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THE PROBLEM OF DETERMINING THE COEFFICIENT AT THE LOWEST TERM IN THE EQUATION OF OSCILLATIONS

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Abstract. In this paper, the problem of determining the coefficient at the lowest term of the equation of string vibration investigated. The problem is reduced to the optimal control problem and investigated by the optimal control methods.

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1 Introduction

Recently, inverse problems are more attracted the attention of experts, because their applied and theoretical significance. The special roles among the inverse problems play the problems of determining the coefficients of the considered equations. In some cases the properties of the investigated medium (coefficients of the equations) are unknown. And then the inverse problems arise that requires reconstructing the coefficients of the equation by given information regarding the solution of the direct problem. A lot of papers devoted to the solution of such problems due their strong relation to the practice (see (Kabanikhin, 2009); (Iskakov et al., 2014); (Yonchev, 2017) and the works cited therein).

2 Statement of the problem and main results

Consider the problem of determining a pair $(u(x,t), \upsilon(x,t)) \in W_2^1(Q)\times L_\infty(Q)$ from the following relations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \upsilon(x,t)u = f(x,t), \qquad (x,t) \in Q = (0,l)\times(0,T), \tag{1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = u_1(x), \qquad 0 \le x \le l, \tag{2}$$

$$u(0,t) = u(l,t) = 0, \qquad 0 \le t \le T, \tag{3}$$

$$u(x_i,t) = g_i(t), \qquad 0 \le t \le T, \quad i = 1,\dots, n, \tag{4}$$

where $f \in L_2(Q)$, $u_0 \in \overset{0}{W}{}_2^1(0,l)$, $u_1 \in L_2(0,l)$, $g_i \in L_2(0,T)$ are given functions, $x_i \in (0,l)$ - various given points.

The problem (1), (2), (3) for a given function $v(x,t)$ is a direct problem in the domain Q, and the problem (1)-(4) an inverse problem to the problem (1)-(3).

Let us reduce the inverse problem (1)-(4) to the following optimal control problem: to find a function $v(x,t)$ from

$$V = \{ v\ (x,t) \in\ L_\infty(Q) :\ v\ (x,t) \in [\alpha, \beta\,]\ \ a.e.\ Q\}\,,$$

that minimizes the functional

$$J_0(v) = \frac{1}{2}\int_0^T \sum_{i=1}^{n} [u(x_i,\, t\,;\, v) - g_i(t)]^{\,2} dt \tag{5}$$

subject to (1)-(3), where $u(x,t;v)$is the solution of the problem (1)-(3), where $v = v\ (x,t)$, $\alpha,\ \beta\,,\ \alpha\ <\ \beta$ are given numbers, $v(x,t)$ - control and V- the class of admissible controls.

Note that if $\min\limits_{v\in V} J_0(v) = 0$, then the additional conditions (4) are satisfied.

Let us regularize the problem (1) - (3), (5) as follows: find a function $v(x,t)$ from V, that minimizes the functional

$$J_\varepsilon(v) = \frac{1}{2}\int_0^T \sum_{i=1}^{n} [u(x_i,\, t\,;\, v) - g_i(t)]^{\,2} dt + \frac{\varepsilon}{2}\int_Q (v-\omega)^2 dx\, dt\,, \tag{6}$$

where $\varepsilon\ >\ 0$ is given number, $\omega(x,t)\ \in\ L_2(Q)$ is a given function. This problem we call a problem (1) - (3), (6).

As a generalized solution of the boundary value problem (1)-(3) from $W_2^1\,(Q)$ for each fixed control $v\ \in\ V$ we assume the function $u\ =\ u\ (x,t\,;\,v)$ from $W_{2,0}^1\,(Q)$, that is of the equal to $u_0(x)$ by $t = 0$ and satisfies the integral identity

$$\begin{aligned} &\int_Q \left[-\tfrac{\partial u}{\partial t}\cdot\tfrac{\partial \eta}{\partial t} + \tfrac{\partial u}{\partial x}\cdot\tfrac{\partial \eta}{\partial x} + v(x,\,t)\,u\eta\right] dx\,dt\; - \int_0^l u_1(x)\,\eta(x,0)\,dx\; = \\ &= \int_Q f(x,t)\,\eta\,dx\,dt, \end{aligned} \tag{7}$$

for all $\eta\ =\ \eta\,(x,t)$ from $W_{2,0}^1\,(Q)$, equal to zero for $t = T$.

It follows from the results Ladijhenskaya (1973) (p.209-215) that by imposed above conditions for each fixed $v\ \in\ V$ problem (1)-(3) has a unique generalized solution from $W_2^1\,(Q)$ and the estimate

$$\|u\|_{W_2^1(Q)}\ \le\ c\ \left[\|u_0\|_{W_2^1(0,\,l)} + \|u_1\|_{L_2(0,\,l)} + \|f\|_{L_2(Q)}\right]. \tag{8}$$

is valid.

Here and later on we denote by c various constants, not depending on the estimated quantities and admissible controls.

Theorem 1. *Suppose that the conditions of the formulation of the problem* (1) - (3), (6) *are satisfied.*

Then there exists a dense subset G of the space $L_2(Q)$, such that for all $\omega\ \in\ G$ with $\varepsilon > 0$ the problem (1) - (3), (6) *has a unique solution.*

Proof. We prove the continuity of the functional

$$J_0(v) = \frac{1}{2}\int_0^T \sum_{i=1}^{n} [u(x_i,\, t\,;\, v) - g_i(t)]^{\,2} dt$$

in the norm of the space $L_2(Q)$ in the set V.

Let $\delta v \in L_\infty(Q)$ be an increment of the control on the element $v \in V$ such that $v+\delta v \in V$. Denote $\delta u(x,t) \equiv u(x,t;\, v+\delta v) - u(x,t;\, v)$. It is clear that the function $\delta u\,(x,t)$ is a generalized solution from $W_2^1\,(Q)$ for the boundary problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \frac{\partial^2 \delta u}{\partial x^2} + (v+\delta v)\,\delta u = -u\,\delta v,\ \ (x,t) \in Q, \tag{9}$$

$$\delta u\,|_{t=0} = 0\,,\quad \frac{\partial \delta u}{\partial t}\,|_{t=0} = 0, \qquad 0 \le x \le l, \tag{10}$$

$$\delta\ u\,(0,t) = \ \delta u\ (l,t)\ = 0\,, \qquad 0 \le t \le T. \tag{11}$$

The generalized solution $W_2^1\,(Q)$of the problem (9)-(11) is equal to zero by $t=0$ and satisfies to the identity

$$\int_Q \left[\frac{\partial \delta u}{\partial t}\cdot\frac{\partial \eta}{\partial t} - \frac{\partial \delta u}{\partial x}\cdot\frac{\partial \eta}{\partial x}\right] dx\,dt\ = \int_Q \left[(v+\delta v)\,\delta u + \,u\,\delta v\right]\ \eta\,dx\,dt, \tag{12}$$

for all $\eta\ = \eta\,(x,t)\,\in W_{2,0}^1\,(Q)$, equal to zero by $t = T$.

Let us prove that for the solution of the problem (9)-(11) the following estimate is valid

$$\|\delta u\|_{W_2^1(Q)}\ \le c\ \|\delta v\|_{L_2(Q)}\,. \tag{13}$$

For this purpose we use Faedo-Galerkin method. Let $\{\varphi_k(x)\}$ be a fundamental system in $\overset{0}{W}{}_2^1(0,l)$ and $\int_0^l \varphi_k(x)\,\varphi_m(x)\ dx\ = \delta_k^m$, where δ_k^m is Cronekker's symbol.

Approximate solution of the problem (9)-(11) we search in the form

$$\delta u^N(x,t) = \sum_{k=1}^N c_k^N(t)\varphi_k(x)$$

from the relations

$$\begin{gathered}\int_0^l \frac{\partial^2 \delta u^N}{\partial t^2}\cdot\varphi_m(x)\,dx\, + \int_0^l \frac{\partial \delta u^N}{\partial x}\cdot\frac{d\varphi_m(x)}{dx}\,dx + \\ + \int_0^l (v+\delta v)\,\delta u^N \varphi_m(x)\,dx = \ -\int_0^l u\,\delta v \varphi_m(x)\ dx\,,\ m = 1,...,N,\end{gathered} \tag{14}$$

$$c_k^N(0)\ = 0,\ \ \frac{d\,c_k^N(t)}{dt}\,|_{t=0} = 0. \tag{15}$$

The equality (14) is a system of linear ordinary differential equations of the second order relatively t for the unknown functions $c_k^N(t),\ k = 1,...,N$ solved with respect to $\frac{d^2 c_k^N}{dt^2}$. This system is uniquely solvable by initial data (15), and $\frac{d^2 c_k^N}{dt^2} \in L_2(0,T)$. Multiplying each equality from (14) by its $\frac{d}{dt}c_m^N(t)$ and summing relatively m from 1 to N we get

$$\begin{gathered}\int_0^l \frac{\partial^2 \delta u^N}{\partial t^2}\cdot\frac{\partial \delta u^N}{\partial t}\,dx\, + \int_0^l \frac{\partial \delta u^N}{\partial x}\cdot\frac{\partial^2 \delta u^N}{\partial t\,\partial x}\,dx\, = \\ = -\int_0^l (v+\delta v)\,\delta u^N \frac{\partial \delta u^N}{\partial t}\,dx - \int_0^l u\,\delta v \frac{\partial \delta u^N}{\partial t}\,dx\,.\end{gathered}$$

From this under the imposed conditions we get

$$\int_0^l \left[(\frac{\partial \delta u^N}{\partial t})^2 + (\frac{\partial \delta u^N}{\partial x})^2\right] dx \le c\int_0^t\int_0^l \left[\left|\delta u^N\right|^2 + \left|\frac{\partial \delta u^N}{\partial t}\right|^2\right] dxds + c\int_0^t\int_0^l |u\delta v|^2\,dxds.$$

Since the fixed solution $u = u(x,t;v)$ of the problem (1)-(3) has a property $u \in C\left([0,T]\,;\overset{0}{W}{}_2^1(0,l)\right)$, (Lions & Madjenes, 1971)(p.307) this function can be assumed continuous on $\bar{Q}$. So it is bounded on $\bar{Q}$. Then from the above inequality follows that

$$\int_0^l \left[\left(\frac{\partial \delta u^N}{\partial t}\right)^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dx \le c\int_0^t\int_0^l \left[\left(\delta u^N\right)^2 + \left(\frac{\partial \delta u^N}{\partial t}\right)^2\right] dxds + c\int_0^t\int_0^l |\delta v|^2\, dxds$$

Considering the equivalency of the norm in $\overset{0}{W}{}_2^1(0,l)$ from the last we obtain

$$\int_0^l \left[\left|\delta u^N\right|^2 + \left|\frac{\partial \delta u^N}{\partial t}\right|^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dx \le$$

$$\le c\int_0^t\int_0^l \left[\left(\delta u^N\right)^2 + \left(\frac{\partial \delta u^N}{\partial t}\right)^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dxds + c\int_0^t\int_0^l |\delta v|^{\,2} dxds.$$

Applying here Gronwall's lemma we get

$$\int_0^l \left[\left|\delta u^N\right|^2 + \left|\frac{\partial \delta u^N}{\partial t}\right|^2 + \left|\frac{\partial \delta u^N}{\partial x}\right|^2\right] dx \le c\int_0^T\int_0^l |\delta v|^{\,2} dxdt, \quad \forall t \in [0,T]\,.$$

Integrating this over t from 0 to T we arrive to

$$\left\|\delta\, u^N\right\|_{W_2^1} \le c\,\|\delta\, v\|_{L_2(Q)}\,.$$

This inequality allows one to choose a subsequence (we denote it also by $\left\{\delta u^N\right\}$) from the sequence $\left\{\delta u^N\right\}, N = 1,2,...,$ that converges weakly in $W_2^1(Q)$ to some element $\delta u \in W_2^1(Q)$.

Since the norm in the Hilbert space is weakly lower semicontinuous, it follows from the last that for the weak limit δu of the sequence $\left\{\delta\, u^N\right\}$ in $W_2^1(Q)$ the following estimation is valid

$$\|\delta\, u\|_{W_2^1(Q)} \le c\, \|\delta\, v\|_{L_2(Q)}\,.$$

Thus the estimation (13) is proved.

Following the embedding theorem (Ladijhenskaya, 1973)(p.70) $W_2^1(Q)$ is bloodedly embedded into $L_2(0,T)$ so from (13) follows that

$$\|\delta\, u\,(x_i, t\,)\|_{L_2(0,T)} \le c\, \|\delta\, u\|_{W_2^1(Q)} \le c\, \|\delta\, v\|_{L_2(Q)}\,, i = 1,...,n\,.$$

Therefore

$$\|\delta\, u\,(x_i, t)\|_{L_{2(0,T)}} \to 0 \;by\; \|\delta\, v\,\|_{L_2(Q)} \to 0. \tag{16}$$

Increment of the functional $J_o(v)$ we can write as

$$\Delta J_0(v) = J_0(v+\delta v) - J_0(v) = \int_0^T \sum_{i=1}^n \left[u(x_i,t;v) - g_i(t)\right]\delta u\,(x_i,t))dt + \frac{1}{2}\int_0^T \sum_{i=1}^n |\delta u\,(x_i,t)|^{\,2}dt\,.$$

From this and (16) one can get the continuity of the functional $J_o(v)$ with respect to the norm of the space $L_2(Q)$ on the set V.

Thus the functional $J_0(v)$ is continuous and lower bounded on V. The set V is closed and bounded in the uniform convex Banach space $L_2(Q)$. Then the statement of the Theorem 1 follows from the known theorem (Goebel, 1979). Theorem is proved. □

Now we study Frechet differentiability of the functional (6).

Let $\psi = \psi(x,t;v)$ be generalized solution from $W_2^1(Q)$of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + v\psi = -\sum_{i=1}^{n}\left[u(x,t;v - g_i(t)\right]\delta(x - x_i), \tag{17}$$

$$\psi\Big|_{t=T} = 0, \frac{\partial\psi}{\partial t}\Big|_{t=T} = 0, \psi(o,t) = \psi(l,t) = 0, \tag{18}$$

where $\delta(x)$ is Dirac function.

As a generalized solution from $W_2^1(Q)$ of the boundary problem (17), (18) by given $v \in V$, we take the function $\psi = \psi(x,t;v)$ from $W_2^1(Q)$ that is equal to zero by $t = T$ and satisfies the integral equality

$$\int_Q\left[-\frac{\partial\psi}{\partial t}\cdot\frac{\partial\mu}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial\mu}{\partial x} + v\psi\mu\right]dxdt = -\sum_{i=1}^{n}\int_0^T\left[u(x_i,t;v) - g_i(t)\right]\mu(x_i,t)dt, \tag{19}$$

for all equal to zero by $t = 0$ functions $\mu = \mu(x,t) \in W_{2,0}^1(Q)$.

Since $u \in W_2^1(Q)$ in the equality (19) $u(x_i,t;v)$ has a sense.

Theorem 2. *Let the conditions imposed on the data of the problem* (1)-(3), (6) *be fulfilled. Then the problem* (17), (18) *has unique generalized solution from* $W_2^1(Q)$.

Proof. We'll use Faedo-Galerkin's method. As a fundamental system $\{\varphi_k(x)\}$ in $\overset{0}{W}{}_2^1(0,l)$ we take $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi k}{l}x\right\}_{k=1}^{\infty}$.

Approximate solution $\psi^N(x,t)$ we search in the form

$$\psi^N(x,t) = \sum_{k=1}^{N}C_k^N(t)\varphi_k(x)$$

from the relation

$$\int_0^l\frac{\partial^2\psi^N}{\partial t^2}\varphi_m dx + \int_0^l\frac{\partial\psi^N}{\partial x}\frac{d\varphi_m}{dx}dx = \sum_{i=1}^{n}\left[u(x_i,t;v) - g_i(t)\right]\varphi_m(x_i), m = 1,...,N, \tag{20}$$

and

$$C_k^N(T) = 0, \frac{dC_k^N(t)}{dt}\,|_{t=T} = 0\ . \tag{21}$$

The equality (20) is a system of linear differential equations of the second order with respect to t for all unknown $C_k^N(t), k = 1,...,N$, solved relatively $\frac{d^2C_k^N}{dt^2}$, and the terms

$$\sum_{i=1}^{n}\left[u\left(x_i,t;v\right) - g(t)\right]\varphi_m(x_i) \in L_2(0,T)$$

. This system is uniquely solvable by initial data (21) and $\frac{d^2C_k^N}{dt^2} \in L_2(0,T)$.

Multiplying each of equalities from (20) by its own $\frac{d}{dt}C_m^N(t)$ and taking a sum with respect to m from 1 to N we arrive to the equality

$$\int_0^l\frac{\partial^2\psi^N}{\partial t^2}\frac{\partial\psi^N}{\partial t}dx + \int_0^l\frac{\partial\psi^N}{\partial x}\cdot\frac{\partial^2\psi^N}{\partial t\partial x}dx =$$

$$= -\sum_{i=1}^{n}\left[u(x_i,t;v) - g_i(t)\right]\frac{\partial\psi^N(x_i,t)}{\partial t} - \int_0^l v\psi^N\frac{\partial\psi^N(x,t)}{\partial t}dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_0^l\left[\left(\frac{\partial\psi^N}{\partial t}\right)^2+\left(\frac{\partial\psi^N}{\partial x}\right)^2\right]dx=$$

$$=-\sum_{i=1}^n\left[u\left(x_i,t;v\right)-g_i(t)\right]\frac{\partial\psi^N(x_i,t)}{\partial t}-\int\limits_0^l v\psi^N\frac{\partial\psi^N(x,t)}{\partial t}dx$$

From this integrating over t from t to T considering (21) we obtain

$$\int_0^l\left[\left(\frac{\partial\psi^N}{\partial t}\right)^2+\left(\frac{\partial\psi^N}{\partial x}\right)^2\right]dx=$$

$$=2\int_t^T\sum_{i=1}^n\left[u\ \left(x_i,s;v\right)-g_i(s)\right]\frac{\partial\psi^N(x_i,s)}{\partial t}ds+2\int\limits_t^T\int\limits_0^l v\psi^N(x,s)\frac{\partial\psi^N(x,s)}{\partial t}dxds.$$

Successively solving the system of ordinary differential equations (20) with respect to the unknown functions $C_1^N(t),...,C_N^N(t)$ under conditions (21) and using some known properties of trigonometric functions we find that $\left|\frac{\partial\psi^N(x_i,t)}{\partial t}\right|\le c$ uniformly relatively N and $t\in[0,T]$, $i=1,...,n$.

Therefore there exists a constant c such that

$$\left|\frac{\partial\psi^N(x_i,t)}{\partial t}\right|^2\le c\int_0^l\left|\frac{\partial\psi(x,t)}{\partial t}\right|^2dt,\ i=1,\ ...,\ n.$$

Then it is clear that

$$\int_0^l\left[\left(\frac{\partial\psi^N}{\partial t}\right)^2+\left(\frac{\partial\psi^N}{\partial x}\right)^2\right]dx\le\int_t^T\sum_{i=1}^n\left|u\left(x_i,s;v\right)-g(s)\right|^2ds+$$

$$+c\int_t^T\int_0^l\left[\left|\psi^N(x,s)\right|^2+\left|\frac{\partial\psi^N(x,s)}{\partial t}\right|^2+\left|\frac{\partial\psi^N(x,s)}{\partial x}\right|^2\right]dxds.$$

Considering the equivalency of the norms in $\overset{0}{W}{}_2^1(0,l)$ we have

$$\int_0^l\left[\left|\psi^N\right|^2+\left|\frac{\partial\psi^N}{\partial t}\right|^2+\left|\frac{\partial\psi^N}{\partial x}\right|^2\right]dx\le c\int_t^T\sum_{i=1}^n\left|u(x_i,s;v)-g_i(s)\right|^2ds+$$

$$+c\int_t^T\int_0^l\left[\left|\psi^N\right|^2+\left|\frac{\partial\psi^N}{\partial t}\right|^2+\left(\frac{\partial\psi^N}{\partial x}\right)^2\right]dxds.$$

From the last applying the Gronwall's lemma we get

$$\int_0^l\left[\left|\psi^N\left(x,t\right)\right|^2+\left|\frac{\partial\psi^N(x,t)}{\partial t}\right|^2+\left|\frac{\partial\psi^N(x,t)}{\partial x}\right|^2\right]dx\le$$

$$\le c\int_0^T\sum_{i=1}^n\left|u(x_i,t;v)-g_i(t)\right|^2dt,\quad\forall t\in[0,T]$$

or

$$\left\|\psi^N\right\|_{W_2^1(Q)}\le c\int_0^T\sum_{i=1}^n\left|u(x_i,t;v)-g_i(t)\right|^2dt.\tag{22}$$

Now doing as in Ladijhenskaya (1973) (pp.214-215) we obtain that the weak limit $\psi(x,t)$ of the sequence $\left\{\psi^N(x,t\right\}$ by $N\to\infty$in $W_2^1(Q)$ is a generalized solution of the problem (17), (18). The uniqueness of the solution of the problem (17), (18) is proved by standard way. Theorem is proved. □

Note that, considering the weak lower semicontinuity of the norm in the Hilbert space, for the limit function $\psi(x,t)$ the inequality (22) hold true, i.e.

$$\|\psi\|_{W_2^1(Q)} \le c\int_0^T \sum_{i=1}^n |u(x_i,t;v)-g_i(t)|^2\,dt. \tag{23}$$

Taking into account here the estimate (8) and the fact that $W_2^1(Q)$ is bounded embedded into $L_2(0,T)$ (Ladijhenskaya, 1973)(p.70) we get

$$\|\psi\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(0,l)}+\|u_1\|_{L_2(0,l)}+\|f\|_{L_2(Q)}+\sum_{i=1}^n \|g_i\|_{L_2(0,T)}\right]. \tag{24}$$

Theorem 3. *Let the conditions of the Theorem 1 be fulfilled. Then the functional* (6) *is continuously differentiable in the Frechet sense on* V *and its differential at the point* $v\in V$ *by the increment* $\delta v\in L_\infty(Q)$ *is defined by the formula*

$$< J_\varepsilon'(v),\delta v >= \int_Q u\psi\delta v dxdt+\varepsilon\int_Q (v-\omega)\delta v dxdt. \tag{25}$$

Proof. Consider the increment of functional (6):

$$\begin{gathered}\Delta J_\varepsilon(v)=J_\varepsilon(v+\delta v)-J_\varepsilon(v)=\int_0^T\sum_{i=1}^n\left[u(x_i,t;v)-g_i(t)\right]\delta u(x_i,t)dt+\\ +\varepsilon\int_Q(v-\omega)\delta v dxdt+\tfrac{1}{2}\int_0^T\sum_{i=1}^T|\delta u(x_i,t)|^2dt+\frac{\varepsilon}{2}\int_Q|\delta v|^2\,dxdt\,.\end{gathered} \tag{26}$$

If take $\eta=\psi(x,t;v)$, in (12) take $\mu=\delta\,u\,(x,t)$ in (19) and add the obtained relations then we get

$$\int_0^T\sum_{i=1}^n\left[u(x_i,t;v)-g_i(t)\right]\delta u(x_i,t)dt=\int_Q u\psi\delta v dxdt+\int_Q\psi\delta v\delta u dxdt.$$

Considering this in (26) we have

$$\Delta J_\varepsilon(v)=\int_Q\left[u\psi\delta v+\varepsilon(v-\omega)\delta v\right]dxdt+R, \tag{27}$$

where

$$R=\int_Q\psi\delta v\delta u\,dxdt+\frac{1}{2}\int_0^T\sum_{i=1}^n|\delta u(x_i,t)|^2dt+\frac{\varepsilon}{2}\int_Q|\delta v|^2\,dxdt\,. \tag{28}$$

It is clear that the expression in the right hand side of (25) by given $v\in V$ defines a linear functional of δv. Moreover

$$\left|\int_Q\left[u\psi+\varepsilon(v-\omega)\right]\delta v dxdt\right|\le\|u\|_{L_2(Q)}\|\psi\|_{L_{2(Q)}}\|\delta v\|_{L_\infty(Q)}+\varepsilon\|v-\omega\|_{L_2(Q)}\cdot\|\delta v\|_{L\infty(Q)}\le$$

$$\le c\cdot\left[\|u\|_{W_2^1(Q)}\cdot\|\psi\|_{W_2^1(Q)}+\varepsilon\|v-\omega\|_{L_2(Q)}\right]\cdot\|\delta v\|_{L_\infty(Q)}\,.$$

If to consider here the estimations (8), (24) we get boundedness of the functional in the right hand side of (25) with respect to $\delta\nu$.

Now we estimate the remainder term R, in (27). Using Cauchy-Bunyakovski inequality we get

$$|R| \leq \|\psi\|_{L_2(Q)} \cdot \|\delta u\|_{L_2(Q)} \cdot \|\delta v\|_{L_\infty(Q)} + \frac{1}{2}\sum_{i=1}^{n} \|\delta u(x_i,t)\|^2_{L_{2(0,T)}} + \frac{\varepsilon}{2}\|\delta v\|^2_{L_\infty(Q)} .$$

Considering here the boundedness of the embedding $W_2^1(Q) \to L_2(0,T)$, (Ladijhenskaya, 1973)(p.70) and (13), we get $R = 0\left(\|\delta v\|_{L_\infty(Q)}\right)$. Then as follows from (27) the functional (6) is differentiable in Freshe sense on V and the formula (25) is valid.

Let us show that the mapping $v \to J'_\varepsilon(v)$ defined by the equality (25) acts continuously from V to the adjoint to $L_\infty(Q)$ space $(L_\infty(Q))^*$.

Let $\delta\psi(x,t) = \psi(x,t;v+\delta v) - \psi(x,t;v)$. From (17), (18) follows that $\delta\psi$ is a generalized solution from $W_2^1(Q)$ for the boundary problem

$$\frac{\partial^2 \delta\psi}{\partial t^2} - \frac{\partial^2 \delta\psi}{\partial x^2} + v\delta\psi = -\sum_{i=1}^{n} \delta u(x,t)\delta(x-x_i), \tag{29}$$

$$\delta\psi\Big|_{t=T} = 0,\ \frac{\partial\delta\psi}{\partial t}\Big|_{t=T} = 0, \delta\psi(0,t) = \delta\psi(l,t) = 0. \tag{30}$$

Similarly to (23) the estimate

$$\|\delta\psi\,\|_{W_2^1(Q)} \leq c\sum_{i=1}^{n} \|\delta u\,(x_i,t)\|_{L_2(0,T)}\,, i = 1, ..., n.$$

is valid.

Due to the boundedness of the embedding $W_2^1(Q) \to L_2(0,T)$ (Ladijhenskaya, 1973)(p.70) from the last follows that

$$\|\delta\psi\|_{W_2^1(Q)} \leq c\,\|\delta u\|_{W_2^1(Q)} . \tag{31}$$

Then (31) and (13) give the estimation

$$\|\delta\psi\|_{W_2^1(Q)} \leq c\,\|\delta v\|_{L_\infty(Q)} . \tag{32}$$

Using (25) and Cauchy-Bunyakovski inequality we get

$$\left\|J'_\varepsilon(v+\delta v) - J'_\varepsilon(v)\right\|_{(L_\infty(Q))^*} \leq$$

$$\leq c\left[\|u\|_{L_2(Q)} \cdot \|\delta\psi\|_{L_2(Q)} + \|\psi\|_{L_2(Q)} \cdot \|\delta u\|_{L_2(Q)} + \|\delta u\|_{L_2(Q)} \cdot \|\delta\psi\|_{l_2(Q)}\right] + \varepsilon\,\|\delta v\|_{L_\infty(Q)} .$$

Due to (13) and (32) the right hand side of this inequality tents to zero by $\|\delta\, v\|_{L_\infty(Q)} \to o$. From this follows that $v \to J'_\varepsilon(v)$ is a continuous mapping from V to $(L_\infty(Q))^*$.
Theorem is proved. □

Theorem 4. *Let the conditions of the Theorem 3 be fulfilled. Then for the optimality of the control* $v_*(x,t)$*in the problem* (1)-(3), (6) *it is necessary the fulfillment of the inequality*

$$\int_Q \left[u_*(x,t)\psi_*(x,t) + \varepsilon(v_*(x,t) - \omega(x,t))\right](v(x,t) - v_*(x.t))dxdt \geq 0\,, \tag{33}$$

For any $v = v(x,t) \in V$, *where* $u_*(x,t) = u\,(x,t;v_*)$, $\psi_*(x,t) = \psi(x,t;v_*)$ *is a solution of the problem* (1)-(3) *and* (17), (18) *correspondingly by* $v = v_*(x,t)$.

Proof. The set V is convex in $L_\infty(Q)$. In addition, according to Theorem 3 functional $J_\varepsilon(v)$ is continuously Frechet differentiable on V and its differential at the point $v \in V$ is defined by (25).

Then by Theorem 5 from (Vasilyev, 1981)(p.28) on the element $v_* \in V$ it is necessary fulfillment of the inequality $< J_\varepsilon^{'}(v_*), v - v_* > \geq o$ for all $v \in V$. From this and (25) follows the validity of (33) for all $v \in V$. Theorem is proved. □

Note. On the basis of the obtained necessary condition an iterative algorithm can be offered for finding the approximate solutions of problem (1)-(3), (6).

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