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t,.)$ is a random variable (Kotrys, 2012).

Definition 2. (Kotrys, 2012) Let (Ω, I, P) be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval. The stochastic process $X: I \times \Omega \to \mathbb{R}$ is called almost everywhere

(i) convex if

$$X(\lambda t + (1-\lambda)s,.) \leq \lambda X(t,.) + (1-\lambda)X(s,.)$$

for all $t, s \in I$ and $\lambda \in [0,1]$,

(ii) λ-convex if

$$X(\lambda t + (1-\lambda)s,.) \leq \lambda X(t,.) + (1-\lambda)X(s,.)$$

for all $t, s \in I$ and λ is a fixed number in $(0,1)$.

(iii) Jensen-convex if

$$X\left(\frac{t+s}{2},.\right) \leq \frac{X(t,.) + X(s,.)}{2}$$

for all $t, s \in I$ (Kotrys,2012).

Let us give some basic definitions and notions about continuity concepts and differentiability for stochastic processes, and a mean-square integral of a stochastic process.

Definition 3. (Kotrys, 2012) Let (Ω, I, P) be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is called

(i) continuous in probability on I if for all $t_0 \in I$ if

$$P - \lim_{t \to t_0} X(t,\cdot) = X(t_0,\cdot),$$

where $P-$ lim denotes limit in probability,

(ii) mean-square continuous on I if for all $t_0 \in I$ if

$$\lim_{t\to t_0} E[X(t,\cdot) - X(t_0,\cdot)]^2 = 0,$$

where $E[X(t,\cdot)]$ denotes expectation value of the random variable $X(t,\cdot)$,

(iii) increasing (decreasing) if for all $t, s \in I$ such that t$< s$ if

$$X(t,\cdot) \le X(s,\cdot)\big(X(t,\cdot) \ge X(s,\cdot)\big),$$

(iv) monotonic if it is increasing or decreasing,

(v) mean-square differentiable at a point if $t \in I$ if there is a random variable $X'(t,\cdot): I \times \Omega \to \mathbb{R}$ such that

$$X'(t) = P - \lim_{t\to t_0} \frac{X(t,\cdot) - X(t_0,\cdot)}{t - t_0}.$$

We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I (Kotrys, 2012).

Definition 4. (Kotrys, 2012) Let (Ω, I, P) be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval and $X: I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t,.)^2] \le \infty$ for all $t \in I$. Let $[a,b] \subset I$,$a = t_0 < t_1 \dots t_n = b$ be a partition of $[a,b]$ and $\Theta_k \in [t_{k-1}, t_k]$ arbitrary for $k = 1, \dots, n$. A random variable $Y: \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,.)$ on $[a,b]$ if the following identity holds:

$$\lim_{n\to\infty} E\left(\sum_{k=1}^{n} X(\Theta_k).(t_k - t_{k-1}) - Y\right)^2 = 0.$$

Then we can write almost everywhere

$$\int_a^b X(t,\cdot)dt = Y(.).$$

The mean-square integral operator is increasingon $[a,b]$ almost everywhere, that is,

$$X(t,\cdot) \le Z(t,\cdot) \Rightarrow \int_a^b X(t,\cdot)dt \le \int_a^b Z(t,\cdot)dt.$$

Now, we recall the well-known Hermite-Hadamard integral inequality for convex stochastic processes:

Theorem 1. (Kotrys, 2012) If $X: I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $I \times \Omega$, then for any $a, b \in I$, $a < b$ we have almost everywhere

$$X\left(\frac{a+b}{2},\cdot\right) \le \frac{1}{b-a}\int_a^b X(t,\cdot)\,dt \le \frac{X(a,\cdot)+X(b,\cdot)}{2}.$$

Also, Set et al (2015) established the following similar inequality of Hadamard's type for co-ordinated convex stochastic processes on a rectangle from the plane $\mathbb{R}^2$:

Definition 5. (Set *et al*., 2015) Let us consider a two-dimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A stochastic process $X: \Delta \times \Omega \to \mathbb{R}$ is said to be convex on $\Delta \times \Omega$ if the following inequality holds almost everywhere

$$X\big((\lambda t_1 + (1-\lambda)t_2, \lambda s_1 + (1-\lambda)s_2),.\big) \leq \lambda X\big((t_1,s_1),\cdot\big) + (1-\lambda)X\big((t_2,s_2),\cdot\big)$$

for all $(t_1,s_1),(t_2,s_2) \in \Delta$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be concave on Δ (Set et al, 2015).

Theorem 2. (Set *et al*., 2015) Suppose that $X: \Delta \times \Omega \to \mathbb{R}$ is co-ordinated convex on Δ. Then the following inequalities are true almost everywhere (Set et al, 2015):

$$X\left(\left(\frac{a+b}{2},\frac{c+d}{2}\right),\cdot\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b X\left(\left(t,\frac{c+d}{2}\right),\cdot\right)dt + \frac{1}{d-c}\int_c^d X\left(\left(\frac{a+b}{2}\right),s),\cdot\right)ds\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d X\big((t,s),\cdot\big)\,dtds$$

$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \Big(X\big((t,c),\cdot\big) + X\big((t,d),\cdot\big)\Big)dt\right.$$

$$\left. + \frac{1}{d-c}\int_c^d \Big(X\big((a,s),\cdot\big) + X\big((b,s),\cdot\big)\Big)ds\right]$$

$$\leq \frac{X\big((a,c),\cdot\big) + X\big((a,d),\cdot\big) + X\big((b,c),\cdot\big) + X\big((b,d),\cdot\big)}{4}.$$

The above inequalities are sharp.

Let us consider the Hermite-Hadamard integral inequality for harmonically convex stochastic processes:

Definition 6. (Okur *et al*., 2018) Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A stochastic process $X: I \times \Omega \to \mathbb{R}$ is said to be a harmonically convex stochastic process almost everywhere, if

$$X\left(\frac{ts}{\lambda t + (1-\lambda)s},\cdot\right) \leq \lambda X(s,\cdot) + (1-\lambda)X(t,\cdot)$$

for all $t,s \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then X is said to be a harmonically concave almost everywhere (Okur *et al*., 2018).

The following result of the Hermite-Hadamard type inequalities holds:

Theorem 3. (Okur *et al*., 2018) Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $X: I \times \Omega \to \mathbb{R}$ be a harmonically convex stochastic process and $a,b \in I^{\circ}$ with $a < b$. If $X \in L[a,b]$ then the following inequalities holdalmost everywhere

$$X\left(\frac{2ab}{a+b},\cdot\right)\leq\frac{ab}{b-a}\int_a^b\frac{X(t,\cdot)}{t^2}dt\leq\frac{X(a,\cdot)+X(b,\cdot)}{2}. \quad (1)$$

The above inequalities are sharp (Okur *et al.*, 2018).

3. Main Results

Motivated by Set et al (2015) and as a contribution to Okur et al (2018), we explore a new concept of convex stochastic processes and introduce particularly harmonically convex stochastic processes on the co-ordinates in present study. By virtue of this new concept, we present Hermite-Hadamard inequalities and an interesting important estimation for these stochastic processes.

Throughout this section, let us consider the two-dimensional interval $\Delta= [a,b]\times[c,d]$ in $(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$ from here.

Definition 7. A stochastic process $X:\Delta\times\Omega\to\mathbb{R}$ is said to be a harmonically convex on Δ, if the following inequality holdsalmost everywhere

$$X\left(\left(\frac{t_1s_1}{\lambda t_1+(1-\lambda)s_1},\frac{t_2s_2}{\lambda t_2+(1-\lambda)s_2}\right),\cdot\right)\leq\lambda X\big((t_1,s_1),\cdot\big)+(1-\lambda)X\big((t_2,s_2),\cdot\big)$$

for all $(t_1,s_1),(t_2,s_2)\in\Delta$ and $\lambda\in[0,1]$. If the above inequality is reversed, then $X$is said to be a harmonically concave on Δ.

Definition 8. A stochastic process $X:\Delta\times\Omega\to\mathbb{R}$ is said to be a harmonically convex on the co-ordinates on Δ if the partial mappings $X_s:[a,b]\times\Omega\to\mathbb{R}, X_s(u,\cdot):=X\big((u,s),\cdot\big)$ and $X_t:[c,d]\times\Omega\to\mathbb{R}, X_t(v,\cdot):=X((t,v),\cdot)$ defined for all $t\in[a,b]$ and $s\in[c,d]$ are harmonically convex

almost everywhere.

Now we give a formal definition of coordinated harmonically convex stochastic processes:

Definition 9. A stochastic process $X:\Delta\times\Omega\to\mathbb{R}$ is said to be a harmonically convex stochastic process on Δ almost everywhere, if

$$X\left(\left(\frac{t_1s_1}{\phi t_1+(1-\phi)s_1},\frac{t_2s_2}{\theta t_2+(1-\theta)s_2}\right),\cdot\right)$$
$$\leq\phi\theta X\big((t_1,s_1),\cdot\big)+\phi(1-\theta)X\big((t_1,s_2),\cdot\big)$$
$$+(1-\phi)\theta X\big((t_2,s_1),\cdot\big)+(1-\phi)(1-\theta)X\big((t_2,s_2),\cdot\big)$$

for all $(t_1,s_1),(t_2,s_2)\in\Delta$ and $\phi,\theta\in[0,1]$.

Lemma 1. Every harmonically convex stochastic process $X:\Delta\times\Omega\to\mathbb{R}$ is harmonically convex on the co-ordinatesalmost everywhere.

Proof. Suppose that $X:\Delta\times\Omega\to\mathbb{R}$ is a harmonically convex stochastic process on Δ. Consider $X_t:[c,d]\times\Omega\to\mathbb{R}$, $X_t(v,\cdot):=X\big((t,v),\cdot\big)$. Then for all $\lambda\in[0,1]$, $s_1,s_2\in[c,d]$, the following inequality holds almost everywhere.

$$X_t((\lambda s_1 + (1-\lambda)s_2),\cdot) = X\big((t, \lambda s_1 + (1-\lambda)s_2),\cdot\big)$$
$$= X\big((\lambda t_1 + (1-\lambda)t_2, \lambda s_1 + (1-\lambda)s_2),\cdot\big)$$
$$\le \lambda X((t_1, s_1),\cdot) + (1-\lambda)X((t_2, s_1),\cdot) = \lambda X_{t_1}(s_1,\cdot) + (1-\lambda)X_{t_2}(s_1,\cdot)$$

which shows the harmonically convexity of X_t. The fact that $X_s\colon [a,b]\times\Omega\to\mathbb{R}$, $X_s(u,\cdot) \coloneqq X(u,s),\cdot)$ is also harmonically convex on $[a,b]$ for all $s\in[c,d]$ goes likewise, and we shall omit the details.

Theorem 3. Suppose that $X\colon \Delta\times\Omega\to\mathbb{R}$ is harmonically convex on the co-ordinates on Δ. Then the following inequalities hold almost everywhere:

$$X\left(\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right),\cdot\right)$$
$$\le \frac{1}{2}\left[\frac{ab}{b-a}\int_a^b \frac{1}{t^2} X\left(\left(t, \frac{2cd}{c+d}\right),\cdot\right)dt + \frac{cd}{d-c}\int_c^d \frac{1}{s^2} X\left(\left(\frac{2ab}{a+b}, s\right),\cdot\right)ds\right]$$
$$\le \frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2} X((t,s),.)dtds$$
$$\le \frac{1}{4}\left[\frac{ab}{b-a}\int_a^b \frac{1}{t^2}\Big(X\big((t,c),\cdot\big) + X\big((t,d),\cdot\big)\Big)dt\right.$$
$$\left.+\frac{cd}{d-c}\int_c^d \frac{1}{s^2}\Big(X\big((a,s),\cdot\big) + X\big((b,s),\cdot\big)\Big)ds\right]$$
$$\le \frac{X\big((a,c),.\big) + X\big((a,d),.\big) + X\big((b,c),.\big) + X\big((b,d),.\big)}{4}. \tag{2}$$

The above inequalities are sharp.

Proof. Since $X\colon \Delta\times\Omega\to\mathbb{R}$ is harmonically convex on the co-ordinates it follows that the mapping $X_t\colon [c,d]\times\Omega\to\mathbb{R}$, $X_t(s,\cdot)\coloneqq X\big((t,s),\cdot\big)$ is harmonically convex on $[c,d]$ for all $t\in[a,b]$. Then almost everywhere

$$X_t\left(\frac{2cd}{c+d},\cdot\right) \le \frac{cd}{d-c}\int_c^d \frac{1}{s^2} X_t(s,\cdot)ds \le \frac{X_t(c,\cdot) + X_t(d,\cdot)}{2}.$$

That is,

$$X\left(\left(t, \frac{2\mathrm{cd}}{c+d},\cdot\right)\right) \le \frac{cd}{c+d}\int_c^d \frac{1}{s^2} X((t,s),\cdot)ds \le \frac{X((t,c),\cdot) + X((t,d),\cdot)}{2}.$$

Integrating this inequality on $[a,b]$, we have

$$\frac{ab}{b-a}\int_a^b \frac{1}{t^2} X\left(\left(t,\frac{2cd}{c+d}\right),\cdot\right)dt \leq \frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2}X((t,s),.)dtds$$

$$\leq \frac{ab}{2(b-a)}\int_a^b \frac{1}{t^2}\Big(X\big((t,c),\cdot\big)+X\big((t,d),\cdot\big)\Big)dt. \tag{3}$$

By a similar argument applied for the mapping $X_s\colon [a,b]\times\Omega\to\mathbb{R}$, $X_s(u,\cdot) \coloneqq X(u,s),\cdot)$ we get

$$\frac{cd}{d-c}\int_c^d \frac{1}{s^2} X\left(\left(\frac{2ab}{a+b},s\right),\cdot\right)ds \leq \frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2}X((t,s),.)dtds$$

$$\leq \frac{cd}{2(d-c)}\int_a^b \frac{1}{s^2}\Big(X\big((a,s),\cdot\big)+X\big((b,s),\cdot\big)\Big)ds. \tag{4}$$

Summing the inequalities (3) and (4), we get the second and the third inequality in (2). By the inequality (1) we also have

$$X\left(\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right),\cdot\right) \leq \frac{ab}{b-a}\int_a^b \frac{1}{t^2}X\left(\left(t,\frac{2cd}{c+d}\right),\cdot\right)dt,$$

$$X\left(\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right),\cdot\right) \leq \frac{cd}{d-c}\int_c^d \frac{1}{s^2}X\left(\left(\frac{2ab}{a+b},s\right),\cdot\right)ds$$

which give, by addition, the first inequality in (2). Finally, by the inequality (1) we also have

$$\frac{ab}{b-a}\int_a^b \frac{1}{t^2}X\big((t,c),\cdot\big)dt \leq \frac{X\big((a,c),.\big)+X\big((b,c),.\big)}{2},$$

$$\frac{ab}{b-a}\int_a^b \frac{1}{t^2}X\big((t,d),\cdot\big)dt \leq \frac{X\big((a,c),.\big)+X\big((b,c),.\big)}{2},$$

$$\frac{cd}{d-c}\int_c^d \frac{1}{s^2}X\big((a,s),\cdot\big)ds \leq \frac{X\big((a,c),.\big)+X\big((a,d),.\big)}{2},$$

$$\frac{cd}{d-c}\int_c^d \frac{1}{s^2}X\big((b,s),\cdot\big)ds \leq \frac{X\big((a,c),.\big)+X\big((a,d),.\big)}{2}$$

which give, by addition, the last inequality in (2). The above inequalities are sharp. Indeed, if $X\colon (0,\infty)^2\times\Omega\to\mathbb{R}, X\big((t,s),\cdot\big)=1$. Thus

$$1 = X\left(\left(\frac{t_1s_1}{\lambda t_1+(1-\lambda)s_1},\frac{t_2s_2}{\lambda t_2+(1-\lambda)s_2}\right),\cdot\right)$$

$$= \lambda X\big((t_1, s_1),\cdot\big) + (1-\lambda)X\big((t_2, s_2),\cdot\big) = 1$$

for all for all (t_1, s_1), $(t_2, s_2) \in (0,\infty)^2$ and $\lambda \in [0,1]$. Therefore X is harmonically convex on $(0,\infty)^2$. We also have

$$X\left(\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right),\cdot\right) = 1,$$

$$\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2}X((t,s),.)dtds = 1,$$

$$\frac{X\big((a,c),.\big) + X\big((a,d),.\big) + X\big((b,c),.\big) + X\big((b,d),.\big)}{4} = 1$$

which shows us the inequalities (2) are sharp.

Lemma 2. Let $X{:}\,\Delta \times \Omega \to \mathbb{R}$ be a partial differentiable mapping on Δ with $a < b$ and $c < d$. If $\frac{\partial^2 X}{\partial\phi\partial\theta} \in L(\Delta)$, then the following equality holds almost everywhere

$$\frac{abcd(b-a)(d-c)}{4}\int_0^1\int_0^1 \frac{(1-2\phi)(1-2\theta)}{(A_\phi B_\theta)^2}\frac{\partial^2 X}{\partial\phi\partial\theta}\left(\left(\frac{ab}{A_\phi}, \frac{cd}{B_\theta}\right),\cdot\right)d\phi d\theta$$
$$= \frac{X\big((a,c),.\big) + X\big((a,d),.\big) + X\big((b,c),.\big) + X\big((b,d),.\big)}{4}$$
$$+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2}X((t,s),.)dtds$$
$$-\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b \frac{1}{t^2}\Big(X\big((t,c),\cdot\big) + X\big((t,d),\cdot\big)\Big)dt\right.$$
$$\left.+\frac{cd}{d-c}\int_c^d \frac{1}{s^2}\Big(X\big((a,s),\cdot\big) + X\big((b,s),\cdot\big)\Big)ds\right]$$

where $A_\phi = \phi b + (1-\phi)a$ and $B_\theta = \theta d + (1-\theta)c$.

Proof. By integration by parts, we have almost everywhere

$$\frac{abcd(b-a)(d-c)}{4}\int_0^1\int_0^1 \frac{(1-2\phi)(1-2\theta)}{(A_\phi B_\theta)^2}\frac{\partial^2 X}{\partial\phi\partial\theta}\left(\left(\frac{ab}{A_\phi}, \frac{cd}{B_\theta}\right),\cdot\right)d\phi d\theta$$

$$= \frac{\mathrm{cd(d-c)}}{4}\int_0^1 \frac{(1-2\theta)}{{B_\theta}^2}\left\{(1-2\phi)\frac{\partial X}{\partial\theta}\left(\left(\frac{ab}{A_\phi}, \frac{cd}{B_\theta}\right),\cdot\right)\right|_0^1$$

$$+2\int_0^1 \frac{\partial X}{\partial \theta}\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)d\phi\Bigg\}d\theta$$

$$=\frac{cd(d-c)}{4}\int_0^1 \frac{(2\theta-1)}{B_\theta{}^2}\left[\frac{\partial X}{\partial \theta}\left(\left(a,\frac{cd}{B_\theta}\right),\cdot\right)+\frac{\partial X}{\partial \theta}\left(\left(b,\frac{cd}{B_\theta}\right),\cdot\right)\right]d\theta \qquad (5)$$

$$+\frac{cd(d-c)}{2}\int_0^1\int_0^1 \frac{(1-2\theta)}{B_\theta{}^2}\frac{\partial X}{\partial \theta}\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)d\phi d\theta$$

$$=\frac{1}{4}(2\theta-1)\left[X\left(\left(a,\frac{cd}{B_\theta}\right),\cdot\right)+X\left(\left(b,\frac{cd}{B_\theta}\right),\cdot\right)\right]\Bigg|_0^1$$

$$-\frac{1}{2}\int_0^1\left[X\left(\left(a,\frac{cd}{B_\theta}\right),\cdot\right)+X\left(\left(b,\frac{cd}{B_\theta}\right),\cdot\right)\right]d\theta$$

$$+\frac{1}{2}\int_0^1(1-2\theta)\,X\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)\Bigg|_0^1 d\phi+\int_0^1\int_0^1 X\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)d\phi d\theta$$

$$=\frac{X((a,c),.)+X((a,d),.)+X((b,c),.)+X((b,d),.)}{4}$$

$$-\frac{1}{2}\left[\int_0^1\left(X\left(\left(a,\frac{cd}{B_\theta}\right),\cdot\right)+X\left(\left(b,\frac{cd}{B_\theta}\right),\cdot\right)\right)d\theta\right.$$

$$\left.+\int_0^1\left(X\left(\left(\frac{ab}{A_\phi},c\right),\cdot\right)+X\left(\left(\frac{ab}{A_\phi},d\right),\cdot\right)\right)d\phi\right]$$

$$+\int_0^1\int_0^1 X\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)d\phi d\theta$$

$$=\frac{X((a,c),.)+X((a,d),.)+X((b,c),.)+X((b,d),.)}{4}$$

$$-\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b \frac{1}{t^2}\Big(X((t,c),\cdot)+X((t,d),\cdot)\Big)dt\right.$$

$$\left.+\frac{cd}{d-c}\int_c^d \frac{1}{s^2}\Big(X((a,s),\cdot)+X((b,s),\cdot)\Big)ds\right]$$

$$+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{(ts)^2}X((t,s),.)dtds$$

which completes the proof.

Theorem 4. Let $X:\Delta\times\Omega\to\mathbb{R}$ be a partial differentiable mapping on Δ. If $\frac{\partial^2 X}{\partial\phi\partial\theta}\in L(\Delta)$ and $\left|\frac{\partial^2 X}{\partial\phi\partial\theta}\right|^q, q>1$, is a harmonically convex stochastic process on the co-ordinates on Δ then the following equality holds almost everywhere:

$$\left|\frac{X((a,c),.)+X((a,d),.)+X((b,c),.)+X((b,d),.)}{4}\right.$$

$$+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{1}{(ts)^2}X((t,s),.)dtds$$

$$-\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}\big(X((t,c),\cdot)+X((t,d),\cdot)\big)dt\right.$$

$$\left.\left.+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}\big(X((a,s),\cdot)+X((b,s),\cdot)\big)ds\right]\right|$$

$$\leq\frac{ac(b-a)(d-c)}{4bd(p+1)^{2/p}}\left(\frac{C_1\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((a,c),\cdot)\right|^q+C_2\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((a,d),\cdot)\right|^q}{4}\right.$$

$$\left.+\frac{C_3\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((b,c),\cdot)\right|^q+C_4\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((b,d),\cdot)\right|^q}{4}\right)^{1/q}$$

where $A_\phi=\phi b+(1-\phi)a$, $B_\theta=\theta d+(1-\theta)c$ and

$$C_1={}_2F_1\left(2q,1;2;1-\frac{a}{b}\right)\times{}_2F_1\left(2q,1;2;1-\frac{c}{d}\right),$$

$$C_2={}_2F_1\left(2q,1;2;1-\frac{a}{b}\right)\times{}_2F_1\left(2q,2;3;1-\frac{c}{d}\right),$$

$$C_3={}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\times{}_2F_1\left(2q,1;2;1-\frac{c}{d}\right),$$

$$C_4={}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\times{}_2F_1\left(2q,2;3;1-\frac{c}{d}\right),$$

β is the Euler Beta function defined by

$$\beta(t,s)=\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}=\int_0^1\phi^{t-1}(1-\phi)^{s-1}d\phi,$$

for all $t,s>0$ and ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;\gamma)=\frac{1}{\beta(b,c-b)}\int_0^1\phi^{b-1}(1-\phi)^{c-b-1}(1-\gamma\phi)^{-a}d\phi,$$

for all $c>b>0, |\gamma|<1$.

Proof. From Lemma 2, we have almost everywhere

$$\left|\frac{X\big((a,c),.\big)+X\big((a,d),.\big)+X\big((b,c),.\big)+X\big((b,d),.\big)}{4}\right.$$
$$+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{1}{(ts)^2}X((t,s),.)dtds$$
$$-\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}\Big(X\big((t,c),\cdot\big)+X\big((t,d),\cdot\big)\Big)dt\right.$$
$$\left.\left.+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}\Big(X\big((a,s),\cdot\big)+X\big((b,s),\cdot\big)\Big)ds\right]\right|$$
$$\leq\frac{abcd(b-a)(d-c)}{4}\int_0^1\int_0^1\frac{|(1-2\phi)(1-2\theta)|}{(A_\phi B_\theta)^2}\left|\frac{\partial^2X}{\partial\phi\partial\theta}\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)\right|d\phi d\theta.$$

By using the well-known Hölder inequality for double integrals, if $X:\Delta\times\Omega\to\mathbb{R}$ is a harmonically convex stochastic process on the co-ordinates on Δ, then the inequalities hold almost everywhere:

$$\left|\frac{X\big((a,c),.\big)+X\big((a,d),.\big)+X\big((b,c),.\big)+X\big((b,d),.\big)}{4}\right.$$
$$+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{1}{(ts)^2}X((t,s),.)dtds$$
$$-\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}\Big(X\big((t,c),\cdot\big)+X\big((t,d),\cdot\big)\Big)dt\right.$$
$$\left.\left.+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}\Big(X\big((a,s),\cdot\big)+X\big((b,s),\cdot\big)\Big)ds\right]\right|$$
$$\leq\frac{abcd(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|(1-2\phi)(1-2\theta)|^p d\phi d\theta\right)^{1/p}$$
$$\times\left(\int_0^1\int_0^1(A_\phi B_\theta)^{-2q}\left|\frac{\partial^2X}{\partial\phi\partial\theta}\left(\left(\frac{ab}{A_\phi},\frac{cd}{B_\theta}\right),\cdot\right)\right|^q d\phi d\theta\right)^{1/q}$$
$$\leq\frac{abcd(b-a)(d-c)}{4bd(p+1)^{2/p}} \tag{6}$$
$$\times\left(\int_0^1\int_0^1(A_\phi B_\theta)^{-2q}\left(\phi\theta\left|\frac{\partial^2X}{\partial\phi\partial\theta}\big((a,c),\cdot\big)\right|^q+\phi(1-\theta)\left|\frac{\partial^2X}{\partial\phi\partial\theta}\big((a,d),\cdot\big)\right|^q\right.\right.$$

$$+(1-\phi)\theta\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((b,c),\cdot)\right|^q$$

$$+(1-\phi)(1-\theta)\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((b,d),\cdot)\right|^q\Bigg)d\phi d\theta\Bigg)^{1/q},$$

where an easy calculation gives

$$\int_0^1\int_0^1\left(A_\phi B_\theta\right)^{-2q}\phi\theta d\phi d\theta \tag{7}$$

$$=\frac{1}{4(bd)^{2q}}\times {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right)\times {}_2F_1\left(2q,1;2;1-\frac{c}{d}\right),$$

$$\int_0^1\int_0^1\left(A_\phi B_\theta\right)^{-2q}\phi(1-\theta)d\phi d\theta \tag{8}$$

$$=\frac{1}{4(bd)^{2q}}\times {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right)\times {}_2F_1\left(2q,2;3;1-\frac{c}{d}\right),$$

$$\int_0^1\int_0^1\left(A_\phi B_\theta\right)^{-2q}(1-\phi)\theta d\phi d\theta \tag{9}$$

$$=\frac{1}{4(bd)^{2q}}\times {}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\times {}_2F_1\left(2q,1;2;1-\frac{c}{d}\right),$$

$$\int_0^1\int_0^1\left(A_\phi B_\theta\right)^{-2q}(1-\phi)(1-\theta)\left|\frac{\partial^2 X}{\partial\phi\partial\theta}((b,d),\cdot)\right|d\phi d\theta \tag{10}$$

$$=\frac{1}{4(bd)^{2q}}\times {}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\times {}_2F_1\left(2q,2;3;1-\frac{c}{d}\right).$$

Hence, if we use (7)-(10) in (6), we obtain the desired result. This completes the proof.

4. Conclusion

In this paper, we define an important extension of convexity for stochastic processes which is called harmonically convex stochastic processes on the co-ordinates. We also obtain Hermite-Hadamard type inequalities for harmonically convex stochastic processes on the co-ordinates. In probabilistic words, it can be interpreted briefly as follows:

$$X(H.O(T),.)\leq_{cx} EX(T,.)\leq_{cx} A.O\big(X(T,.)\big), X\in HC$$

where X is a stochastic process having the harmonically convexity on the co-ordinates; $T\subset\mathbb{R}^2$; $H.O$, E and $A.O$ are respectively defined as harmonically mean, expectation value, arithmetically mean of the process X; $\leq_{cx}$ stands for the so called convex order of stochastic processes; HC is the set of all harmonically convex stochastic processes on the co-ordinates.

Therefore, we obtain estimation for harmonically convex stochastic processes on the co-ordinates. This result can be interpreted probabilistically as follows:

$$\left|A.O\big(X(T,.)\big)+EX(T,.)\right| \leq_{cx} K\left(A.O\left(\left|\frac{\partial^2 X}{\partial\phi\partial\theta}(\mathrm{T},\cdot)\right|^q\right)\right)^{1/q}, K \in \mathbb{R}$$

These inequalities are necessary to compare some values of a stochastic process with its expected value. These concepts may be particularly interesting from optimization view point, since it provides a broader setting for studying optimization and mathematical programming problems.

As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes. Applying some type of inequalities for stochastic processes is another promising direction for future research.

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