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A NOTE ON THE OSCILLATORY CHARACTER OF SOME NON CONFORMABLE GENERALIZED LIÉNARD SYSTEM

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Abstract. In this work, we study the oscillatory character of a non-conformable differential equation of order 2α, under suitable conditions, which contains as a particular case the iconic Liénard Equation. The analysis is carried out using Phase Plan tools, in this way, known results are extended in the entire and fractional Caputo case, for non-conformable derivatives.

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1 Preliminaries

Fractional calculus concerns the generalization of differentiation and integration to non-integer (fractional) orders. The subject has a long mathematical history being discussed for the first time already in the correspondence of Leibniz with L'Hopital when this replied "What does $\frac{d^n}{dx^n}f(x)$ mean if n=$\frac{1}{2}$?" in September 30 of 1695. Over the centuries many mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature (Podlybny, 1999). The nature of many systems makes that they can be more precisely modeled using fractional differential equations. The differentiation and integration of arbitrary orders have found applications in diverse fields of science and engineering like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals (Kilbas et al., 2006; Lakshmikantham et al., 2009; Podlybny, 1999).

We know that the fractional derivative of a non-integer function can be conceived in two branches: global (classical) and local. The former are often defined by means of integral transforms, Fourier or Mellin, which means in particular that their nature is not local, has "memory", in the second case, if they are defined locally by a certain incremental quotients. The first are associated with the emergence of the Fractional Calculation itself, with the pioneering works of Euler, Laplace, Lacroix, Fourier, Abel, Liouville, ... until the establishment of the classical definitions of Riemann-Liouville and Caputo. For various reasons, only very little were considered global derivatives, that is, defined in terms of an integral. We must point out that these derivatives have a group of inconsistencies, the main ones are:

1) Most of the fractional derivatives except Caputo-type, do not satisfy $D^\alpha(1) = 0$, if α is not a natural number.

2) All fractional derivatives do not satisfy the familiar Product Rule for two functions $D^\alpha(fg) = gD^\alpha(f) + fD^\alpha(g)$.

3) All fractional derivatives do not satisfy the familiar Quotient Rule for two functions $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$ with $g \neq 0$.

4) All fractional derivatives do not satisfy the Chain Rule for composite functions $D^\alpha(f \circ g)(t) = D^\alpha(f(g))D^\alpha g(t)$.

5) The fractional derivatives do not have a corresponding "calculus".

6) All fractional derivatives do not satisfy the Indices Rule $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$.

7) It is known that in systems of differential equations of integer order, that satisfy the conditions of existence and uniqueness, two different trajectories do not intercect each other in finite time, however, fractional systems do not satisfy this property.

However, in Khalil et al. (2014) the authors define a new well-behaved simple fractional derivative called the conformable fractional derivative, depending just on the basic limit definition of the derivative. Namely, for a function $f : (0, +\infty) \to \mathbb{R}$ the conformable fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by (see also Abdeljawad (2015))

$$T_\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

A little more recently, the authors defined a local derivative, called non conformable (see Guzmán et al. (2018)), which we present below as $N_1^\alpha f(t)$.

Definition 1. *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by $N_1^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}$ for all $t > 0$, $\alpha \in (0, 1)$. If f is $\alpha-$ differentiable in some $(0, a)$, and $\lim_{t \to 0^+} N_1^{(\alpha)} f(t)$ exists, then define $N_1^{(\alpha)} f(0) = \lim_{t \to 0^+} N_1^{(\alpha)} f(t)$.*

Theorem 1. *(see Guzmán et al. (2018)) Let f and g be N-differentiable at a point $t > 0$ and $\alpha \in (0, 1]$. Then*

a) $N_1^\alpha(af + bg)(t) = aN_1^\alpha(f)(t) + bN_1^\alpha(g)(t)$.

b) $N_1^\alpha(t^p) = e^{t^{-\alpha}} p t^{p-1},\ p \in \mathbb{R}$.

c) $N_1^\alpha(\lambda) = 0,\ \lambda \in \mathbb{R}$.

d) $N_1^\alpha(fg)(t) = fN_1^\alpha(g)(t) + gN_1^\alpha(f)(t)$.

e) $N_1^\alpha(\frac{f}{g})(t) = \frac{gN_1^\alpha(f)(t) - fN_1^\alpha(g)(t)}{g^2(t)}$.

f) If, in addition, f is differentiable then $N_1^\alpha(f) = e^{t^{-\alpha}} f'(t)$.

g) Being f differentiable and $\alpha = n$ integer, we have $N_1^n(f)(t) = e^{t^{-n}} f'(t)$.

Remark 1. *The relations a), c), d) and (e) are similar to the classical results mathematical analysis, these relationships are not established (or do not occur) for fractional derivatives of global character (see Kilbas et al. (2006) and Podlybny (1999) and bibliography there). The relation c) is maintained for the fractional derivative of Caputo. Cases c), f) and g) are typical of this non conformable local fractional derivative.*

Remark 2. *The N-derivative solves almost all the insufficiencies that are indicated to the classical fractional derivatives. In particular we have the following result.*

Theorem 2. *(Guzmán et al., 2018) Let $\alpha \in (0,1]$, g N-differentiable at $t > 0$ and f differentiable at $g(t)$ then $N_1^\alpha(f \circ g)(t) = f'(g(t))N_1^\alpha g(t)$.*

This result is the equivalent, for N_1^α, of the well-known chain rule of classic calculus and that is basic in the Second Method of Lyapunov, for the study of stability of perturbed motion.

Definition 2. *The non conformable fractional integral of order α is defined by the expression ${}_N J_{t_0}^\alpha f(t) = \int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds$.*

The following statement is analogous to the one known from the Ordinary Calculus.

Theorem 3. *Let f be N-differentiable function in (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

a) If f is differentiable ${}_N J_{t_0}^\alpha (N_1^\alpha f(t)) = f(t) - f(t_0)$.

b) $N_1^\alpha \left({}_N J_{t_0}^\alpha f(t)\right) = f(t)$.

Following the same procedure of the ordinary calculus, we can proved the following result.

Theorem 4. *(Nápoles et al., 2018) Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies:*

i) f is continuous on $[a,b]$,

ii) f is N-differentiable for some $\alpha \in (0,1)$.

Then, we have that if $N_1^\alpha f(t) \geq 0$ (≤ 0) then f is a non-decreasing (increasing) function.

2 Problem Statement

In this paper we will study the oscillatory character of the following non conformable differential equation of order 2α

$$\begin{aligned} N_1^\alpha x &= \frac{1}{h(x)}(A(y) - F(x)), \\ N_1^\alpha y &= -h(x)g(x, N_1^\alpha x). \end{aligned} \tag{1}$$

If $h(x) \equiv 1$, $A(y) = y$, $F(x) =_{N_1} J_0^\alpha f(s)(t)$, $g(x, N_1^\alpha x) = g(x)$ then system (1) is reduced to the non conformable Liénard equation

$$N_1^\alpha(N_1^\alpha x) + f(x)N_1^\alpha x + g(x) = 0, \tag{2}$$

studied in Guzmán et al. (2019) within the framework of the Caputo derivative.

Our central result is the following.

Theorem 5. *Suppose that*

1) i) $xg(x, N_1^\alpha x) > 0$, for all $x \neq 0$;

ii) $0 < \alpha \leq h(x) \leq \beta$;

iii) $A(y)$ is a continuous and strictly increasing in $\mathbb{R}$ with $A(0) = 0$ and $A(\pm\infty) = \pm\infty$.

2) these exists a constant $c > 0$ and sequences $\{x_n\}, \{x'_n\}$ such that

$$F(x_n) \geq -c, \quad x_n \xrightarrow{n \to +\infty} +\infty,$$

$$F(x'_n) \leq c, \quad x'_n \xrightarrow{n \to +\infty} -\infty;$$

3) these exist sequences $\{z_n\}, \{z'_n\}$ *such that*

$F(z_n) \leq 0$ *with* $z_n > 0, \quad z_n \xrightarrow{n\to+\infty} 0,$

$F(z'_n) \geq 0$ *with* $z_n < 0, \quad z'_n \xrightarrow{n\to+\infty} 0,$

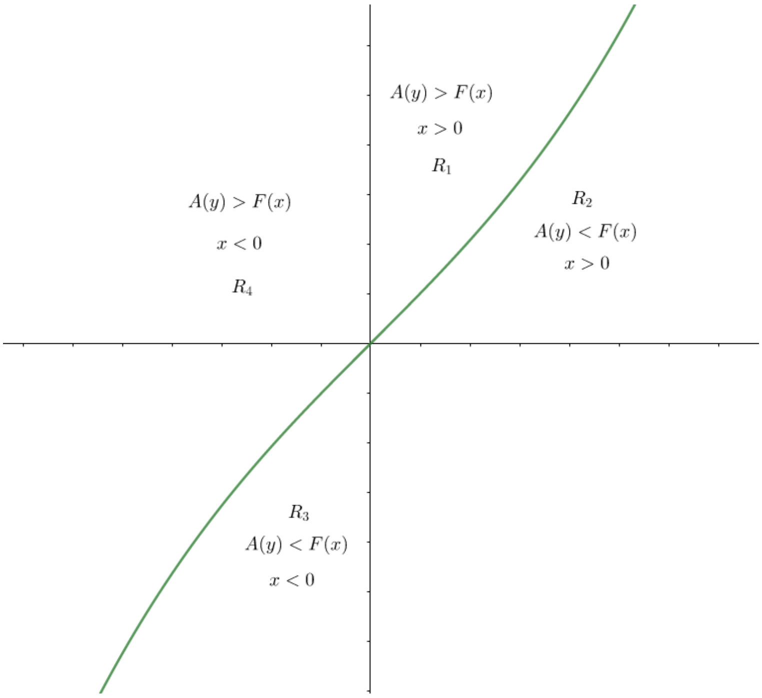
then all solutions of (1) *oscillate iff*

$$\lim_{x\to\infty} sup \quad \left[F(x) +_{N_1} J_0^{\alpha} \frac{g(s, N_1 s)}{1 + F_-(s)}(x)\right] = +\infty, \tag{3}$$

$$\lim_{x\to\infty} sup \quad \left[-F(x) +_{N_1} J_0^{\alpha} \frac{g(s, N_1 s)}{1 + F_+(s)}(x)\right] = +\infty, \tag{4}$$

where $F_+(x) = \{maxF(x), 0\}$ *and* $F_-(x) = max\{-F(x), 0\}$.

Proof. An analysis of the phase velocities of the system (1), allows us to distinguish the following four regions in dependence on the signs of the N-derivatives of x and y.



Sufficiency.
Suppose that (3) and (4) hold and $(x(t), y(t))$ is a solution of (1) in $(x(t_0), y(t_0)) = (x_0, y_0)$ which does not oscillate. There are no problems if we assume $x(t) > 0$ for all $t \geq T \geq t_0 \geq 0$. First, we show that if (3) holds then the solution $(x(t), y(t))$ of system (1) departing from the point (x_0, y_0), with $x_0 \geq 0$, $A(y_0) > F(x_0)$ at $t = T$, must intersect the curve $A(y) = F(x)$ and will be in the region R_2 as t increasing.

For the take of obtaining a contradiction we assume that

$$A(y(t)) > F(x(t)), t > T. \tag{5}$$

Then

$$\begin{aligned} N_1^{\alpha} x(t) &= \frac{A(y(t)) - F(x(t))}{h(x(t))} > 0, \\ N_1^{\alpha} y(t) &= -h(x(t)) g(x, N_1^{\alpha} x) < 0. \end{aligned}$$

From this fact and the Theorem 4, $x(t)$ is increasing and $y(t)$ is decreasing with $x(t) \geq x_0$ and $y(t) \leq y_0$, for all $t > T$. Now consider the following two cases:

I) $\lim_{x\to+\infty} sup F(x) = +\infty$, then exists a $x_1 > 0$ such that $F(x_1) > A(y_0)$. Since $F(x(t)) < A(y(t)) \le A(y_0)$, it follows that $x(T) \le x(t) \le x_1$ for all $t \ge T$. Choose $L > 0$, such that $N_1^\alpha y(t) = -h(x(t))g(x, N_1^\alpha x) \le -L$ for all $T \le t \le +\infty$. This yields $y(t) \le y_0 - L(t-t_0) \le y_0 - L(t-T) \xrightarrow{t\to+\infty} -\infty$ and $A(y(t)) - F(x(t)) \le A(y(t)) + \underset{x_0\le h\le x_1}{sup} (F(h)) \xrightarrow{t\to+\infty} -\infty$ which contradicts (5).

II) If ${}_{N_1}J_0^\alpha \frac{g(s)}{1+F_-(s)}(x) = +\infty$, let $\lim_{t\to+\infty} x(t) = k, \quad 0 < k \le +\infty$ then $k = +\infty$. Otherwise, these will be a contradiction similar to I) by lefting $x_1 = k$. If then follows that $F(x(t)) < A(y(t)) \le A(y_0), t \ge T$ and

$$\begin{aligned} y(t) &= y_0 -{}_{N_1} J_{t_0}^\alpha \quad h(x(s))g(x(s), N_1^\alpha x(s))(t) \\ &\le y_0 - \alpha^2_{N_1} J_{t_0}^\alpha \quad \left[\frac{g(x(s))}{A(y_0) + F_-(x(s))} N_1^\alpha x(s)\right](t) \\ &\le y_0 - \alpha^2_{N_1} J_{x(t_0)}^\alpha \quad \left[\frac{g(u)}{A(y_0) + F_-(u))}\right](x(t)) \\ &\le y_0 - \frac{\alpha^2}{(1 + |A(y_0)|)} \qquad {}_{N_1}J_{x(t_0)}^\alpha \quad \left[\frac{g(u)}{1 + F_-(u))}\right](x(t)) \xrightarrow{t\to+\infty} -\infty. \end{aligned}$$

Since $x(t) \xrightarrow{t\to+\infty} +\infty$, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that $x(t_n) = x_n$ defined in 2). Thus

$$-c \le \lim_{n\to+\infty} F(x_n) < \lim_{n\to+\infty} A(y_n) = -\infty$$

which is a contradiction. Taking into consideration the two previous cases, we have $x = x(t), \quad y = y(t)$ are decreasing for all $t > t_0$ in the region R_2 then exists $x_2 \ge 0$ such that $\lim_{t\to+\infty} x(t) = x_2, \quad x_2 \le x(t) \le x_0$ for all $t \ge t_0$. If $\lim_{t\to+\infty} y(t) = -\infty$ then

$$N_1^\alpha x = \frac{A(y(t)) - F(x(t))}{h(x(t))} \le \frac{1}{\beta}\left[A(y(t)) - \quad \underset{x_2\le u\le x_0}{inf} F(u)\right] \xrightarrow{t\to+\infty} -\infty$$

which contradicts $x_2 \le x(t) \le x_0$. Now suppose that

$$\lim_{t\to+\infty} y(t) = y_1 > -\infty.$$

Integrate the second equation of (1)

$$N_1^\alpha y = -h(x(t))g(x, N_1^\alpha x)$$

we have

$$y(t) = y_0 -{}_{N_1} J_{t_0}^\alpha \quad h(x(s))g(x(s), N_1^\alpha x(s))(t).$$

Then $\lim_{t\to+\infty} x(t) = 0$ since $x(t)$ is decreasing, $z_m < x_0$, z_m is given in 3). Let $x(t_m) = z_m$ in some $t_m > t_0$.

$$\begin{aligned} A(y_1) &= \lim_{t\to+\infty} A(y(t)) \le A(y(t_m)) \\ &\le F(x(t_m)) = F(z_m) \le 0 \end{aligned}$$

hence $t_1 > t_0$ implies that $A(y(t)) \le \frac{1}{2}A(y_1)$ and $|F(x(t))| \le \frac{1}{4}|A(b_1)|$. From here we have

$$N_1^\alpha x = \frac{A(y(t)) - F(x(t))}{h(x)} \le \frac{1}{\beta}\left[\frac{1}{2}A(b_1) + A(b_1)\right] = \frac{1}{4\beta}A(b_1). \tag{6}$$

Integrating the above inequality we obtain the contradiction

$$x(t) \leq x(t_1) + \frac{1}{4\beta} A(b_1)(t - t_1) \xrightarrow{t \to +\infty} -\infty$$

Therefore, these exist $t_2 > t_1 > t_0$ such that $x(t_2) = 0$ and $y(t_2) < 0$, $x(t_3) = 0$ and $y(t_3) > 0$. A similar argument allows us to obtain that if $A(y_0) < F(x_0)$ and $x_0 < 0$, there exists $t_2 > t_0$ such that $x(t_2) = 0$ and $y(t_0) > 0$ whenever (4) holds. We can conclude that if (3) and (4) hold then all solutions of (1) oscilled and the proof of sufficiency is complete.

Necessity.
Suppose that (3) or (4) fail. For example (3). We assume that

$$\lim_{x \to +\infty} sup F(x) < +\infty$$

and

$$_{N_1}J_0^\alpha \quad \frac{g(x)}{1 + F_-(x)}(+\infty) < +\infty.$$

Let $M > 0$ such that $F(x) \leq M$ for all $x \geq 0$. For any $x_0 > 0$,

$$y_0 = h\beta g(x_0) + \beta^2 \quad _{N_1}J_0^\alpha \quad \frac{g(x)}{1 + F_-(x)}(+\infty) + A^{-1}(M + 1)$$

claim the $A(y(t)) > M+1$ for all $t \geq 0$. Otherwise, there exists $\varphi \geq 0$ such that $A(y(\varphi)) = M+1$ and $A(y(s)) > M + 1$ for all $s \in [0, \varphi)$. It then follows that, for $t \in [0, \varphi)$

$$\begin{aligned} N_1^\alpha x &= \frac{A(y(t)) - F(x(t))}{h(x(t))} \geq \frac{M + 1 - (M - F_-(x))}{h(x(t))} \\ &= \frac{1 + F_-(x)}{h(x(t))} \end{aligned}$$

Thus $x = x(t)$ is increasing on $[0, \varphi)$. Consider now two cases

I) Suppose that $\varphi \leqslant h$, integrate the second equation of (1) $N_1^\alpha y(t) = -h(x(t))g(x, N_1^\alpha x)$ we have the contradiction

$$y(\varphi) = y_0 -_{N_1} J_0^\alpha a(x(s))g(x(s), N_1^\alpha x(s))(\varphi) \geq y_0 - h\beta g(x_0) > A^{-1}(M + 1)$$

II) $\varphi > h$, then

$$\begin{aligned} y(\varphi) &= y_0 - \int_0^h a(x(s))g(x(s), N_1^\alpha x(s))ds - \int_h^\varphi a(x(s))g(x(s), N_1^\alpha x(s))ds \\ &\geq y_0 - h\beta g(x_0) - \beta^2 \int_{x_0}^{+\infty} \frac{g(x)}{1 + F_-(x)}dx \\ &> A^{-1}(M + 1) \end{aligned}$$

is contradiction.

In view of $I)$ and $II)$ we get our affirmation. Hence, $A(y(t)) > M + 1$ and

$$N_1^\alpha = \frac{A(y(t)) - F(x(t))}{h(x(t))} > \frac{1}{\beta}(M + 1 - F(x)) > \frac{1}{\beta}$$

for all $t \geq 0$. Consequently we have $x(t) \geqslant 1\beta(t - t_0)x_0 \quad \to +\infty$ as $t \to +\infty$. This implies that $(x(t), y(t))$ does not oscilates and the proof of necessity is complete.
In this way the proof of the theorem is completed. $\square$

3 Conclusions

In this work, we study the oscillatory character of a non-conformable equation of order 2α using the analysis of the phase plane, in this way, we extend known results for ordinary second order differential equations and fractional differential equations with the classical derivative of Caputo.

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