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PERIODIC AND SINGULAR KINK SOLUTIONS OF THE HAMILTONIAN AMPLITUDE EQUATION

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Abstract. In this paper, the improved $\tan(\Phi(\xi)/2)$-expansion and $\tanh(\Phi(\xi)/2)$-expansion methods are proposed to seek more general exact solutions of the Hamiltonian amplitude equation. These methods are applied to the new Hamiltonian amplitude equation. The exact particular solutions contain five types hyperbolic function solution, trigonometric function solution, exponential solution, logarithmic solution and rational solution. Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions have been found. It is shown that the proposed method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving the nonlinear partial differential equations.

Keywords: Improved $\tan(\Phi(\xi)/2)$-expansion method, Hamiltonian amplitude equation, solitons, kink, periodic and rational solutions.

AMS Subject Classification: 34A34.

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1 Introduction

The study of nonlinear evolution equations (NLEEs) appear everywhere in applied mathematics and theoretical physics including engineering sciences and biological sciences. These equations play a key role in describing key scientific phenomena. For this reason, the search of exact travelling wave solutions to NLEEs plays very important role in the study of these physical phenomena. In every phenomenon in real life, there are many parameters and variables related to each other under the imperious law on that phenomenon. When the relations between the parameters and variables are presented in mathematical language we usually derive a mathematical model of the problem, which may be an equation, a differential equation, an integral equation, a system of integral equations and etc. We first present an applicable analytical method for solving the new Hamiltonian amplitude equation in which are called the improved $\tanh(\Phi(\xi)/2)$-expansion method. In fact, it has been discovered that many models in mathematics and physics are described by nonlinear partial differential equations. With the rapid development of nonlinear sciences based on computer algebraic system, many effective methods have been presented, such as, the homotopy analysis method (Dehghan et al., 2010a,b), the variational iteration method, (He, 1999; Dehghan et al., 2010c; Jafari et al., 2014), the homotopy perturbation method (Dehghan et al., 2010c; Dehghan & Manafian, 2009), the sine-cosine method (Wazwaz, 2006), the tanh-coth method (Manafian Heris & Zamanpour, 2014; Abdou & Soliman, 2006; El-Wakil & Abdou, 2007), the modified extended tanh-function method (Abdou & Soliman, 2006; El-Wakil & Abdou, 2007), the Exp-function method (Dehghan et al., 2011a,b; Manafian

Heris & Bagheri, 2010), the exp(-$\Phi(\xi)$)-expansion method (Roshid & Rahman, 2014; Hafez et al., 2015), the $(\frac{G'}{G})$-expansion method (Aghdaei & Manafian Heris, 2011; Naher & Abdullah, 2013), the modified simple equation method (Jawad et al., 2010), the novel $(\frac{G'}{G})$-expansion method (Alam et al., 2014; Abazari, 2013), the new approach of the generalized $(\frac{G'}{G})$-expansion method (**?**), the Jacobi elliptic function method (Chen & Wang, 2005), the homogeneous balance method (Zhao & Wang, 2006) and so on. In this paper, we consider the Hamiltonian amplitude equation as follows

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0, \tag{1}$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$. The current equation was recently introduced by Wadati et al. (1992). This is an equation which governs certain instabilities of modulated wave trains, with the additional term u_{xt} overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinski equation which arises in dissipative systems and is apparently not integrable. Here, we use the improved $\tanh(\Phi(\xi)/2)$-expansion method for constructing a range of exact solutions for the following ordinary partial differential equations that in this article we developed solutions as well. In this paper, we put forth the new approaches of improved $\tanh(\Phi(\xi)/2)$-expansion method to construct exact travelling wave solutions including solitons, kink, periodic and rational solutions to the Hamiltonian amplitude equation. The purpose of this paper is to obtain exact solutions of the new Hamiltonian amplitude equation and to determine the accuracy of the improved $\tanh(\Phi(\xi)/2)$-expansion method in solving this kind of problems. The paper is organized as follows: In Section 2, we describe the improved $\tanh(\Phi(\xi)/2)$-expansion method. In section 3, we examine the new Hamiltonian amplitude equation with method introduced in Sections 2 and offer the physical interpretations of the solutions. Also conclusion is given in Section 4. Finally some references are given at the end of this paper.

2 Methodology

2.1 *Description of improved* $\tanh(\Phi(\xi)/2)$*-expansion technique*

The objective of this section is to outline the use of the $\tanh(\Phi(\xi)/2)$-expansion for solving certain nonlinear PDE.

Step 1. We suppose that given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{2}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u'', \mu^2 u'', ...) = 0, \tag{3}$$

by the transformation $\xi = \mathrm{x} - \mu\mathrm{t}$ is the wave variable. Also, μ is constant to be determined later.

Step 2. Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} A_k \left[p + \tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{m} B_k \left[p + \tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-k}, \tag{4}$$

where A_k $(0 \le k \le m)$ and B_k $(1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation:

$$\Phi'(\xi) = a\sinh(\Phi(\xi)) + b\cosh(\Phi(\xi)) + c. \tag{5}$$

We will consider the following special solutions of equation (5):

Family 1: When $a^2+c^2-b^2<0$ and $b-c\neq 0$, then

$$\Phi(\xi)=2tanh^{-1}\left[-\frac{a}{b-c}+\frac{\sqrt{b^2-a^2-c^2}}{b-c}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)\right].$$

Family 2: When $a^2+c^2-b^2>0$ and $b-c\neq 0$, then

$$\Phi(\xi)=-2tanh^{-1}\left[\frac{a}{b-c}+\frac{\sqrt{a^2+c^2-b^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)\right].$$

Family 3: When $a^2+c^2-b^2<0$, $b\neq 0$ and $c=0$, then

$$\Phi(\xi)=2tanh^{-1}\left[-\frac{a}{b}+\frac{\sqrt{b^2-a^2}}{b}\tan\left(\frac{\sqrt{b^2-a^2}}{2}(\xi+C)\right)\right].$$

Family 4: When $a^2+c^2-b^2>0$, $c\neq 0$ and $b=0$, then

$$\Phi(\xi)=2tanh^{-1}\left[\frac{a}{c}+\frac{\sqrt{a^2+c^2}}{c}\tanh\left(\frac{\sqrt{a^2+c^2}}{2}(\xi+C)\right)\right].$$

Family 5: When $a^2+c^2-b^2<0$, $b-c\neq 0$ and $a=0$, then

$$\Phi(\xi)=2tanh^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tan\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right].$$

Family 6: When $a=0$ and $c=0$, then $\Phi(\xi)=\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]$.

Family 7: When $b=0$ and $c=0$, then $\Phi(\xi)=\ln\left[-\tanh\left(\frac{a}{2}(\xi+C)\right)\right]$.

Family 8: When $a^2+b^2=c^2$, then

$$\Phi(\xi)=2tanh^{-1}\left[\frac{a}{-b+\sqrt{a^2+b^2}}+\frac{\sqrt{2}a}{-b+\sqrt{a^2+b^2}}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)\right].$$

Family 9: When $a=b=c=ka$, then $\Phi(\xi)=2tanh^{-1}\left[e^{ka(\xi+C)}-1\right]$.

Family 10: When $a=c=ka$ and $b=-ka$, then $\Phi(\xi)=2tanh^{-1}\left[\frac{e^{ka(\xi+C)}}{-1+e^{ka(\xi+C)}}\right]$.

Family 11: When $b=a$, then $\Phi(\xi)=-2tanh^{-1}\left[\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right]$.

Family 12: When $b=c$, then $\Phi(\xi)=2tanh^{-1}\left[\frac{e^{a(\xi+C)}-c}{a}\right]$.

Family 13: When $a=-c$ and $b=c$, then $\Phi(\xi)=2tanh^{-1}\left[1+e^{-c(\xi+C)}\right]$.

Family 14: When $b=-b$ and $c=-b$, then $\Phi(\xi)=2tanh^{-1}\left[\frac{b+e^{a(\xi+C)}}{a}\right]$.

Family 15: When $b=-b$, $a=-b$, and $c=b$, then $\Phi(\xi)=2tanh^{-1}\left[\frac{1}{e^{b(\xi+C)}-1}\right]$.

Family 16: When $b=-c$, then $\Phi(\xi)=2tanh^{-1}\left[\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]$.

Family 17: When $a=0$ and $b=c$, then $\Phi(\xi)=2tanh^{-1}\left[c(\xi+C)\right]$.

Family 18: When $a = 0$ and $b = -c$, then $\Phi(\xi) = 2tanh^{-1}\left[\dfrac{1}{c(\xi + C)}\right]$.

Family 19: When $b = 0$ and $a = c$, then $\Phi(\xi) = 2tanh^{-1}\left[1 + \sqrt{2}\tanh\left(\frac{\sqrt{2}c}{2}(\xi + C)\right)\right]$,
where $A_k, B_k (k = 1, 2, ..., m), a, b$ and c are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (5). If m is not an integer, then a transformation formula should be used to overcome this difficulty.

Step 3. Substituting (4) into Eq. (3) with the value of m obtained in Step 2. Collecting the coefficients of $\tanh\left(\frac{\Phi(\xi)}{2}\right)^k$, $\coth\left(\frac{\Phi(\xi)}{2}\right)^k$ $(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_0, A_k, B_k (k = 1, 2, ..., m)$ a, b, c and p with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, ..., A_m, B_m, \mu, p$ in (4).

3 The Hamiltonian amplitude equation

In this section, we present the improved $\tanh(\Phi(\xi)/2)$-expansion methods to solve the Hamiltonian amplitude equation where introduced in Sections 2.

3.1 *The improved* $\tanh(\Phi(\xi)/2)$*-expansion method*

We consider the Hamiltonian amplitude equation as follows

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0, \tag{6}$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$. This is an equation which governs certain instabilities of modulated wave trains, with the additional term u_{xt} overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinski equation which arises in dissipative systems and is apparently not integrable. By make the transformation

$$u(x,t) = e^{i\eta}v(\xi), \quad \eta = \alpha x + \beta t, \quad \xi = \mu(x - st), \tag{7}$$

Eq. (7) is carried to an ODE

$$(\mu s^2 + \varepsilon\mu^2 s)v'' + i(\mu - 2\beta\mu s - \varepsilon\beta\mu + \varepsilon\alpha\mu s)v' - (\alpha + \beta^2 - \varepsilon\alpha\beta)v + 2\sigma v^3 = 0. \tag{8}$$

If we take

$$s = \frac{1 - \varepsilon\beta}{2\beta - \alpha\varepsilon}, \tag{9}$$

then Eq. (8) transform into

$$(\mu s^2 + \varepsilon\mu^2 s)v'' - (\alpha + \beta^2 - \varepsilon\alpha\beta)v + 2\sigma v^3 = 0. \tag{10}$$

By balancing the v$''$ and v^3, using homogenous principle, in Eq. (8) we get

$$M + 2 = 3M, \qquad \Rightarrow M = 1. \tag{11}$$

Then by using section 3, the trail solution will be as

$$v(\xi) = A_0 + A_1\left[p + \tanh\left(\frac{\Phi(\xi)}{2}\right)\right] + B_1\left[p + \tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \tag{12}$$

Substituting (12) and (5) into Eq. (10), we obtain the following results

Set I:

$$s = s,\ A_0 = 0,\ A_1 = 0,\ B_1 = B_1,\ \mu = \pm\frac{2(b-c)}{\Delta}\sqrt{\frac{-\sigma}{s^2 + s\varepsilon}}B_1,\ p = \frac{a}{b-c}, \tag{13}$$

$$\Delta = a^2 + b^2 - c^2, \quad \beta = \beta, \;\; \alpha = \frac{\Delta\beta^2 - 2\sigma(b-c)^2 B_1^2}{\Delta(\varepsilon\beta - 1)}, \;\; u(\xi) = B_1 \left[\frac{a}{b-c} + \tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1} e^{i\eta}. \tag{14}$$

By using of the (14) and **Families 1, 2, 6, 8, 10, 11** and **15-19** get respectively as

$$u_1(\xi) = \frac{(b-c)B_1}{\sqrt{-\Delta}} \cot\left(\frac{\sqrt{-\Delta}}{2}(\xi + C)\right) e^{i\eta}, u_2(\xi) = -\frac{(b-c)B_1}{\sqrt{\Delta}} \coth\left(\frac{\sqrt{\Delta}}{2}(\xi + C)\right) e^{i\eta}, \tag{15}$$

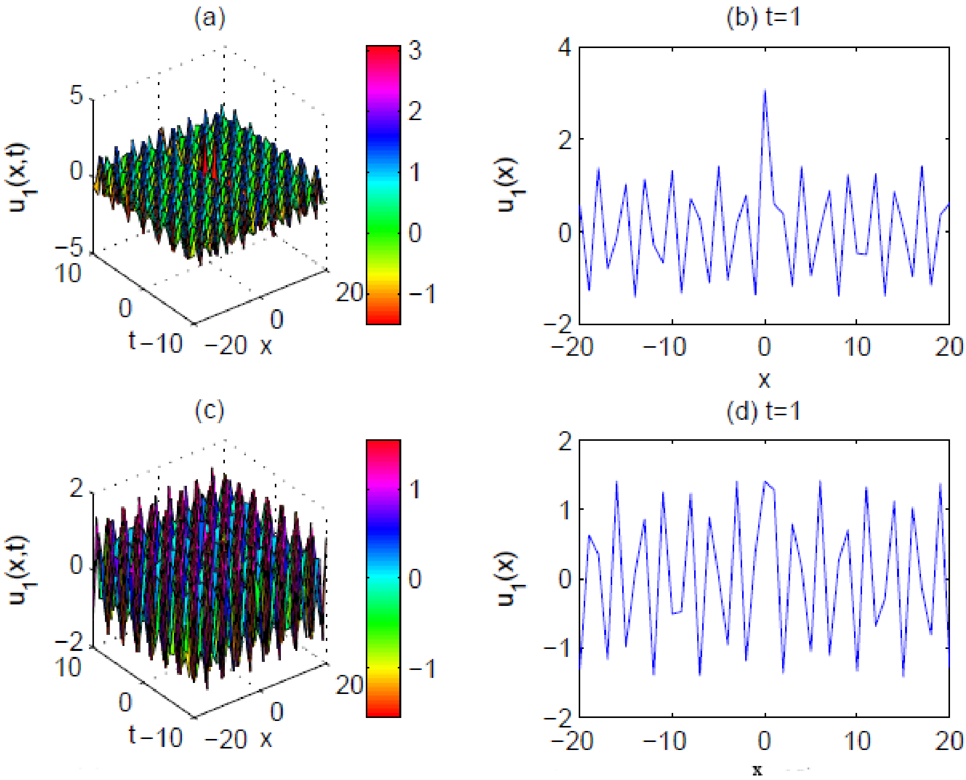

Figure 1: Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_1 (15) are demonstrated at $a = 1,\ b = 1,\ c = 2,\ B_1 = 2, \varepsilon = \frac{1}{3},\ \beta = 2,\ \sigma = 1$ and by considering the values (a) and (c) $-20 < x < 20,\ -10 < t < 10$ and (b) and (d) $-20 < x < 20,\ t = 1$.

$$\begin{aligned} u_3(\xi) &= B_1 \coth\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi + C)\right)\right]\right) e^{i\eta}, \\ u_4(\xi) &= \frac{\sqrt{2}aB_1}{-b + \sqrt{a^2 + b^2}} \coth\left(\frac{\sqrt{2}a}{2}(\xi + C)\right) e^{i\eta}. \end{aligned} \tag{16}$$

$$u_5(\xi) = B_1 \left[-\frac{1}{2} + \frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)} - 1\right]}\right]^{-1} e^{i\eta}, \;\; u_6(\xi) = B_1 \left[\frac{a}{a-c} - \frac{(a+c)e^{c(\xi+C)} - 1}{(a-c)e^{c(\xi+C)} - 1}\right]^{-1} e^{i\eta}, \tag{17}$$

$$u_7(\xi) = B_1 \left[\frac{1}{2} + \frac{1}{e^{b(\xi+C)} - 1}\right]^{-1} e^{i\eta}, \;\; u_8(\xi) = B_1 \left[-\frac{a}{2c} + \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)} - 1}\right]^{-1} e^{i\eta},$$

$$u_9(\xi) = \frac{B_1}{c(\xi + C)} e^{i\eta}, \;\; u_{10}(\xi) = cB_1(\xi + C)e^{i\eta}, \;\; u_{11}(\xi) = \frac{\sqrt{2}B_1}{2} \coth\left(\frac{\sqrt{2}}{2}(\xi + C)\right) e^{i\eta}, \tag{18}$$

where $\xi = \pm\frac{2(b-c)}{a^2+b^2-c^2}\sqrt{\frac{-\sigma}{s^2+s\varepsilon}}B_1(x-st)$, $\eta = \left(\frac{(a^2+b^2-c^2)\beta^2-2\sigma(b-c)^2B_1^2}{(a^2+b^2-c^2)(\varepsilon\beta-1)}\right)x+\beta t$ and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set II:

$$A_0=0,\ A_1=0,\ B_1=\pm\frac{1}{(b-c)\varepsilon}\sqrt{\frac{a^2+c^2-b^2}{2\sigma}},\ \mu=\pm\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+c^2-b^2)(s^2+s\varepsilon)}},\tag{19}$$

$$s=s,\ p=\frac{a}{b-c},\ \beta=\frac{1}{\varepsilon},\ \alpha=\alpha,$$
$$u(\xi)=\pm\frac{1}{(b-c)\varepsilon}\sqrt{\frac{a^2+c^2-b^2}{2\sigma}}\left[\frac{a}{b-c}+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}e^{i\eta}.\tag{20}$$

By using of the (20) and **Families 1, 2, 6, 10, 11, 15** and **16** give respectively as

$$u_{12}(\xi)=\pm\frac{1}{\varepsilon\sqrt{-2\sigma}}\cot\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)e^{i\eta},$$
$$u_{13}(\xi)=\mp\frac{1}{\varepsilon\sqrt{2\sigma}}\coth\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)e^{i\eta},\tag{21}$$

$$u_{14}(\xi)=\pm\frac{1}{\varepsilon\sqrt{-2\sigma}}\coth\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)e^{i\eta},$$
$$u_{15}(\xi)=\mp\frac{1}{2\varepsilon\sqrt{2\sigma}}\left[-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1}e^{i\eta},\tag{22}$$

$$u_{16}(\xi)=\pm\frac{c}{(a-c)\varepsilon\sqrt{2\sigma}}\left[\frac{a}{a-c}-\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right]^{-1}e^{i\eta},\tag{23}$$

$$u_{17}(\xi)=\mp\frac{1}{2\varepsilon\sqrt{2\sigma}}\left[\frac{1}{2}+\frac{1}{e^{b(\xi+C)}-1}\right]^{-1}e^{i\eta},\ u_{18}(\xi)=\mp\frac{a}{2c\varepsilon\sqrt{2\sigma}}\left[-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}e^{i\eta},$$

where $\xi=\pm\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+c^2-b^2)(s^2+s\varepsilon)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$ and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set III:

$$A_0=\pm\mu(a+(c-b)p)\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}},\ A_1=0,$$
$$B_1=\mp\mu(2ap+(c-b)p^2-b-c)\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}},\tag{24}$$

$$\mu=\mu,\ s=s,\ p=p,\ \beta=\beta,\ \ \alpha=\frac{(a^2+c^2-b^2)(s^2\mu^2+\varepsilon s\mu^2)+\beta^2}{2(\varepsilon\beta-1)},\tag{25}$$

$$u(\xi)=\pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(c-b)p]-(2ap+(c-b)p^2-b-c)\left[p+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}.$$

By using of the (25) and **Families 1, 2** and **6-19** can be written respectively as

$$u_{19}(\xi)=\pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(c-b)p]-[2ap+(c-b)p^2-b-c]\left[p-\frac{a}{b-c}+\right.\right.\tag{26}$$
$$\left.\left.\frac{\sqrt{b^2-a^2-c^2}}{b-c}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta},$$

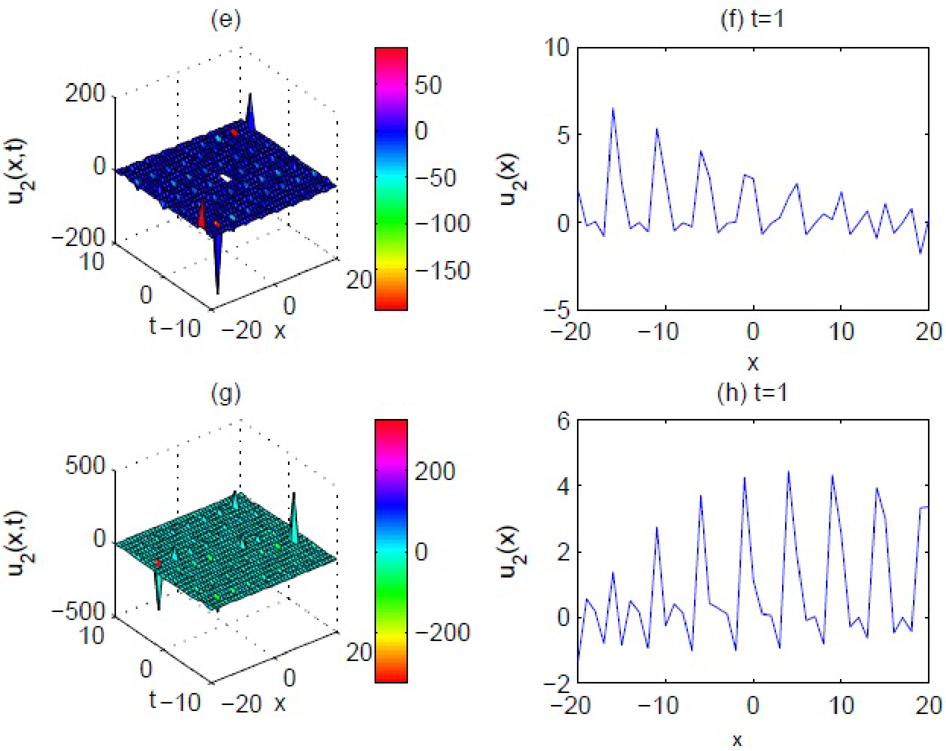

Figure 2: Graphs of (e) and (f) real values and (g) and (h) imaginary values of u_2 (15) are demonstrated at $a = 2, b = 3, c = 1, B_1 = 2, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (e) and (g) $-20 < x < 20, -10 < t < 10$ and (f) and (h) $-20 < x < 20, t = 1$.

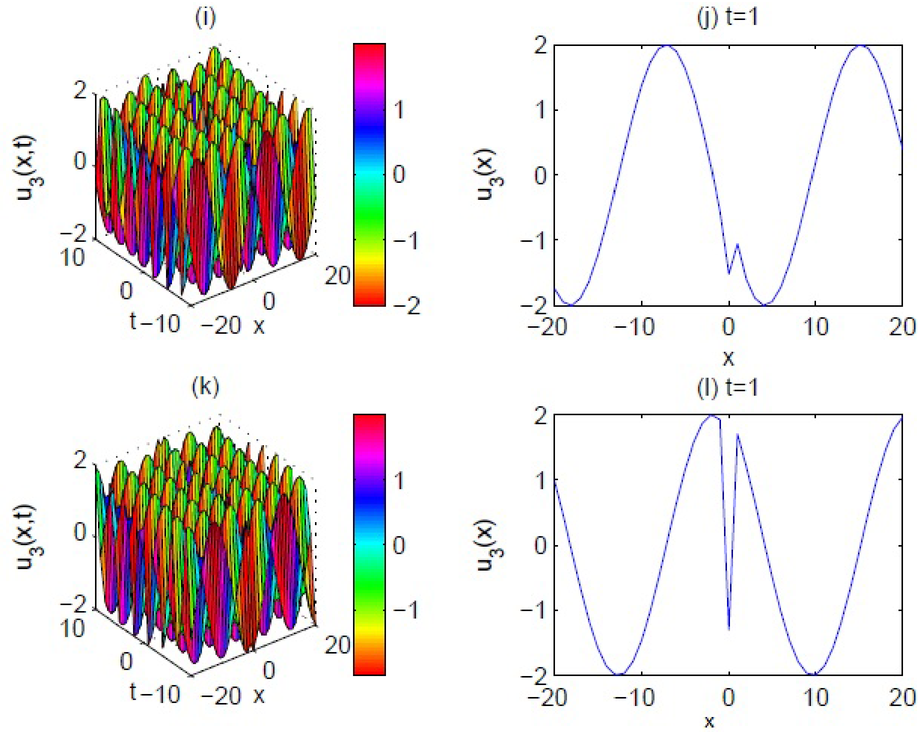

Figure 3: Graphs of (i) and (j) real values and (k) and (l) imaginary values of u_3 (16) are demonstrated at $a = 0, b = 3, c = 0, B_1 = 2, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (i) and (k) $-20 < x < 20, -10 < t < 10$ and (j) and (l) $-20 < x < 20, t = 1$

$$u_{20}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(c-b)p]-[2ap+(c-b)p^2-b-c]\left[p+\frac{a}{b-c}+\frac{\sqrt{a^2+c^2-b^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta}, \quad (27)$$

$$u_{21}(\xi) = \mp b\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{p-(p^2+1)\left[p+\tanh\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)\right]^{-1}\right\}e^{i\eta}, \quad (28)$$

$$u_{22}(\xi) = \pm a\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{1-2p\left[p+\tanh\left(\frac{1}{2}\ln\left[-\tanh\left(\frac{a}{2}(\xi+C)\right)\right]\right)\right]^{-1}\right\}e^{i\eta},$$

$$u_{23}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(c-b)p]-[2ap+(c-b)p^2-b-c]\left[p+\frac{a}{-b+\sqrt{a^2+b^2}}+\frac{\sqrt{2}a}{-b+\sqrt{a^2+b^2}}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta},$$

$$u_{24}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}ka\left\{1-2(p-1)\left[p+e^{ka(\xi+C)}-1\right]^{-1}\right\}e^{i\eta}, \quad (29)$$

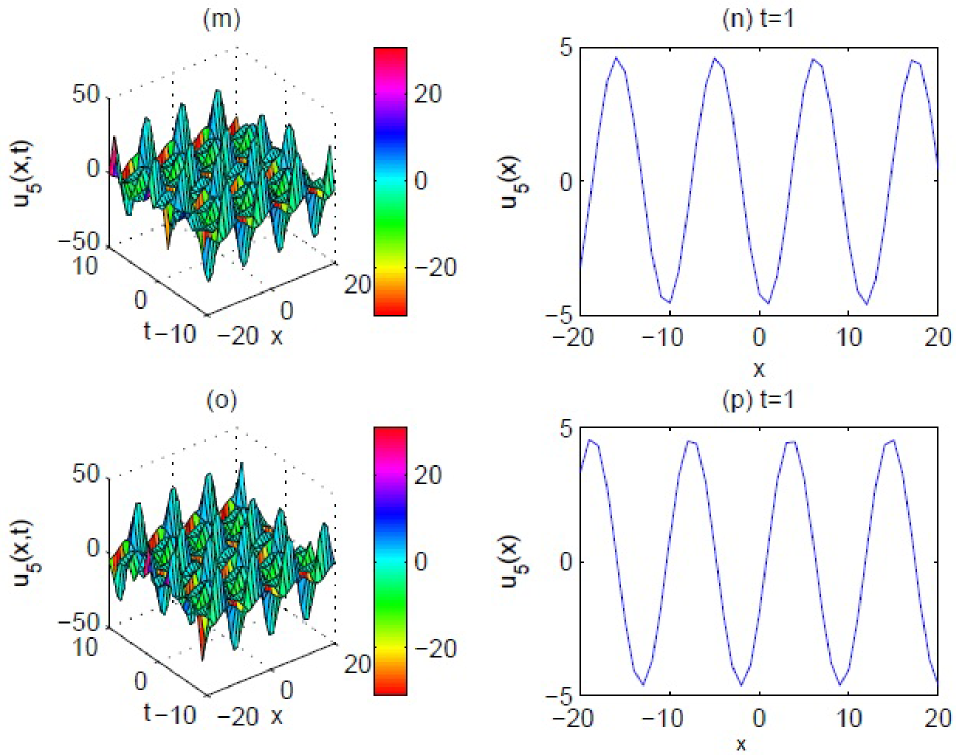

Figure 4: Graphs of (m) and (n) real values and (o) and (p) imaginary values of u_5 (17) are demonstrated at $a = 2, b = -2, c = 2, k = 2, B_1 = 2, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (m) and (o) $-20 < x < 20, -10 < t < 10$ and (n) and (p) $-20 < x < 20, t = 1$

$$u_{25}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}ka\left\{1+2p-2p(1+p)\left[p+\frac{e^{a(\xi+C)}-1}{e^{a(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{26}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\times$$

$$\times\left\{[a+(c-a)p]-[2ap+(c-a)p^2-b-c]\left[p-\frac{(a+c)e^{b(\xi+C)}+1}{(a-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{27}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{a-[2ap-2c]\left[p+\frac{e^{a(\xi+C)}-c}{a}\right]^{-1}\right\}e^{i\eta},$$

$$u_{28}(\xi) = \mp\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{c+[2ap+2c]\left[p+1+e^{-c(\xi+C)}\right]^{-1}\right\}e^{i\eta},$$

$$u_{29}(\xi) = \pm b\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[-1+2p]+[2p-2p^2]\left[p+\frac{1}{e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{30}(\xi) = \pm b\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+2pc]-2p[a+cp]\left[p+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta}.$$

$$u_{31}(\xi) = \pm 2c\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\,[p+c(\xi+C)]^{-1}\,e^{i\eta},$$

$$u_{32}(\xi) = \pm 2cp\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{1-p\left[p+\frac{1}{c(\xi+C)}\right]^{-1}\right\}e^{i\eta},$$

$$u_{33}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[c+cp]-[2cp+cp^2-c]\left[p+1+\sqrt{2}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta},$$

where $\xi=\mu(x-st)$, $\eta=\left(\frac{(a^2+c^2-b^2)(s^2\mu^2+\varepsilon s\mu^2)+\beta^2}{2(\varepsilon\beta-1)}\right)x+\beta t$ and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set IV:

$$\beta=\beta,\quad s=s,\quad p=0,\quad \mu=\pm\frac{2A_1}{b-c}\sqrt{-\frac{\sigma}{s^2+\varepsilon s}},\qquad A_0=\frac{aA_1}{b-c},\qquad A_1=A_1,\qquad B_1=0, \tag{30}$$

$$\alpha=-\frac{2(a^2+b^2-c^2)\sigma A_1^2-\beta^2(b-c)^2}{(b-c)^2(\varepsilon\beta-1)},\quad u(\xi)=A_1\left[\frac{a}{b-c}+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]e^{i\eta}. \tag{31}$$

By using of the (31) and **Families 1, 2, 6, 8, 10, 11** and **15-18** get respectively as

$$u_{34}(\xi) = A_1\frac{\sqrt{b^2-a^2-c^2}}{b-c}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)e^{i\eta}, \tag{32}$$

$$u_{35}(\xi) = -A_1\frac{\sqrt{a^2+c^2-b^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)e^{i\eta}, \tag{33}$$

$$\begin{aligned} u_{36}(\xi) &= A_1\tanh\left(\tfrac{1}{2}\ln\left[\tan\left(\tfrac{b}{2}(\xi+C)\right)\right]\right)e^{i\eta},\\ u_{37}(\xi) &= \tfrac{A_1a\sqrt{2}}{-b+\sqrt{a^2+b^2}}\tan\left(\tfrac{\sqrt{2}a}{2}(\xi+C)\right)e^{i\eta}, \end{aligned} \tag{34}$$

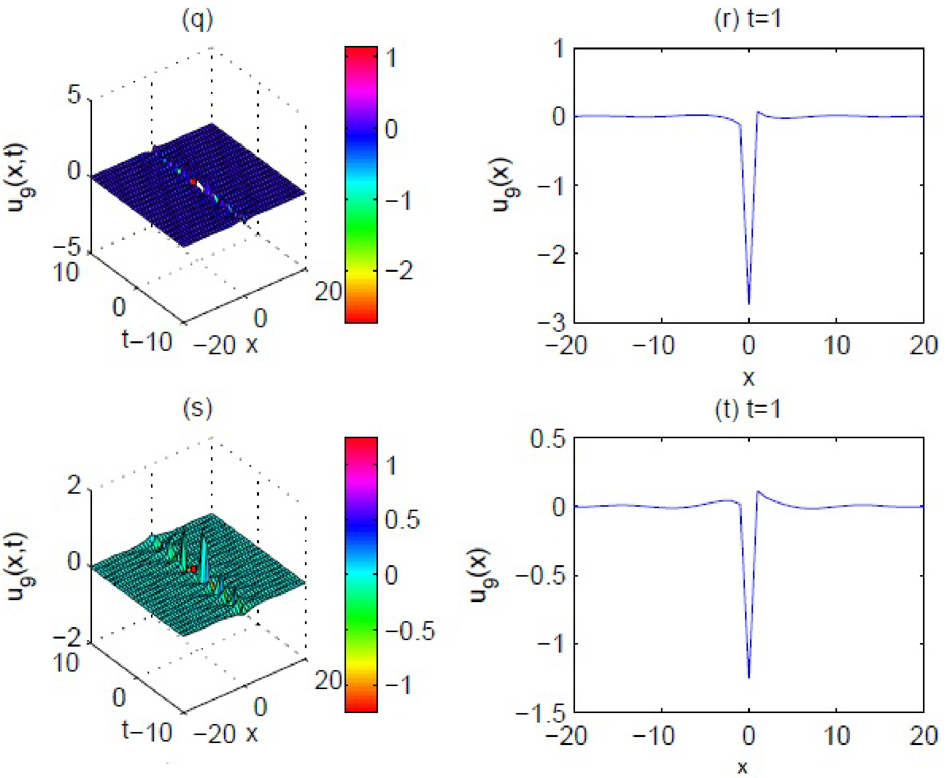

Figure 5: Graphs of (q) and (r) real values and (s) and (t) imaginary values of u_9 (18) are demonstrated at $a = 0, b = 2, c = 2, B_1 = 2, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (q) and (s) $-20 < x < 20, -10 < t < 10$ and (r) and (t) $-20 < x < 20, t = 1$.

$$u_{38}(\xi) = A_1\left[-\frac{1}{2} + \frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]e^{i\eta}, \quad u_{39}(\xi) = A_1\left[\frac{a}{a-c} - \frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right]e^{i\eta}, \qquad (35)$$

$$u_{40}(\xi) = A_1\left[\frac{1}{2} + \frac{1}{e^{b(\xi+C)}-1}\right]e^{i\eta}, \qquad u_{41}(\xi) = A_1\left[-\frac{a}{2c} + \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]e^{i\eta},$$

$$u_{42}(\xi) = A_1 c(\xi+C)e^{i\eta}, \qquad u_{43}(\xi) = \frac{A_1}{c(\xi+C)}e^{i\eta}, \quad u_{44}(\xi) = A_1\sqrt{2}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)e^{i\eta},$$

where $\xi = \pm\frac{2A_1}{b-c}\sqrt{-\frac{\sigma}{s^2+\varepsilon s}}(x-st)$, $\eta = \left(-\frac{2(a^2+c^2-b^2)\sigma A_1^2-\beta^2(b-c)^2}{(b-c)^2(\varepsilon\beta-1)}\right)x + \beta t$ and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set V:

$$\mathrm{p} = \frac{1}{\mathrm{b}-\mathrm{c}}\left(\sqrt{\frac{\mathrm{a}^2+\mathrm{c}^2-\mathrm{b}^2}{3}}+\mathrm{a}\right), \quad \mu = \frac{1}{\varepsilon}\sqrt{\frac{-2}{(\mathrm{a}^2+\mathrm{c}^2-\mathrm{b}^2)(\mathrm{s}^2+\varepsilon\mathrm{s})}}, \qquad \mathrm{A}_0 = \pm\frac{1}{\varepsilon\sqrt{6\sigma}}, \qquad (36)$$

$$s = s, \quad \beta = \frac{1}{\varepsilon}, \quad A_1 = 0, \quad B_1 = \pm\frac{1}{(b-c)\varepsilon}\sqrt{\frac{2(a^2+c^2-b^2)}{9\sigma}}, \qquad \alpha = \alpha, \qquad (37)$$

$$u(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}} \pm \frac{1}{(b-c)\varepsilon}\sqrt{\frac{2(a^2+c^2-b^2)}{9\sigma}}\left[\frac{1}{b-c}\left(\sqrt{\frac{a^2+c^2-b^2}{3}}+a\right)+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta},$$

By using of the (37) and **Families 1, 2, 6, 10, 11, 15, 16, 18** and **19** give respectively as

$$u_{45}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}} \pm \frac{1}{\varepsilon}\sqrt{\frac{2(a^2+c^2-b^2)}{9\sigma}}\times\right.$$
$$\left.\times\left[\frac{a^2+c^2-b^2)}{3} + \sqrt{b^2-a^2-c^2}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta}, \qquad (38)$$

$$u_{46}(\xi)=\Bigg\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{\frac{2(a^2+c^2-b^2)}{9\sigma}}\times \\ \times\left[\frac{a^2+c^2-b^2)}{3}-\sqrt{a^2+c^2-b^2}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)\right]^{-1}\Bigg\}e^{i\eta}, \tag{39}$$

$$u_{47}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{-\frac{2}{9\sigma}}\left[\frac{1}{\sqrt{-3}}+\tanh\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)\right]^{-1}\right\}e^{i\eta}, \tag{40}$$

$$u_{48}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\mp\frac{1}{\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[-\frac{1}{2\sqrt{3}}-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1}\right\}e^{i\eta}, \tag{41}$$

$$u_{49}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{c}{(a-c)\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{c}{(a-c)\sqrt{3}}+\frac{a}{a-c}-\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{50}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\mp\frac{1}{2\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{1}{2\sqrt{3}}-\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{51}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{b}{(b+a)\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{b}{(b+a)\sqrt{3}}-\frac{a}{b+a}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{52}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\mp\frac{a}{2c\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[-\frac{a}{2c\sqrt{3}}-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{53}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\mp\frac{a}{2c\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[-\frac{a}{2c\sqrt{3}}-\frac{a}{2c}+\frac{1}{c(\xi+C)}\right]^{-1}\right\}e^{i\eta},$$

$$u_{54}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\mp\frac{a}{2c\varepsilon}\sqrt{\frac{4}{9\sigma}}\left[-\sqrt{\frac{2}{3}}-1+\sqrt{2}\tan\left(\frac{\sqrt{2}c}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta},$$

where $\xi=\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$ and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set VI:

$$s=s,\quad \beta=\frac{1}{\varepsilon},\quad p=\frac{a}{b-c},\quad \mu=\pm\frac{1}{\varepsilon\sqrt{(a^2+c^2-b^2)(s^2+\varepsilon s)}},\qquad A_0=0, \tag{42}$$

$$A_1=\pm\frac{b-c}{\varepsilon\sqrt{-4\sigma(a^2+c^2-b^2)}},\qquad B_1=\pm\frac{\sqrt{a^2+c^2-b^2}}{\varepsilon\sqrt{-4\sigma}(b-c)},\qquad \alpha=\alpha, \tag{43}$$

$$u(\xi)=\left\{A_1\left[\frac{a}{b-c}+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[\frac{a}{b-c}+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}.$$

By using of the (43) and **Families 1, 2, 6, 10, 11, 15** and **16** can be written respectively as

$$u_{55}(\xi)=\pm\frac{1}{\varepsilon\sqrt{4\sigma}}\left\{\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)+\cot\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{44}$$

$$u_{56}(\xi)=\mp\frac{1}{\varepsilon\sqrt{4\sigma}}\left\{\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{45}$$

$$u_{57}(\xi) = \pm\frac{1}{\varepsilon\sqrt{4\sigma}}\left\{\tanh\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right) + \coth\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)\right\}e^{i\eta}, \tag{46}$$

$$u_{58}(\xi) = \mp\frac{1}{\varepsilon\sqrt{-4\sigma}}\left\{2\left(-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)+\frac{1}{2}\left(-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)^{-1}\right\}e^{i\eta}, \tag{47}$$

$$u_{59}(\xi) = \pm\frac{1}{\varepsilon\sqrt{-4\sigma}}\left\{\frac{a-c}{c}\left(\frac{a}{a-c}-\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right)+\right.$$

$$\left.+\frac{c}{a-c}\left(\frac{a}{a-c}-\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{60}(\xi) = \mp\frac{1}{\varepsilon\sqrt{-4\sigma}}\left\{2\left(\frac{1}{2}+\frac{1}{e^{b(\xi+C)}-1}\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{e^{b(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{61}(\xi) = \sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{-4\sigma}}\left\{\frac{2c}{a}\left(-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)+\frac{a}{2c}\left(-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

where $\xi = \pm\frac{1}{\varepsilon\sqrt{(a^2+c^2-b^2)(s^2+\varepsilon s)}}(x-st)$, $\eta = \alpha x + \frac{1}{\varepsilon}t$ and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set VII:

$$\alpha=\alpha,\ \beta=\frac{1}{\varepsilon},\ p=p,\ \mu=\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+c^2-b^2)(s^2+\varepsilon s)}},\ A_0=\pm\frac{a+(c-b)p}{\varepsilon\sqrt{2\sigma(a^2+c^2-b^2)}}, \tag{48}$$

$$s=s,\ A_1=\pm\frac{b-c}{\varepsilon\sqrt{2\sigma(a^2+c^2-b^2)}},\ B_1=0,\ u(\xi)=A_0+A_1\left[p+\tanh\left(\frac{\Phi(\xi)}{2}\right)\right]e^{i\eta}. \tag{49}$$

By using of the (20) and **Families 1, 2, 6, 10, 11, 15** and **18** give respectively as

$$u_{62}(\xi) = \pm\frac{e^{i\eta}}{\varepsilon\sqrt{-2\sigma}}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right),$$
$$u_{63}(\xi) = \mp\frac{e^{i\eta}}{\varepsilon\sqrt{-2\sigma}}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right), \tag{50}$$

$$u_{64}(\xi) = \pm\frac{e^{i\eta}}{\varepsilon\sqrt{-2\sigma}}\left[p+\tanh\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)\right],$$
$$u_{65}(\xi) = \mp\frac{e^{i\eta}}{\varepsilon\sqrt{2\sigma}}\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}, \tag{51}$$

$$u_{66}(\xi) = \mp\frac{2a-c}{\varepsilon\sqrt{2\sigma}}\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}e^{i\eta},\quad u_{67}(\xi) = \mp\frac{1}{\varepsilon\sqrt{2\sigma}}\left(1+\frac{2}{e^{b(\xi+C)}-1}\right)e^{i\eta}, \tag{52}$$

$$u_{68}(\xi) = \mp\frac{1}{\varepsilon\sqrt{2\sigma}}\left(1-\frac{2c}{a}\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)e^{i\eta},\ u_{69}(\xi) = \mp\frac{2}{a\varepsilon\sqrt{2\sigma}(\xi+C)},$$

where $\xi = \frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+c^2-b^2)(s^2+\varepsilon s)}}(x-st)$, $\eta = \alpha x + \frac{1}{\varepsilon}t$ and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Set VIII:

$$p=\frac{a}{b-c},\ \ \mu=\pm\frac{1}{\varepsilon}\sqrt{\frac{-1}{2(a^2+c^2-b^2)(s^2+\varepsilon s)}},\ \ A_0=0,$$
$$A_1=\pm\frac{(b-c)}{\varepsilon\sqrt{8\sigma(a^2+c^2-b^2)}},\ \ \beta=\frac{1}{\varepsilon},\ \ \alpha=\alpha, \tag{53}$$

$$B_1=\pm\frac{\sqrt{a^2+c^2-b^2}}{\sqrt{8\sigma}\varepsilon(b-c)},\ \ s=s,$$
$$u(\xi)=\left\{A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}. \tag{54}$$

By using of the (53) and **Families 1, 2, 6, 10, 11, 15, 16** and **19** can be written respectively as

$$u_{70}(\xi)=\pm\frac{1}{\sqrt{-8\sigma}}\left\{\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)+\cot\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{55}$$

$$u_{71}(\xi)=\mp\frac{1}{\sqrt{8\sigma}}\left\{\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{56}$$

$$u_{72}(\xi)=\pm\frac{1}{\sqrt{-8\sigma}}\left\{\tanh\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)+\coth\left(\frac{1}{2}\ln\left[\tan\left(\frac{b}{2}(\xi+C)\right)\right]\right)\right\}e^{i\eta}, \tag{57}$$

$$u_{73}(\xi)=\mp\frac{1}{\sqrt{8\sigma}}\left\{2\left(-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)+\frac{1}{2}\left(-\frac{1}{2}+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)^{-1}\right\}e^{i\eta}, \tag{58}$$

$$u_{74}(\xi)=\pm\frac{1}{\sqrt{8\sigma}}\left\{\frac{a-c}{c}\left(\frac{a}{a-c}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)+\right.$$
$$\left.+\tfrac{c}{a-c}\left(\frac{a}{a-c}-\frac{(a+c)e^{c(\xi+C)}-1}{(a-c)e^{c(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{75}(\xi)=\mp\frac{1}{\sqrt{8\sigma}}\left\{2\left(\frac{1}{2}+\frac{1}{\left[e^{b(\xi+C)}-1\right]}\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{\left[e^{b(\xi+C)}-1\right]}\right)^{-1}\right\}e^{i\eta},$$

$$u_{76}(\xi)=\mp\frac{1}{\sqrt{8\sigma}}\left\{\frac{2c}{a}\left(-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{\left[ce^{a(\xi+C)}-1\right]}\right)+\frac{a}{2c}\left(-\frac{a}{2c}+\frac{ae^{a(\xi+C)}}{\left[ce^{a(\xi+C)}-1\right]}\right)^{-1}\right\}e^{i\eta},$$

$$u_{77}(\xi)=\mp\frac{1}{\sqrt{\sigma}}\left\{\frac{1}{4}\left(-\frac{a\sqrt{2}}{2c}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)\right)+\frac{1}{2}\left(-\frac{a\sqrt{2}}{2c}\tanh\left(\frac{\sqrt{2}a}{2}(\xi+C)\right)\right)^{-1}\right\}e^{i\eta},$$

where $\xi=\pm\frac{1}{\varepsilon}\sqrt{\frac{-1}{2(a^2+c^2-b^2)(s^2+\varepsilon s)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$ and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

3.2 *Physical Interpretations of the Solutions*

In this section, we depict the graph and signify the obtained solutions to the Hamiltonian amplitude equation. Now, we will discuss all possible physical significance for parameter. The crucial advantage of the new approaches presented in this paper against the generalized and improved (G'/G)-expansion method is that the method provides more general and abundant exact travelling wave solutions with much real parameter. The exact solutions of ODEs and PDEs have its important significance to disclose the internal mechanism of the complex physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

Remark 1. In Figures 1-5, we plot three dimensional graphics of real and imaginary values of (15), (16), (17) and (18) respectively, which denote the dynamics of solutions with appropriate parametric selections. We plot three dimensional graphics of Figs 1-5, when $-20 < x < 20, -10 < t < 10$. To the best of our knowledge, these optical soliton solutions have not been submitted to literature in advance. The analytical solutions and figures obtained in this paper give us a different physical interpretation for the Hamiltonian amplitude equation. Solution u_1 (fig. 1) of the Hamiltonian amplitude equation represents the exact periodic traveling wave solution. Periodic solutions are traveling wave solutions that are periodic, such as $cos(x+t)$.

Remark 2. Figures 2 and 3 (u_2, u_3), represent the exact soliton solutions of the Hamiltonian amplitude equation. Solitons are special kinds of solitary waves. Solitons have a remarkable property that keeps its identity upon interacting with other solitons. Soliton solutions have particle-like structures, for example, magnetic monopoles, and extended structures, like, domain walls and cosmic strings, that have implications in cosmology of the early universe. The other figures are ignored for simplicity.

Remark 3. Figures 4 and 5 (u_5, u_9), represents the singular kink-type traveling wave solution of the Hamiltonian amplitude equation. For convenience the other figures are omitted.

• **Note that**: All the obtained results have been checked with Maple 13 by putting them back into the original equation and found correct.

4 Conclusions

In this article, the new approaches of the improved $\tanh(\Phi(\xi)/2)$-expansion method has successfully been implemented to investigate the nonlinear partial differential equation, namely, the Hamiltonian amplitude equation. The exact particular solutions containing five types hyperbolic function solution, trigonometric function solution, exponential solution, logarithmic solution and rational solution. Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions are attained. It is worth mentioning that some of newly obtained solutions are identical to already published results. It has been shown that the applied methods are effective and more wide-ranging than the generalized and improved (G'/G)-expansion method because it gives many new solutions. Therefore, this method can be applied to study many other nonlinear partial differential equations which frequently arise in engineering, mathematical physics and other scientific real time application fields.

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