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FLOW OF A NON-NEWTONIAN FLUID PAST A POROUS SPHERE AT SMALL REYNOLDS NUMBER

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Abstract. The flow of a second-order fluid past a porous sphere has been discussed by expanding Stokes' stream function in powers of non-Newtonian dimensionless parameter and Reynolds number. Their first order effects have been taken. The flow is divided in three zones. Zone I is the region of porous sphere, zone II is the region of Stokes flow of clear second-order fluid near the surface of the sphere and zone III is the region of Oseen's flow far away from the sphere. It is found that non-Newtonian terms completely modify the stream lines and velocity components but do not affect the drag on the sphere.

Keywords: *Non-Newtonian, Porous sphere, Stokes flow, Oseen's flow.*

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1. Introduction

The flow of viscous fluid over and through a porous medium has been subject of intensive studies because if its importance in many engineering problems. In many cases the moving fluid is non-Newtonian, such as the flow of crude oil through porous rocks, the motion of synovial fluid bounded by porous cartilage in joints of a human body and the flow of muddy water near the banks of a river. The behavior of non-Newtonian fluids is explained by taking the following relations of Rivlin-Ericksen (1955) as well as that of Coleman and Noll (1960):

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu_1 \mathbf{A}_1 + \mu_2 \mathbf{A}_2 + \mu_3 \mathbf{A}_1^2, \tag{1}$$

where $\boldsymbol{\tau}$ is the stress tensor, p is the pressure, $\mathbf{I}$ is the unit matrix of order 3, μ_1, μ_2, μ_3 are the material moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical tensors defined by

$$\mathbf{A}_1 = (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T, \tag{2}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1, \tag{3}$$

where $\mathbf{V}$ is the velocity vector. The data of Philippoff (1956) on a 15% solution of polyisobutylene (B -100) in decalin at 30°C gives

$\mu_1 = 9320$ poise, $\mu_3 = 45000$ dyn cm^{-2} sec^2 , and $\mu_2 = -\frac{1}{2}\mu_3$

Proudman and Pearson (1957) have discussed the flow of an impervious solid sphere for small Reynolds number. They have considered two expansions- an inner expansion (Stokes' expansion) valid close to the sphere and an outer expansion (Oseen expansion) valid far away from its surface. Following their method Caswell and Schwarz (1961) have solved the problem of the creeping motion of the Ericksen and Rivlin fluid (1955) past a sphere at small Reynolds number. They have shown that the outer Oseen's solution far away from the surface of the sphere is not affected by non-Newtonian terms in the constitutive equations. Hence, in this paper we have assumed that Oseen's solution is valid even when the fluid is of second order.

The flow of a viscous Newtonian fluid past a porous sphere has been discussed by Srivastava and Srivastava (2005). They have divided the flow in three zones. The zone I is the region inside the porous sphere in which the flow is governed by Brinkman equation. In zone II and III, clear fluid flows and Stokes' and Oseen's are valid respectively. In all the three zones, the stream functions are expanded in powers of Reynolds number. In this paper we have discussed the flow of a non-Newtonian fluid, governed by (1)-(3), past a porous sphere following the method of Srivastava and Srivastava (2005). This type of coupled boundary value problems have been solved by Joseph and Tao (1966), Jones (1973), Srivastava and Sharma (1992), Padmavati et al (1993), Srivastava (1999) and Srivastava and Saxena (2006).

2. Formulation of the problem

Consider the flow of an incompressible non-Newtonian fluid, governed by constitutive equations (1), (2) and (3), past a porous sphere 'a' with a uniform velocity U. The sphere is fully saturated with the fluid. Let (r^*, θ, ϕ) be a set of spherical polar coordinates with origin at the centre of a sphere so that its surface is $r^* = a$. The flow in the free flow region $r^* \geq a$ is governed by the equation

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V}\right] = \nabla.\boldsymbol{\tau}, \tag{4}$$

where ρ is the density of the fluid at any point. The flow in the porous region $r^* \leq a$ inside the sphere being a slow seepage flow is governed by the following Brinkman equation (1947)

$$\rho\frac{\partial \mathbf{V}}{\partial t} = -\nabla p + \mu_1\nabla^2\mathbf{V} - \frac{\mu_1}{k}\mathbf{V}, \tag{5}$$

where μ_1 is the coefficient of viscosity, k is the permeability of the porous medium and p is the pressure at any point in the region.

Let u, v be the velocity components in the directions of r^*, θ respectively. Then the Stokes' stream function ψ in spherical polar coordinates is given by

$$u = \frac{1}{(r^*)^2\sin\theta}\frac{\partial \Psi}{\partial \theta},\ v = -\frac{1}{r^*\sin\theta}\frac{\partial \Psi}{\partial r^*}, \tag{6}$$

and this form satisfies the equation of continuity.

The problem is discussed by dividing the flow in three zones. The zone I is the region inside the porous sphere governed by the (5), zone II is the region of clear fluid flow near the surface of the sphere where Stokes' approximation holds good and zone III is the region of clear fluid far away from the sphere where Oseen's approximation holds good. Let the index in the superscript under the bracket of an entity $\chi^{(i)}, i=1,2,3$ indicates the zone in which entity is represented, then under this notation, the boundary conditions of the problem can be written as:

$$u^{(3)} = U\cos\theta,\ v^{(3)} = -U\sin\theta \text{ as } r^* \to \infty. \tag{7}$$

At the interface of the porous medium and clear fluid $r^* = a$, we assume that the velocity components and the pressure are continuous and the jump in the shearing stress $\tau_{r^*\theta}$ is given by the equation suggested by Ochao-Tapia and Whitaker (1995). These in our notation are given by:

$$u^{(1)} = u^{(2)}, v^{(1)} = v^{(2)} \text{ at } r^* = a, \tag{8}$$

$$p^{(1)} = p^{(2)} \quad \text{at} \quad r^* = a, \tag{9}$$

$$\tau^{(1)}_{r^*\theta} - \tau^{(2)}_{r^*\theta} = \frac{\beta\mu_1}{\sqrt{k}} v^{(1)} \text{ at } r^* = a, \tag{10}$$

where β is a constant of order one and sign of β may either be positive or negative, $\tau^{(i)}_{r^*\theta}$ is the shearing stress and $p^{(i)}$ is the pressure.

3. Solution of equations

Brinkman equation and Stokes' equation being similar, we choose the following Stokes' variable for zones I and II

$$\Psi = a^2\psi^{(i)},\ p = \frac{\mu_1 U}{a} P^{(i)}, i = 1,2 \tag{11}$$

$$r^* = ar, \eta = \cos\theta. \tag{12}$$

Using these variables, the Brinkman (5) for zone I can be written as:

$$D^4\psi^{(1)} - \sigma^2 D^2\psi^{(1)} = 0, \tag{13}$$

where $\sigma = \frac{a}{\sqrt{k}}$ and D is the dimensionless operator defined as

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial \eta^2}. \tag{14}$$

The equation of motion (4) governing the flow in the zone II near the surface of the sphere in terms of $\psi^{(2)}$ after eliminating the pressure can be written as [see equation (2.5) of Caswell and Schwarz (1961)]

$$\frac{1}{r^2}\frac{\partial\left(\psi^{(2)}, D^2\psi^{(2)}\right)}{\partial(r,\eta)} + \frac{2D^2}{r^2}\psi^{(2)}\left[\frac{\eta}{1-\eta^2}\frac{\partial\psi^{(2)}}{\partial r} + \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\eta}\right]$$

$$=\frac{1}{\mathrm{Re}}D^4\psi^{(2)}+\frac{1}{\mathrm{Re}}\left[\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}[r^3(1-\eta^2)^{1/2}\tau_{r\eta}]-(1-\eta^2)^{1/2}\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}\tau_{\eta\eta}]-\eta\tau_{\varphi\varphi}\right\}\right.$$
$$\left.-\frac{(1-\eta^2)}{r}\frac{\partial}{\partial\eta}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right)+\frac{\partial}{\partial\eta}\left[(1-\eta^2)^{1/2}\tau_{r\eta}\right]-\tau_{\eta\eta}-\tau_{\varphi\varphi}\right\}\right], \qquad (15)$$

where $\mathrm{Re}=\frac{Ua}{\nu}$ and $\nu=\frac{\mu_1}{\rho}$ is the kinematical coefficient of viscosity based on μ_1. The stress components $\tau_{r\eta}$, $\tau_{\eta\eta}$ etc. represent non-Newtonian part of the dimensionless stress $\left(\frac{a\tau}{\mu_1 U}\right)$.

For zone III, we introduce following Oseen's variables:

$$\xi=(\mathrm{Re})r=\mathrm{Re}\frac{r^*}{a}=\frac{Ur^*}{\nu}, \qquad (16)$$

$$\psi^{(3)}=(\mathrm{Re})^2\psi^{(2)}=\frac{(\mathrm{Re})^2\psi}{a^2U}=\frac{U}{\nu^2}\psi. \qquad (17)$$

Writing Navier – Stokes equation in terms of $\psi^{(3)}(r,\eta)$ [see, Langlois (1964), page 148, (4.7)], the expression for $\psi^{(3)}(r,\eta)$ can be expressed as:

$$\psi^{(3)}=\frac{1}{2}a^2U\left[\left\{\frac{a}{2r^*}\left(\frac{r^*}{a}\right)^2\right\}\left(1-\eta^2\right)-\frac{4B}{\mathrm{Re}}(1+\eta)\left(1-e^{-\{(r^*\mathrm{Re})/2a\}(1-\eta)}\right)\right]. \qquad (18)$$

The expression of $\psi^{(3)}(r,\eta)$ is the solution of Eq. (15) when written in Oseen's variables and non-Newtonian terms are omitted because non-Newtonian terms do not influence the flow at a large distance from the sphere. The constant B, written in the second term in the bracket, has to be calculated so that (18) matches with the solution given by Stokes' for the zone II which in turn matches with Brinkman's solution for the zone I at the surface of the sphere. It may be observed that when porous sphere is replaced by the impervious solid sphere of radius 'a' the constant $B=3/4$ and Eq.(18) becomes an exact solution of Oseen's equation (see, Langlois (1964)). It has been shown by Caswell and Schwarz (1961) that the solution (18) holds good even for the case of non-Newtonian fluids because far away from the boundary of the sphere non-Newtonian terms are negligible. Substituting Oseen's variables in Oseen's solution (18) and expanding in powers of Re, we obtain:

$$\psi^{(3)}=\frac{1}{2}\zeta^2\left(1-\eta^2\right)-B\,\mathrm{Re}\left[\zeta\left(1-\eta^2\right)-\frac{1}{4}\zeta^2\left(1-\eta^2\right)(1-\eta)+O\left(\zeta^3\right)\right]. \qquad (19)$$

For matching this stream function with the Stokes' stream function $\psi^{(2)}$, we rewrite it in Stokes' variables as:

$$(\mathrm{Re})^{-2}\psi^{(3)}=\frac{1}{2}\left(r^2-2Br\right)\left(1-\eta^2\right)+\frac{B}{4}(\mathrm{Re})\;r^2\left(1-\eta^2\right)(1-\eta)+O\left(\mathrm{Re}^2\right) \qquad (20)$$

To solve (13) and (15) in zone I and II we expand the stream function $\psi^{(i)}$ and pressure $P^{(i)}$ in powers of Re and $\lambda\left(=\mu_3U/\mu_1a\right)$ and retain here only their first power:

$$\psi^{(i)}=\psi_{00}^{(i)}+\mathrm{Re}\,\psi_{01}^{(i)}+\lambda\psi_{10}^{(i)},\; i=1,2 \qquad (21)$$

The stream function $\psi_{00}^{(1)}, \psi_{00}^{(2)}, \psi_{01}^{(1)}, \psi_{01}^{(2)}$ have been calculated by Srivastava and Srivastava (2005) which we shall present here briefly and then discuss the non-Newtonian contributions by calculating $\psi_{10}^{(1)}, \psi_{10}^{(2)}$.

4. Flow of Newtonian fluid

(i) Calculation of $\psi_{00}^{(1)}$ and $\psi_{00}^{(2)}$:

The differential equations governing these functions are given by (13) and (15) as:

$$D^4\psi_{00}^{(1)} - \sigma^2 D^2 \psi_{00}^{(1)} = 0, \tag{22}$$

$$D^4\psi_{00}^{(2)} = 0. \tag{23}$$

The solution of (22) and (23) are:

$$\psi_{00}^{(1)} = \left(1-\eta^2\right)\left[Kr^2 + C\left\{\frac{\sinh(\sigma r)}{r} - \sigma\cosh(\sigma r)\right\}\right], \tag{24}$$

$$\psi_{00}^{(2)} = \left(1-\eta^2\right)\left[\frac{A}{r} - Br + \frac{1}{2}r^2\right]. \tag{25}$$

where K, C, A and B are constants.
Substituting (24) and (25) into (8)-(10), we get following equations:

$$A - B + \frac{1}{2} = K + C[\sinh\sigma - \sigma\cosh\sigma]- \tag{26}$$

$$-A - B + 1 = 2K + C\left[\sigma\cosh\sigma - \left(\sigma^2+1\right)\sinh\sigma\right] \quad , \tag{27}$$

$$C\left\{\left(\sigma^3 - 12\sigma\right)\cosh\sigma - \left(\sigma^4 - 3\sigma^2 - 12\right)\sinh\sigma\right\} - \sigma^2\left\{2K + C\left[\sigma\cosh\sigma - \left(\sigma^2+1\right)\sinh\sigma\right]\right\} = 6(2A - B), \tag{28}$$

$$C\left[3\left(2+\sigma^2\right)\sinh\sigma - \sigma\left(6+\sigma^2\right)\cosh\sigma\right] - 6A = \beta\sigma(1 - B - A). \tag{29}$$

For studying the effect of variation of permeability of the porous medium inside the sphere the constants A, B, C and K had been calculated by Srivastava and Srivastava (2005) for $\sigma = 5,6,7,8,9,10$ and $\beta = -0.5, 0.5$ which have been reproduced here Table 1.

Table 1. The values of A, B, C and K for various values of β and σ.

	$\beta = -0.5$				$\beta = 0.5$			
σ	A	B	$-C$	K	A	B	$-C$	K
5	0.17527	0.62110	0.00009	0.02484	0.06073	0.45765	0.00028	0.01831
6	0.18471	0.64514	0.00002	0.01792	0.08004	0.50338	0.00006	0.01399
7	0.18793	0.66014	0.000004	0.01135	0.09614	0.53707	0.00001	0.01096
8	0.19812	0.67449	0.000001	0.01053	0.10968	0.56283	0.000003	0.00879
9	0.20253	0.68352	0.0000003	0.00844	0.12118	0.58315	0.0000009	0.00720
10	0.20681	0.69117	0.00000009	0.00690	0.13101	0.59954	0.0000003	0.00599

(i) Calculation of $\psi_{01}^{(1)}$ and $\psi_{01}^{(2)}$:

The differential equations governing these functions are:

$$D^4\psi_{01}^{(1)} - \sigma^2 D^2 \psi_{01}^{(1)} = 0, \tag{30}$$

$$D^4\psi_{01}^{(2)} = -2B\left(\frac{2A}{r^5} - \frac{2B}{r^3} + \frac{1}{r^2}\right)\eta\left(1-\eta^2\right). \tag{31}$$

The solutions of (30) and (31) are:

$$\psi_{01}^{(1)} = \frac{B}{2}\left\{Kr^2 + C\left[\frac{\sinh(\sigma r)}{r} - \sigma\cosh(\sigma r)\right]\right\}\left(1-\eta^2\right)$$
$$+\left\{Mr^2 + N\left[\frac{\left(3+\sigma^2 r^2\right)}{r^2}\sinh(\sigma r) - \frac{3\sigma}{r}\cosh(\sigma r)\right]\right\}\eta\left(1-\eta^2\right), \tag{32}$$

$$\psi_{01}^{(2)} = \frac{B}{4}\left(\frac{2A}{r} - 2Br + r^2\right)\left(1-\eta^2\right) + \frac{B}{4}\left(\frac{2A}{r} + 2Br - r^2 + m + \frac{n}{r^2}\right)\eta\left(1-\eta^2\right). \tag{33}$$

The four constants m, n, M and N are given by

$$\frac{B}{4}\left(2A + 2B - 1 - m - n\right) = M + N\left\{\left(3+\sigma^2\right)\sinh\sigma - 3\sigma\cosh\sigma\right\}, \tag{34}$$

$$\frac{B}{2}\left(-A + B - 1 + n\right) = 3M + N\left\{\left(6+\sigma^2\right)\sigma\cosh\sigma - 3\left(2+\sigma^2\right)\sinh\sigma\right\}, \tag{35}$$

$$\frac{B}{2}\left\{(10-S)A + (4+S)B - (2+S) - 3m - (8-S)n\right\}$$
$$= \gamma^2\left[6M + N\left\{\left(48 + 21\sigma^2 + \sigma^4\right)\sinh\sigma - \left(48\sigma + 5\sigma^3\right)\cosh\sigma\right\}\right], \tag{36}$$

$$3B\left[-1 + 7A + 4B - 3m - 8n\right] = \gamma^2\left[6M + N\left\{\left(\sigma^5 - 36\sigma - 72\right)\cosh\sigma - \left(3\sigma^4 - 36\sigma^2 - 108\right)\sinh\sigma\right\}\right] - \sigma^2\left[3M + N\left\{\left(6+\sigma^2\right)\sigma\cosh\sigma - 3\left(2+\sigma^2\right)\sinh\sigma\right\}\right], \tag{37}$$

where $S = \beta\sigma$. The constants m, n, M and N for $\sigma = 5,6$ and $\beta = -0.5, 0.0, 0.5$ are given in Table 2.

Table 2. The values of m, n, M and N for $\sigma = 5,6$ and $\beta = -0.5; 0.0; 0.5$.

σ	β	m	n	$-M$	$-N$
5	- 0.5	0.46433	0.28331	0.01960	0.00004
	0.0	0.73637	0.09375	0.02482	0.000012
	0.5	0.85082	0.01399	0.02702	0.000015
6	- 0.5	0.44313	0.32122	0.00845	0.0000013
	0.0	0.52641	0.25536	0.01016	0.0000021
	0.5	0.68162	0.13261	0.01110	0.0000038

5. The flow due to non-Newtonian terms:

Equation (1) in dimensionless form can be written as

$$\left(\frac{a\boldsymbol{\tau}}{\mu_1 U}\right) = -P\mathbf{I} + \left(\frac{a}{U}\right)\mathbf{A}_1 + \left(\frac{a}{U}\right)^2\left[\alpha\lambda\mathbf{A}_2 + \mathbf{A}_1{}^2\right], \tag{38}$$

$$\text{where } P = \frac{ap}{\mu_1 U} \text{ and } \alpha = \frac{\mu_2}{\mu_3}. \tag{39}$$

From (38), we get the expression for the dimensionless non-Newtonian part of the stress tensor $\mathbf{t}$ as

$$\mathbf{t} = \alpha\lambda\mathbf{A}_2 + \mathbf{A}_1{}^2 \tag{40}$$

Substituting (21) and (40) into (15) and equating coefficient of λ on both sides of the equation, we get

$$0 = D^4\psi_{10}^{(2)} + \frac{1}{\lambda}\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\left(1-\eta^2\right)t_{r\eta} - \left(1-\eta^2\right)^{1/2}\frac{\partial}{\partial\eta}\left(1-\eta^2\right)^{1/2}t_{\eta\eta}\right] - \eta t_{\varphi\varphi}\right\}$$
$$-\frac{\left(1-\eta^2\right)}{r}\frac{\partial}{\partial\eta}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 t_{rr}\right) + \frac{\partial}{\partial\eta}\left[\left(1-\eta^2\right)t_{r\eta}\right] - t_{\eta\eta} - t_{\varphi\varphi}\right\}. \tag{41}$$

In (40) tensor $\mathbf{t}$ should be evaluated by putting $\psi_{00}^{(2)}$ as it already contains λ as multiplier. The expressions of the components of $\mathbf{t}$ are given by the following equations:

$$t_{rr} = \frac{\lambda\eta^2}{r^3}\left[8\alpha\left(-B + \frac{4B^2}{r} + \frac{6A}{r^2} - \frac{26AB}{r^3} + \frac{30A^2}{r^5}\right) + \frac{16}{r}\left(-B + \frac{3A}{r^2}\right)^2\right]$$
$$+\frac{\lambda\left(1-\eta^2\right)}{r^3}\left[4\alpha\left(B - \frac{B^2}{r} - \frac{6A}{r^2} + \frac{8AB}{r^3} + \frac{15A^2}{r^5}\right) + 36\left(\frac{A^2}{r^5}\right)\right], \tag{42}$$

$$t_{\eta\eta} = \frac{\lambda\eta^2}{r^3}\left[4\alpha\left(B - \frac{B^2}{r} - \frac{6A}{r^2} + \frac{8AB}{r^3} - \frac{3A^2}{r^5}\right) + \frac{4}{r}\left(B - \frac{3A}{r^2}\right)^2\right]$$
$$+\frac{\lambda\left(1-\eta^2\right)}{r^3}\left[2\alpha\left(-B + \frac{B^2}{r} + \frac{9A}{r^2} - \frac{14AB}{r^3} + \frac{9A^2}{r^5}\right) + 36\left(\frac{A^2}{r^5}\right)\right], \tag{43}$$

$$t_{\varphi\varphi} = \frac{\lambda\eta^2}{r^3}\left[4\alpha\left(B - \frac{B^2}{r} - \frac{6A}{r^2} + \frac{8AB}{r^3} - \frac{3A^2}{r^5}\right) + \frac{4}{r}\left(B - \frac{3A}{r^2}\right)^2\right]$$
$$+\frac{2\alpha\left(1-\eta^2\right)}{r^3}\left(-B - \frac{B^2}{r} + \frac{3A}{r^2} - \frac{2AB}{r^3} - \frac{3A^2}{r^5}\right), \tag{44}$$

$$t_{r\eta} = \eta\left(1-\eta^2\right)^{1/2}\left[6\alpha\left(-B + \frac{2B^2}{r} + \frac{8A}{r^2} - \frac{16AB}{r^3} + \frac{10A^2}{r^5}\right) + \frac{12A}{r^3}\left(-B + \frac{3A}{r^2}\right)\right]. \tag{45}$$

Substituting (42)-(45) into (41), we get

$$D^4\psi_{10}^{(2)} = \frac{48(1+\alpha)}{r^5}B\left(B - \frac{6A}{r^2}\right)\eta\left(1-\eta^2\right). \tag{46}$$

The solution of (46) is

$$\psi_{10}^{(2)} = -2(1+\alpha)B\left(\frac{B}{r}+\frac{A}{r^3}+q_1+\frac{q_2}{r^2}\right)\eta\left(1-\eta^2\right), \tag{47}$$

where q_1 and q_2 are constants of integration. Other constants of integration which make the stream function infinite have been omitted. For many fluids $\alpha = -1/2$, hence we consider this case here.

In the zone I substituting (21) into (13), we get

$$D^4\psi_{10}^{(1)} - \sigma^2 D^2\psi_{10}^{(1)} = 0. \tag{48}$$

The solution of (48) is given by

$$\psi_{10}^{(1)} = \left[Q_1 r^3 + Q_2\left\{\frac{3+\sigma^2 r^2}{r^2}\sinh(\sigma r) - \frac{3\sigma}{r}\cosh(\sigma r)\right\}\right]\eta\left(1-\eta^2\right), \tag{49}$$

where Q_1 and Q_2 are constants of integration. Other constants of integration which make the stream function infinite have been omitted. The constants q_1, q_2, Q_1 and Q_2 are calculated by satisfying four matching conditions (8)-(10). These conditions give the following four equations:

$$Q_1 + Q_2\left\{\left(3+\sigma^2\right)\sinh(\sigma) - 3\sigma\cosh(\sigma)\right\} + 2(q_1+q_2)B = -2B(B+A), \tag{50}$$

$$3Q_1 + Q_2\left\{\left(6+\sigma^2\right)\sigma\cosh(\sigma) - 3\left(2+\sigma^2\right)\sinh(\sigma)\right\} - 4Bq_2 = 2B(B+3A), \tag{51}$$

$$\left(4-3\sigma^2\right)Q_1 + Q_2\left\{\left(4\sigma^3+48\sigma\right)\cosh(\sigma) - \left(48+20\sigma^2\right)\sinh(\sigma)\right\} = -6B+14B^2-36AB+36A^2, \tag{52}$$

$$6Q_1 + Q_2\left\{\left(48+21\sigma^2+\sigma^4\right)\sinh(\sigma) - \left(48\sigma+5\sigma^3\right)\cosh(\sigma)\right\} + 4B\sigma\beta q_2 = 6\left(B-2B^2-8A+12AB+2A^2\right) - 2\beta\sigma\left(B^2+3AB\right). \tag{53}$$

Given A, B, σ and β, the four constants q_1, q_2, Q_1 and Q_2 can be calculated from four Eqs.(50)-(53).Taking the values of A and B form Table 1, we have calculated q_1, q_2, Q_1 and Q_2 and Q_2 for $\sigma = 5,6,7,8,9,10$; $\beta = -0.5, 0.5$ which are given in Table 3.

Table 3. The values of constants q_1, q_2, Q_1 and Q_2 for different values of β and σ.

	$\beta = -0.5$				$\beta = 0.5$			
σ	Q_1	$-Q_2$	$-q_1$	$-q_2$	Q_1	$-Q_2$	$-q_1$	$-q_2$
5	- 0.00436	0.000102	0.09192	0.80343	0.00586	0.00002	0.13892	0.36623
6	- 0.00102	0.000020	- 0.00808	0.76961	0.00363	0.000004	0.14747	0.42250
7	0.00016	0.0000028	- 0.06767	0.74498	0.00269	0.0000009	0.16136	0.46057
8	0.00132	0.0000005	- 0.11615	0.73354	0.00221	0.0000002	0.17467	0.48889
9	0.00159	0.0000001	- 0.14557	0.72443	0.00191	0.00000004	0.19599	0.51135
10	0.00163	0.00000002	- 0.16790	0.71922	0.00169	0.00000001	0.19541	0.52966

The stream function $\psi_{10}^{(1)}$ and $\psi_{10}^{(2)}$ are now fully determined. Hence, differentiating these, velocity components due to non-Newtonian terms are known. It may be noted that both stream functions have multiplier $\eta\left(1-\eta^2\right)$ i.e. $\cos\theta$ and $\sin\theta$ which shows that the non-Newtonian terms completely modify the stream lines and velocity components. Since non-Newtonian parameters of the higher order approximations do not affect the drag, hence non-Newtonian terms up to this approximation will not contribute to the drag on the surface of the sphere. This has also been mentioned by Caswell and Schwarz (1961).

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