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STABILITY OF A NEW CONTINUOUS QUADRATIC RECURRENT NEURAL NETWORK

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Abstract. In this paper we construct a new continuous recurrent neural network from the fixed points given a priori of quadratic polynomials. Our goal is to provide a criterion for the assignment of synaptic weights of the network, which will allow us to demonstrate the stability of the neural network, without using the energy function used in Hopfield networks. In addition, we give an application to recognition of a pattern.

Keywords: Continuous discrete Neural network, recurrent neural network, stability, fixed point, Hopfield network.

AMS Subject Classification: 37D05, 37D25, 37D40, 37D45.

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1 Introduction

Since 1982 when J. Hopfield introduced the first neural network (Hopfield, 1982, 1984), different generalizations of this type of neural network have been developed, due to the multiple applications in signal processing, associative memories, pattern recognition, optimization, etc. This also motivated the analysis of the dynamics of a neural network, which is mainly reflected in the concept of stability of a neural network.

A continuous Hopfield neural network of dimension "N" is a neural network totally connected with N units of continuous value (Shihuan et al., 2004; Hopfield, 1984; Talaván & Yáñez, 2005), whose dynamics is given by:

$$C_i \frac{du_i}{dt} = \sum_{j=1}^{N} T_{ij} v_j - \frac{u_i}{R_i} + I_i, \qquad v_i = g_i(u_i), \qquad \forall i = 1, ..., N,$$

where $C_i > 0$, $R_i > 0$ and $I_i > 0$ are capacity, resistance and bias, respectively; and T_{ij} are the synaptic interconnection weights of the j-th neuron with the i-th neuron.

In this paper, a continuous fully connected recurrent neural network is constructed, using quadratic polynomial functions constructed by (Rubio & Hernández, 2015, 2017, 2017; Rubio et al., 2017, 2019); that have fixed attractor points given a priori. Furthermore, the stability of this continuous neural network is guaranteed, and an application to pattern recognition is given.

2 Quadratic function

In this section are given some concepts about a fixed point application and some conclusions obtained by Rubio & Hernández (2015).

Definition 1. *A fixed point of the application* $F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, *is a point* $a \in U$ *such that* $F(a) = a$, *Lages (2014).*

Now, it is necessary to formulate the theorem of uniqueness and existence of a fixed point Burden (1985).

Theorem 1. *Let* $U = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i, \forall i = \overline{1,n}\}$, *such that* $a_i, b_i \in \mathbb{R}$ $\forall i = \overline{1,n}$, *are constants and* $F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is an application* $C^1(U)$, $F(x) \in U$, $\forall x \in U$, *then* F *have a fixed point in* U, *also assume the existence of a constant* $k < 1$ *with*

$$\left|\frac{\partial F_i(x)}{\partial x_j}\right| \leq \frac{k}{n}, \quad \forall x \in U, \quad \forall i, j = \overline{1,n},$$

then the sequence $\{X^{(t)}\}_{t=0}^{\infty}$ *which is defined by* $X^{(0)} \in U$ *and*

$$X^{(t)} = F(X^{(t-1)}), \quad t \geq 1$$

it converge to a unique fixed point $a \in U$ *and*

$$\left\|X^{(t)} - a\right\|_{\infty} < \frac{k^t}{1-k}\left\|X^{(1)} - X^{(0)}\right\|_{\infty}.$$

According to Feigenbaum (1980), the fixed points, which are the limit of a convergent sequence, these are called attractor fixed points; otherwise they are designate repellent fixed points.

Some results obtained by Rubio & Hernández (2015) are presented, let x_0, $x_1 \in \mathbb{R}$ two points, $x_0 < x_1$ these are fixed points given a priori and the quadratic function is determined by

$$f(x) = Ax^2 + Bx + C \tag{1}$$

where:

$$\begin{cases} A = \dfrac{y_m - x_m}{(x_m - x_1)(x_m - x_0)} \\[2ex] B = \dfrac{y_m(x_0 + x_1) - x_0 x_1 - x_m^2}{(x_1 - x_m)(x_m - x_0)} \\[2ex] C = \dfrac{x_0 x_1 (y_m - x_m)}{(x_m - x_1)(x_m - x_0)} \end{cases} \tag{2}$$

The point (x_m, y_m) is given in such a way that (x_0, y_0), (x_1, y_1) and (x_m, y_m) are non-collinear.

By using theorem (5.1) in (Rubio & Hernández, 2015), with $x_m = x_0 - \epsilon$, $y_m = x_0$, $\epsilon = 0.1$, we have:

(a) x_0 is an attractor fixed point.

(b) x_1 is a repellent fixed point. (3)

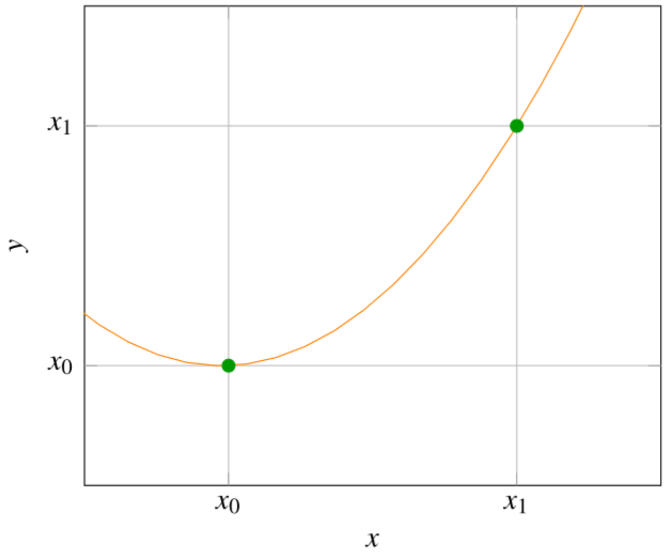

Figure 1: Fixed point x_0, attractor

By using theorem (5.4) in (Rubio & Hernández, 2015), with $x_m = x_1 + \epsilon$, $y_m = x_1$, $\epsilon = 0.1$, we have:

(a) x_0 is a repellent fixed point.

(b) x_1 is an attractor fixed point. (4)

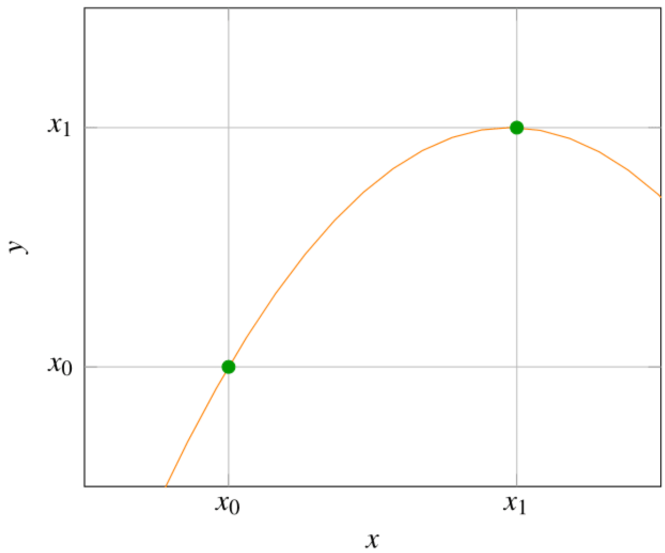

Figure 2: Fixed point x_1, attractor

3 Construction of the Continuous Quadratic Recurrent Neural Network

Using the quadratic functions given in the previous section, a new Continuous Quadratic Recurrent neural network is constructed for N neurons, whose dynamics are given by the System of Ordinary Differential Equations:

$$\frac{dx_i(t)}{dt} = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N, \tag{5}$$

where: $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, f_j are functions given by (1) and (2); w_{ij} are the synaptic weights, which are determined according to (Rubio & Hernández, 2017).

Actually, for our study, we are considering the space of Hamming $\mathbb{H}^n = \{-1, 1\}^n$ and $X_p \in \mathbb{H}^n$. Now, let $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$ such that:

$$f_j(x_p^j) = x_p^j, \quad \forall j = 1, ..., N. \tag{6}$$

the critical points of the system (5) are given by:

$$\frac{dx_i(t)}{dt} = 0, \quad \forall i = 1, ..., N,$$

then

$$0 = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N$$

$$\frac{x_i(t)}{a} = b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N \tag{7}$$

Now consider the curve

$$x_j(t) = x_p^j, \quad \forall j = 1, ..., N$$

then, by (Rubio & Hernández, 2015, 2017, 2017):

$$\sum_{j=1}^{N} w_{ij} f_j(x_p^j) = \sum_{j=1}^{N} w_{ij} x_p^j = x_p^i, \quad \forall i = 1, ..., N$$

Since (7):

$$\frac{x_i(t)}{a} = b x_p^i,$$

Therefore:

$$x_i(t) = ab x_p^i, \quad \forall i = 1, ..., N \tag{8}$$

Theorem 2. *Let $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, where x_p^i are fixed points of f_i given by (1) and (2), $a, b \in \mathbb{R}$, $a > 0$, $b > 0$. If $ab = 1$, then $X_p \in \mathbb{H}^N$ is a critical point of recurrent neural network quadratic continuous:*

$$\frac{dx_i(t)}{dt} = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N.$$

Proof. In particular, if $a = 2$ and $b = \frac{1}{2}$, then $ab = 1$. By (8) we have

$$x_i(t) = x_p^i, \quad \forall i = 1, ..., N.$$

Therefore, X_p is a critical point of (5). □

4 Stability

In this section we study the stability of the continuous quadratic recurrent neural network:

$$\frac{dx_i(t)}{dt} = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N.$$

Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be an application, which is defined by $F(x) = (F_1(x), F_2(x), ..., F_N(x))$, where:

$$F_i(x(t)) = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij} f_j(x_j(t)), \quad \forall i = 1, ..., N \tag{9}$$

and

$$f_j(x_j) = A_j x_j^2 + B_j x_j + C_j, \quad \forall j = 1, ..., N$$

are functions given by (1).

Moreover, note that the application F previously defined is differentiable of class $C^\infty(\mathbb{R}^N)$.

Theorem 3. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ defined by (9). Then, the Jacobian matrix of F is given by:*

$$JF(x) = (\frac{\partial F_i(x)}{\partial x_k})_{NxN}$$

where:

$$\frac{\partial F_i(x)}{\partial x_k} = \begin{cases} -\frac{1}{a} + (2A_i x_i + B_i) w_{ii} b, & i = k \\ (2A_k x_k + B_k) w_{ik} b, & i \neq k \end{cases} \tag{10}$$

Proof. By (9):

$$F_i(x(t)) = -\frac{x_i(t)}{a} + b\sum_{j=1}^{N} w_{ij}[A_j x_j^2 + B_j x_j + C_j], \quad \forall i = 1, ..., N$$

Then:

1. To $i = k$:

$$\frac{\partial F_i(x)}{\partial x_i} = -\frac{1}{a} + b\sum_{j=1}^{N} w_{ij}\frac{\partial}{\partial x_i}[A_j x_j^2 + B_j x_j + C_j]$$

$$\frac{\partial F_i(x)}{\partial x_i} = -\frac{1}{a} + (2A_i x_i + B_i) w_{ii} b$$

2. To $i \neq k$:

$$\frac{\partial F_i(x)}{\partial x_k} = \frac{\partial}{\partial x_k}(-\frac{x_i}{a}) + b\sum_{j=1}^{N} w_{ij}\frac{\partial}{\partial x_i}[A_j x_j^2 + B_j x_j + C_j]$$
$$= b w_{ik}(2A_k x_k + B_k)$$

By (1) and (2):

$$\frac{\partial F_i(x)}{\partial x_k} = \begin{cases} -\frac{1}{a} + (2A_i x_i + B_i) w_{ii} b, & i = k \\ (2A_k x_k + B_k) w_{ik} b, & i \neq k \end{cases}$$

□

Theorem 4. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *defined by (9),* $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, $f_j(x_p^j) = x_p^j$, $\forall j = 1, ..., N$, *attractor fixed points. Then:*

$$JF(X_p) = -\frac{1}{a} I_N + bM \tag{11}$$

where I_N *is the identity matrix of order N, and:*

$$M = \begin{bmatrix} (2A_1 x_p^1 + B_1) w_{11} & \cdots & (2A_N x_p^N + B_N) w_{1N} \\ \vdots & \ddots & \vdots \\ (2A_1 x_p^1 + B_1) w_{N1} & \cdots & (2A_N x_p^N + B_N) w_{NN} \end{bmatrix} \tag{12}$$

Proof. By (10):

$$\frac{\partial F_i(x)}{\partial x_k} = \begin{cases} -\frac{1}{a} + (2A_i x_i + B_i) w_{ii} b, & i = k \\ \\ (2A_k x_k + B_k) w_{ik} b, & i \neq k \end{cases}$$

Then:

$$JF(X_p) = \begin{bmatrix} -\frac{1}{a} + (2A_1 x_p^1 + B_1) w_{11} b & \cdots & (2A_N x_p^N + B_N) w_{1N} b \\ \vdots & \ddots & \vdots \\ (2A_1 x_p^1 + B_1) w_{N1} b & \cdots & -\frac{1}{a} + (2A_N x_p^N + B_N) w_{NN} b \end{bmatrix}$$

$$= -\frac{1}{a} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + b \begin{bmatrix} (2A_1 x_p^1 + B_1) w_{11} & \cdots & (2A_N x_p^N + B_N) w_{1N} \\ \vdots & \ddots & \vdots \\ (2A_1 x_p^1 + B_1) w_{N1} & \cdots & (2A_N x_p^N + B_N) w_{NN} \end{bmatrix}$$

Therefore:

$$JF(X_p) = -\frac{1}{a} I_N + bM$$

□

In this paper will be use the matricial norm

$$\|A\|_\infty = \max_{1 \le j \le n} \sum_{k=1}^{n} |A_{jk}|, \quad \textit{with} \quad A = (A_{ij})_{n \times n}$$

Theorem 5. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *defined by (9),* $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, $f_j(x_p^j) = x_p^j$, $\forall j = 1, ..., N$, *attractor fixed points. Then:*

$$||M||_\infty < ||W||_\infty. \tag{13}$$

Proof. By (12):

$$\sum_{j=1}^{N} |(2A_j x_p^j + B_j) w_{ij}| = \sum_{j=1}^{N} |2A_j x_p^j + B_j| |w_{ij}|$$

$$< \sum_{j=1}^{N} |w_{ij}|, \quad \forall i = 1, ..., N$$

because x_p^j is an attractor point of f_j, $\forall j = 1, ..., N$.

Therefore:

$$||M||_\infty < ||W||_\infty.$$

□

Corollary 1. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *defined by (9);* $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, $f_j(x_p^j) = x_p^j$, $\forall j = 1, ..., N$, *attractor fixed points. Then:*

$$||M||_\infty < 1. \tag{14}$$

Proof. By (Rubio & Hernández (2017)), the matrix of synaptic weights W fulfills: $||M||_\infty = 1$. And by theorem (5) we have:

$$||M||_\infty < ||W||_\infty.$$

Therefore:

$$||M||_\infty < 1.$$

□

Theorem 6. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *defined by (8);* $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, $f_j(x_p^j) = x_p^j$, $\forall j = 1, ..., N$, *attractor fixed points. Then:*

$$||JF(X_p)||_\infty < \frac{1}{a} + b \tag{15}$$

Proof. By (11):

$$JF(X_p) = -\frac{1}{a} I_N + bM$$

Then:

$$\begin{aligned} ||JF(X_p)||_\infty &= || - \frac{1}{a} I_N + bM||_\infty \\ &\leq | - \frac{1}{a}|||I_N||_\infty + |b|||M||_\infty \\ &= \frac{1}{a} + b||M||_\infty \\ &< \frac{1}{a} + b, \quad \text{por (14)} \end{aligned}$$

Therefore:

$$||JF(X_p)||_\infty < \frac{1}{a} + b.$$

□

Next we show a result that ensures that X_p is an attractor fixed point of the continuous quadratic recurrent neural network.

Theorem 7. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *defined by (9),* $X_p = (x_p^1, x_p^2, ..., x_p^N) \in \mathbb{H}^N$, $f_j(x_p^j) = x_p^j$, $\forall j = 1, ..., N$, *attractor fixed points,* $a, b \in \mathbb{R}$, *con* $a \geq 2, ab = 1$. *Then:*

$$||JF(X_p)||_\infty < 1 \tag{16}$$

Proof. From theorem (6) we have:

$$||JF(X_p)||_\infty < \frac{1}{a} + b.$$

without loss of generality, to values $a = 2$ and $b = \frac{1}{a}$, we have:

$$||JF(X_p)||_\infty < \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore:

$$||JF(X_p)||_\infty < 1.$$

□

5 Examples

This section we gives some examples of our continuous quadratic recurrent neural network. Using (1), (2), (3) and (4), with $x_0 = -1$, $x_1 = 1$, we construct the quadratic functions:

(a) $f_+(x) = -0.4762x^2 + x + 0.4762$, with $x_1 = 1$, attractor fixed point.

(b) $f_-(x) = 0.4762x^2 + x - 0.4762$, with $x_0 = -1$, attractor fixed point.

Example 1. *Let $X_p = (-1, 1)$ be the fixed point given previously. Then the synaptic weight matrix is:*

$$W = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and the system of ordinary differential equations (5) that determines the dynamics of the neural network is:

$$\begin{cases} \frac{dx_1}{dt} = -\frac{x_1}{2} + \frac{1}{4}f_1(x_1) - \frac{1}{4}f_2(x_2) \\ \frac{dx_2}{dt} = -\frac{x_2}{2} - \frac{1}{4}f_1(x_1) + \frac{1}{4}f_2(x_2) \end{cases} \tag{17}$$

where:

$$f_1(x) = f_-(x), \quad f_2(x) = f_+(x)$$

Now, let's consider the initial value problem:

$$\begin{cases} \frac{dx_1}{dt} = -\frac{x_1}{2} + \frac{1}{4}f_1(x_1) - \frac{1}{4}f_2(x_2) \\ \frac{dx_2}{dt} = -\frac{x_2}{2} - \frac{1}{4}f_1(x_1) + \frac{1}{4}f_2(x_2) \\ Initial\ Condition: \\ x_1(0) = -0.75, \quad x_2(0) = -2 \end{cases} \tag{18}$$

and using the Runge - Kutta scheme of 4th order, the numerical solution to problem (18) is obtained.

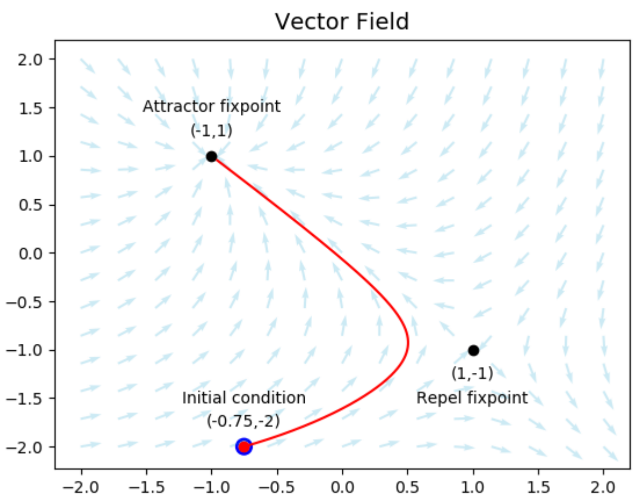

Figure 3: Numerical solution converges to the attractor fixed point X_p.

Example 2. *Let $X_p = (1, -1, 1)$ be the fixed point given previously. Then the synaptic weight matrix is:*

$$W = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and the system of ordinary differential equations (5) that determines the dynamics of the neural network is:

$$\begin{cases} \frac{dx_1}{dt} = -\frac{x_1}{2} + \frac{1}{6}f_1(x_1) - \frac{1}{6}f_2(x_2) + \frac{1}{6}f_3(x_3) \\ \frac{dx_2}{dt} = -\frac{x_2}{2} - \frac{1}{6}f_1(x_1) + \frac{1}{6}f_2(x_2) - \frac{1}{6}f_3(x_3) \\ \frac{dx_3}{dt} = -\frac{x_3}{2} + \frac{1}{6}f_1(x_1) - \frac{1}{6}f_2(x_2) + \frac{1}{6}f_3(x_3) \end{cases} \tag{19}$$

where:

$$f_1(x) = f_+(x), \quad f_2(x) = f_-(x), \quad f_3(x) = f_+(x)$$

Now, let's consider the initial value problem:

$$\begin{cases} \frac{dx_1}{dt} = -\frac{x_1}{2} + \frac{1}{6}f_1(x_1) - \frac{1}{6}f_2(x_2) + \frac{1}{6}f_3(x_3) \\ \frac{dx_2}{dt} = -\frac{x_2}{2} - \frac{1}{6}f_1(x_1) + \frac{1}{6}f_2(x_2) - \frac{1}{6}f_3(x_3) \\ \frac{dx_3}{dt} = -\frac{x_3}{2} + \frac{1}{6}f_1(x_1) - \frac{1}{6}f_2(x_2) + \frac{1}{6}f_3(x_3) \\ \textit{Initial Condition} \\ \quad x_1(0) = -0.75\,, \quad x_2(0) = -2\,, \quad x_3(0) = -1 \end{cases} \tag{20}$$

and using the Runge - Kutta scheme of 4th order, the numerical solution to problem (20) is obtained.

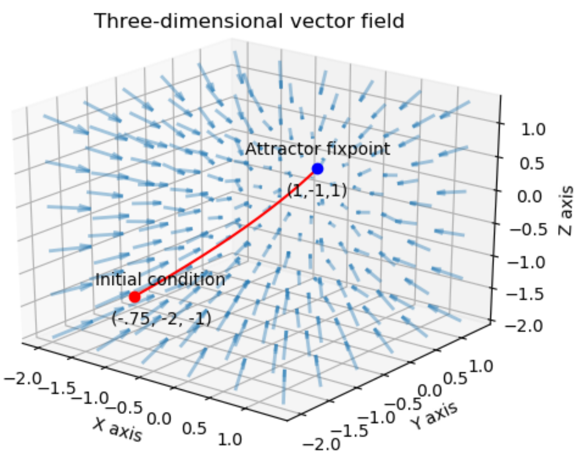

Figure 4: Numerical solution converges to the attractor fixed point X_p.

Example 3. *In this example we give an application of our continuous quadratic recurrent neural network to the recognition of a pattern.*

Consider the pattern given by the figure

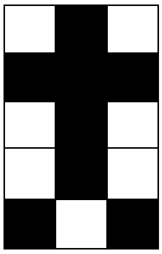

Figure 5: Pattern

which is represented using 15 neurons:

$$X_p = [1, -1, 1, -1, -1, -1, 1, -1, 1, 1, -1, 1, -1, 1, -1]$$

Synaptic weights are given by:

1. $w_{ii} = \dfrac{1}{15}$, $\forall i = 1, 2, \dots, 15$.
2. If $\dfrac{x_p^j}{x_p^i} > 0$, then $w_{ij} = \dfrac{1}{15}$.
3. If $\dfrac{x_p^j}{x_p^i} < 0$, then $w_{ij} = -\dfrac{1}{15}$.

Now, let's consider a disturbance of the pattern, given by the figure:

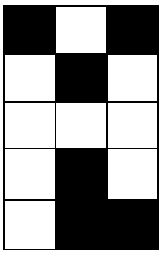

Figure 6: Disturbed Pattern

which is represented using 15 neurons:

$$X_0 = [-1, 1, -1, 1, -1, 1, 1, 1, 1, 1, -1, 1, 1, -1, -1]$$

Initial value problem is given by:

$$\begin{cases} \dfrac{dx_i}{dt} = -\dfrac{x_i}{2} + \dfrac{1}{2}\displaystyle\sum_{j=1}^{15} w_{ij} f_j(x_j), \quad \forall i = 1, \dots, 15 \\ \textit{Initial Conditions}: \\ \qquad x(0) = X_0 \end{cases} \tag{21}$$

where:

$$x(t) = (x_1(t), \ldots, x_{15}(t))$$

$$f_1(x) = f_3(x) = f_7(x) = f_9(x) = f_{10}(x) = f_{12}(x) = f_{14}(x) = f_+(x)$$

$$f_2(x) = f_4(x) = f_5(x) = f_6(x) = f_8(x) = f_{11}(x) = f_{13}(x) = f_{15}(x) = f_-(x)$$

Using the Runge - Kutta scheme of 4th order, we obtain the numerical solution to the problem (21); that allows to restore the initial pattern, see figure:

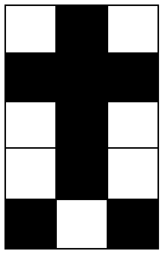

Figure 7: Result of the continuous recurrent quadratic neural network.

6 Conclusion

In this paper we construct a new continuous quadratic recurrent neural network with a fixed point attractor given previously, using the fixed points of quadratic functions given by (1) – (4). Using (8), values are assigned to the elements of the matrix W that guarantee that $\|W\|_\infty = 1$.

In theorem 5 it is proved that the norm of the Jacobian matrix associated with the neural network at fixed point X_p, $\|JF(X_p)\|_\infty < 1$, which guarantees the stability of the fixed point; methodology different from that used by Hopfield (1984), which makes use of the energy function associated to the system.

This new continuous quadratic recurrent neural network behaves as auto-associative memory; allowing to restoration objects from certain information; as in the recognition of images, sounds; as in the application example to the recognition of a pattern.

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