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CONVEX FUNCTIONS: ARIADNE'S THREAD OR CHARLOTTE'S SPIDERWEB?

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Abstract. In this work we present different notions of convexity used in different works of recent times, both theoretical and applied, and based on them, we establish various relationships between them and make some methodological recommendations on the study of integral inequalities of the Hermite-Hadamard-Fejer type.

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1 Introduction

Throughout the History of Mathematics, we find concepts that are the center of various theories and theoretical developments, it may even happen that this initial concept has received innumerable extensions and generalizations that make a researcher who begins in a certain area that it involves, you may be overwhelmed by such theoretical sublimation.

We must add that sometimes in the face of such a multiplicity of results, these same concepts may be the solution to obtain new results in this direction.

One of these concepts is that of convex function, present today in multiple mathematical disciplines ranging from Optimization to Function Theory and center of, possibly, the most fruitful nucleus in the study of integral inequalities, mainly linked to the estimation of the integral mean value of a certain function over a given interval.

Hence the title of this work.

Therefore, in our work, we make a brief historical overview of the concept of convex function, the development of the now-known Hermite-Hadamard-Fejer type Inequality and present, on this basis, new results involving these areas and generalized integral operators of recent date.

2 A little bit of history

The significant role of inequalities in the development and evolution of Mathematics is well known. Some basic notions related to them were already in use by the ancient Greeks, such as triangular inequality and isoperimetric inequalities. However, the inequalities were not used in arithmetic or in any other type of number manipulation. The formalization of the mathematical theory of inequalities essentially begins in the XVIII century with studies by Gauss. It was continued by Cauchy and Chebishev, who came up with the idea of appying some inequalities to mathematical analysis. Later, the Russian mathematician Bunyakovsky, proved in 1859 the well-known Cauchy-Schwarz inequality, for the case of infinite dimensions. We must point out that Hardy's research on this topic must be recognized as particularly significant, as it went beyond particular inequalities. Hardy managed to gather the best mathematicians of the moment to solve problems related to inequalities. In addition, he founded the Journal of the London Mathematical Society, a magazine especially suited to publishing articles on inequalities. Along with renowned mathematicians such as Littlewood and Polya, he developed the famous volume entitled "Inequalities" (Hardy et al., 1934), which was the first monograph on this topic. The book became a landmark in the field of inequalities, achieving the goal of giving structure, systematization, and formalization to an apparently isolated set of results, and in doing so made them a theory. Currently, inequalities have reached an outstanding theory and applied development and are the methodological basis of processes of approximation, estimation, dimensioning, interpolation, etc. In general, they are central to every modeling problem.

To help the problem statement, we must remember some basic definitions.

Definition 1. A function $f : I \to \mathbb{R}$ is said to be **convex** on interval $I \subset \mathbb{R}$, if the inequality $tf(x) + (1-t)f(y) \leq t(x) + (1-t)f(y)$, for $x, y \in I$ is fulfilled.

We say that f is concave if $-f$ is convex. On the other hand, the average value of an integrable function over a compact interval $[a, b]$ is known to all, which is given by the value $V_m(f) = \frac{1}{b-a}\int_a^b f(x)dx$, since it turns out that between of the many important inequalities that involve convex functions, there is one in particular that allows us to limit this mean value $f\left(\frac{a+b}{2}\right) \leq V_m(f) \leq \frac{f(a)+f(b)}{2}$, with $a, b \in I$, the inverse inequalities are maintained if the function f is concave in said interval. This seminal result was demonstrated at Hadamard (1893) and is known as the Hermite-Hadamard inequality (see Dragomir et al. (2000) and Khan et al. (2018) for details). Since its discovery, this inequality has received considerable attention. In recent years, this inequality has been generalized to different fractional integral operators (Huang et al. (2019); Khan et al. (2018); Li et al. (2017); Kórus (2019); Mohammed & Hamasalh (2019); Portilla & Tourís (2009); Qi et al. (2019) in the conformable case; Nápoles Valdés et al. (2019) in the non-conformable case and Basci & Baleanu (2018a,b); Sarikaya & Yildirim (2017); Wang et al. (2013); Zhu et al. (2012) for the global case). Convexity is a basic notion in geometry but also is widely used in other areas of mathematics.

We will begin this work by giving a short description of the history of the concept of convexity according to Di Giorgi (2014) and Dwilewicz (2009).

- **Five Platonic Solids**. It was known to the ancient Greeks that there are only five regular convex polyhedra. Each regular polyhedron is made of congruent regular polygons. These five regular convex polyhedra (Tetrahedron, Octahedron, Icosahedron, Cube, Dodecahedron) are called The *Five Platonic solids* (Plato, 427 - 347 B.C.) because Plato mentioned them in the Timaeos, but they were already known before, even in prehistoric times .

- **Archimedes**. With him the notion of convexity appears because he notice that the perimeter of a convex figure it is smaller than the perimeter of any other convex figure that surrounds it.

- **Otto Ludwig Hölder**. In his work entitled *Über einen Miitelwertsatz* in 1889 proved the way moderated from the now known Jensen inequality, under the hypothesis that the second derivative is not negative $f''(x) > 0$, in your domain.

- **Otto Stolz**. In his work entitled *Grundzüje der Differential und Integrabrechnug* in 1893 that if $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and satisfies the inequality:

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)), x, y \in \mathbb{R} \tag{1}$$

then f has a left and right derivative at each point of $[a, b]$.

- **Jaques Hadamard and Charles Hermite**. As we noted, in his 1893 work (see Hadamard (1893)), obtained an inequality for integrals of functions that has a growing derivative in $[a, b]$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(s)ds \leq \frac{f(a)+f(b)}{2} \tag{2}$$

for all $a, b \in I$, $a < b$. The left side of the inequality (2) was proved by Hadamard, for the case in which f functions with increasing distance in a closed interval of the real line. At that time the notion of convex functions was in the process of construction. Already today this inequality is known as the inequality of Hermite-Hadamard, because the right side of inequality (2) is attributed to Charles Hermite in 1883 (see Hermite (1883)).

- **Johan Ludwig William Valdemar Jensen**. In Jensen (1905, 1906) values the importance of this notion and consider equation (1) to define convex functions and say the first of a long series of important results which (1) together with inequality implies the continuity of f.

- **Lipót Fejér**. In Fejér (1906) established the following inequality which is the weighted generalization of Hermite-Hadamard inequality (2). If if $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(s)ds \leq \frac{1}{b-a}\int_a^b f(s)g(s)ds \leq \frac{f(a)+f(b)}{2}\int_a^b f(s)ds \tag{3}$$

holds, where $g : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $x = \frac{a+b}{2}$.

- **Stanislaw M. Ulam and Donald H. Hyers**. In 1952 (see Hyers & Ulam (1945)) they proved that if a given function f is $\epsilon - convex$ defined in a convex open subset then this can be approximated by a convex $\phi(x)$ function.

One of the problems of interest is to determine conditions necessary and sufficient about f and g for there to be a function h that separates f and g ($f \leq h \leq g$) and that hold a certain condition, for example: continuity, convexity, quasi-convexity, quasi-convexity, quasi-concavity, monotony, linearity, etc.

- **Karol Baron, Janusz Matkowski and Kazimierz Nikodem**. In Baron et al. (1994) they show that two real functions f and g defined on a interval $I \subseteq \mathbb{R}$ and that satisfy the inequality

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y); x, y \in I, t \in [0; 1]; \tag{4}$$

they can be separated by a convex function.

3 Basic definitions of convex functions

3.1 Convex Set

The most common mathematical definition of a convex set (for simplicity restrict ourselves to the Euclidean space $\mathbb{R}^n$) is

Definition 2. (Convex sets). A set S in $\mathbb{R}^n$ is convex if with any two points p and q belonging to S the entire segment joining p and q lies in S.

We know that the points in the segment are of the form $tp + (1-t)q$, where $0 \leq t \leq 1$. The above definition geometrically is very clear, but in analytical applications not very useful. Later on we give other definitions, more convenient in analysis.

Intimately related to convexity of sets is convexity of functions, however this notion appeared much later than the first one.

Definition 3. (Convex set) Let V be a vector space over $\mathbb{R}$. A subset S of V is called convex if every line intersects S in an interval.

3.2 Convex Function

In this section we present different variables of the convex function taking different authors and then analysing the relationships between them.

In Di Giorgi (2014) we have the following definition:

Definition 4. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is **convex** if satisfies:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{5}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality is in the opposite direction it is said that the function f is concave.

The author considers a subset of $\mathbb{R}$ for the domain of the function, that is, it is a function defined for one variable. This subset cannot be any subset, but must necessarily be an interval. That is why this definition has as hypothesis that $I \subset \mathbb{R}$ is an interval, since in this way this subset is a convex set.

Particulary, if $t = 1/2$ we have the well known **midpoint convex function** or Jensen's inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{6}$$

for all $x, y \in I$.

Now we will see the definition of convex function found in Chemali et al. (2013):

Definition 5. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if dom f is a **convex** set and if for all $x, y \in$ dom f, and t with $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{7}$$

In this definition we observe that the first set of the function is $\mathbb{R}^n$, that is, this function is defined for n variables.

If $n = 1$, then we could say that the difference between Definition (4) and Definition (5) is that the second one specifies that the domain must be a convex set, while the first one considers a real interval as a function domain, which we know is a convex set.

Proposition 1. A function is convex if and only if when restricted to any line that intersects its domain is convex. That is $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff $g(t) = f(tx + tv)$ is convex, $Domg = t|x + tv \in Domf$ for all $x \in domf$, for all $v \in \mathbb{R}^n$.

Proof. $\Rightarrow$) If $f(x)$ is convex, then $f(y)$ is also convex for any $y = x + tv$ that belongs to the same domain as x. $\Leftarrow$) Let us take $x_1, x_2 \in dom\ f$. We need to show that for every $\theta \in [0,1]$

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2).$$

Now, since $g(t) = f(x + vt)$ is convex for all $x \in domf$ and for all v, for every $\theta \in [0,1]$:

$$\begin{aligned} &\theta g(t_1) + (1-\theta)g(t_2)\theta f(x + vt_1) + (1-\theta)f(x + vt_2) \geq g(\theta t_1 + (1-\theta)t_2) \\ &\geq f(x + v(\theta t_1 + (1-\theta)t_2)) \end{aligned}$$

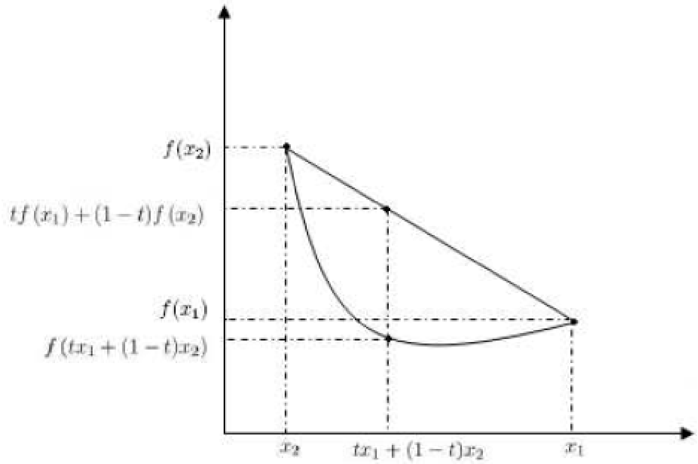

Figure 1: Convex Function, $I = [x_2; x_1]$

let us take $x = x_1$, $v = x_2 - x_1$, $t_1 = 0$ and $t_2 = 1$, and assign them to the last inequality:

$$\theta f(x_1) + (1-\theta)f(x_2) \geq f(\theta x_1 + (1-\theta)x_2)$$

and therefore f is convex. □

This property is useful because it allows us to reduce the problem of checking the convexity of a multivariate function to checking the convexity of a uni-variate function, for which we can use much simpler criteria.

Let's see at the definition of convex function that we have in Dupraz (2014):

Definition 6. We consider a function $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, then f is **convex** iff S is a convex subset of $\mathbb{R}^n$ and:

$$\forall x, y \in S, \forall t \in [0,1], f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{8}$$

This definition, like the Definition (5) is defined for functions of several real variables, since the elements must belong to a subset of $\mathbb{R}^n$. This subset is precisely the domain of the function f, that is, if we call S, the domain of f these two definitions are the same.

We can also find that works Chemali et al. (2013); Di Giorgi (2014); Dupraz (2014) define a strictly convex functions by requiring in addition that the inequalities in the Definition (4), Definition (5) and Definition (6) be strict whenever $x \neq y$ and $t \in (0,1)$. If $x = y$ or $t = 0$ or $t = 1$, then the inequality is $f(x) \leq f(x)$, which is always true but never strict. A strictly convex function is obviously convex .

Further, Di Giorgi (2014) and Dupraz (2014) talk about the geometric interpretation of a convex function, where the first one says that if $f : I \to \mathbb{R}$ is a convex function then the segment that joins the points $(x_1, f(x_1)), (x_2, f(x_2)) \in Graf(f)$ is never below $Graf(f)$ (see Figure 1).

While the second says that a convex function is defined as a function whose area above the curve $\{(x,y)/y \geq f(x)\}$ (called the epigraph of f) is a convex set.

Finally, in Dwilewicz (2009) we find the following definitions:

Definition 7. (Convex functions) Let V be a vector space and $S \subset V$ be a convex set. A function $f : S \to \mathbb{R}$ is called **convex** if

$$f(\lambda p + \mu q) \leq \lambda f(p) + \mu f(q), \tag{9}$$

for $\lambda, \mu \geq 0$, $\lambda + \mu = 1$, $p, q \in S$.

Until now we have seen that the notion of convex function is defined for real values, that is, they are functions whose domain belongs to $\mathbb{R}^n$. But this definition also works by taking any vector space instead of $\mathbb{R}^n$ (but most results and applications concern functions with arguments in $\mathbb{R}^n$). That is why the Definition (7) considers the vector space V in its hypothesis. In the same way, the notion of convexity for functions defined in a vector space remains similar, since it considers a subset of V which should be convex as a domain of the function f. We see immediately, that if $V = \mathbb{R}$ and S is any interval in $\mathbb{R}$, then the Definition (7) is equivalent to the Definition (4) in Basci & Baleanu, (2018) and if $V = \mathbb{R}^n$, as S is already a convex subset, we have that is equivalent to the Definition (6) in Dupraz (2014).

Also note that in the Definition (7) there are two parameters, λ and μ. But taking $\mu = 1 - \lambda$, this definition becomes dependent only on λ as in the previous definitions where a single parameter appears.

Now we can see other definitions which ones are equivalent definitions of convex functions in Dwilewicz (2009).

Definition 8. (Convex functions of one variable) A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is convex if for every compact interval $J \subset I$ with boundary ∂J, and every linear function $L = L(x) = ax$ we have

$$\sup_J (f - L) = \sup_{\partial J} (f - L)$$

Definition 9. (Convex functions of several variables) A function f defined on a convex set $S \subset V$ is convex if for any line ℓ the function f restricted to $\ell \cap S$ is convex.

In all these definitions, to define convexity they used linear functions. The linear functions are the simplest non-trivial functions.

Note that the Definition (9) says the same as the Property of Proposition (1) in Chemali et al. (2013).

The definition of a convex function has its origins in Jensen's results and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

The definition of a convex function relies on the value of f at convex combinations of two points x and y. What can we say about the value of f convex at convex combinations of more than two points? The Jensen inequality offers an answer. Now we extended it to convex combinations of more than two points.

Proposition 2. Jensen's Inequality Let $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ be a convex function. For any integer n, for any $x_1, ..., x_n \in S$, for any positive reals $\lambda_1, ..., \lambda_n$ such that $\sum \lambda_i = 1$,

$$f(\lambda_1 x_1 + ... \lambda_n x_n) \leq \lambda_1 f(x_1) + ... + \lambda_n f(x_n). \tag{10}$$

4 Generalized Convexity

In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions to many areas of Mathematics. The concept of convexity has been generalized depending on the problem and applications studied. In this section we introduce some of these generalizations.

4.1 Convexity with respect to another function

Let I $\subset \mathbb{R}$ is an interval and

$$G = \{g : \mathbb{R} \longrightarrow \mathbb{R}; t \leq g(t), \quad for\ all\ t \in \mathbb{R}\} \tag{11}$$

Definition 10. Let $g \in G$. A function $f : I \longrightarrow \mathbb{R}$ is **convex with respect to g**, if

$$f(tx + (1-t)y) < tg(f(x)) + (1-t)g(f(y)), \tag{12}$$

for all $t \in [0, 1]$ and $x, y \in I$.

If g is the identity function, $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(z) = z$ for all $z \in \mathbb{R}$, then g is a the classical convex function.

4.2 Strongly Convex Function with modulus c

Definition 11. Let D be a convex subset of $\mathbb{R}$ and $c > 0$. A function $f : D \longrightarrow \mathbb{R}$ is called **strongly convex with modulus c** if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \tag{13}$$

for all $x, y \in D$ and $t \in [0, 1]$.

The usual notion of convex function correspond to the case $c = 0$.

4.3 Harmonically Convex Function

Imdat Iscan gave the definition of harmonically convex functions:

Definition 12. Let I be an interval in $\mathbb{R} \setminus 0$. A function $f : I \longrightarrow \mathbb{R}$ is said to be **harmonically convex** on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{14}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 13. Let I be an interval in $\mathbb{R}$ 0 and let $c \in \mathbb{R}_+$. A function $f : I \longrightarrow \mathbb{R}$ is said to be **harmonically strongly convex with modulus c** on I, if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) - xt(1-t)(x-y)^2 \tag{15}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

The symbol SHC will denote the class of functions that satisfy the inequality (15).

If $I \subseteq (0, +\infty)$ and $f \in SHC$ then f is harmonically convex. If $I \subseteq (0, +\infty)$ and $f \in SHC$ and nonincreasing, then f is a strongly convex function with modulus c. If $I \subseteq (0, +\infty)$ and f is strongly convex with modulus c and noncreasing, the $f \in SHC$.

4.4 P- Convex Functions

Definition 14. We say that $f : I \to \mathbb{R}$ is a $P - function$, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I, \alpha \in [0, 1]$, we have

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y). \tag{16}$$

4.5 Godunova-Levin Function

Definition 15. Le I be an interval in $\mathbb{R}$. Les us recall definitions of some special classes of functions (see Varosanec (2007)).

We say that $f : I \to \mathbb{R}$ is a **Godunova-Levin** function or that f belongs to the class $Q(I)$ if f is non-negative and for all x,$y \in I$ and $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}. \tag{17}$$

The class $Q(I)$ was firstly described by Godunova and Levin.

4.6 s- Convex Functions

An $s-convex$ function was introduced in Breckner's paper and a number of properties and connections with $s-convexity$.

Definition 16. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be **s-convex in the first sense** if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \tag{18}$$

for $x, y, \alpha, \beta \in (0,\infty)$ and $\alpha^s + \beta^s = 1$

Definition 17. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be **s-convex in the second sense** if

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y). \tag{19}$$

for all $x, y \in (0,\infty)$ and $\alpha \in [0,1]$.

Remark 1. If f is s-convex in the second sense and $f(0) = 0$ then f is s-convex in the first sense.

Of course, $s-convexity$ means just convexity when $s = 1$.

As we can see, definitions of convex, s-convex, P-functions and Godunova-Levin functions have a similar form: the term on the left-hand side of the inequality is the same in all definitions while the right-hand side of all inequalities has a form $h(\alpha)f(x)+h(1-\alpha)f(y)$. This observation leads us to the unified treatment of these several varieties of convexity.

4.7 h- Convex Functions

Definition 18. Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an **h-convex** function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \tag{20}$$

If inequality (41) is reversed, then f is said to be $h-concave$, i.e. $f \in SV(h, I)$. Obviously, if $h(\alpha) = \alpha$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(\alpha) = 1$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Remark 2. Let h be a non-negative function such that $h(\alpha) \geq \alpha$ for all $\alpha \in (0,1)$. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

So, $f \in SX(h, I)$.

We can see, from this definition, that this class of functions contains the class of Godunova-Levin functions. If $h(\alpha) = 1$ then an h-convex function f is a P-function. If $h(\alpha) = \alpha^s, s \in (0,1]$ then an h-convex function f is an s-function. If $h(\alpha) = \alpha^s$ with $s = -1$ then an h-convex function f is a Godunova-Levin function.

4.8 η - Convex Function

Definition 19. Let I be an interval in real line $\mathbb{R}$. A function $f : I = [x, y] \to \mathbb{R}$ is said to be generalized convex with respect to an arbitrary bi-function $\eta(.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, or **$\eta$-convex** function, if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)), \tag{21}$$

$\forall x, y \in I$, $t \in [0, 1]$.

If $\eta(x, y) = x - y$, then the generalized convex function reduces to a convex function. Every convex function is a generalized convex function, but the converse is not true.

4.9 Geometrically Convex Function

The concept of *geometrically convex* functions was introduced as follows (see Xi et al. (2012) and Zhang et al. (2012)).

Definition 20. A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a **geometrically convex** if

$$f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t} \tag{22}$$

for all $x, y \in I$and $t \in [0, 1]$.

Definition 21. A function $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be **geometric arithmetically convex** on I, if

$$f(y^{1-t} x^t) \leq (1-t)f(y) + tf(x), \tag{23}$$

$\forall x, y \in I$, $t \in [0, 1]$, where $(y^{1-t}x^t)$ and $(1-t)f(y) + tf(x)$ are the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$, respectively.

We now introduce a new class of generalized convex functions on the geometrically convex set with respect to an arbitrary bi-function $\eta(.,.)$, which is called the generalized geometrically convex function.

Definition 22. A function $f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be **generalized geometric arithmetically with respect to a bi-function** $\eta(.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$f(y^{1-t} x^t) \leq (1-t)f(y) + tf(x) + \eta(f(x), f(y)), \tag{24}$$

$\forall x, y \in I$, $t \in [0, 1]$.

If $\eta(x, y) = x - y$, then the generalized geometrically convex functions reduce to geometrically convex functions given in (21).

If $t = \frac{1}{2}$ in (24), then

$$f(\sqrt{xy}) \leq f(y) + \frac{1}{2}\eta(f(x), f(y)), \tag{25}$$

$\forall x, y \in I$, $t \in [0, 1]$, which is called a generalized Jensen geometrically convex function.

4.10 m-Convex Function

In 1984, G. Toader defines the class of m-convex functions as follows:

Definition 23. Let f be a real valued function on $[0, b]$. We will say that it is **m-convex**, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{26}$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$. Also, f is m-concave if $-f$ is m-convex. With $K_m(b)$ will denote the class of all m-convex functions over $[0, b]$ wich $f(0) \leq 0$.

Remark 3. Crearly, 1-convex functions are the classical convex functions, and 0-convex functions are the "starshaped" functions, that is, those functions f that satisfies the inequality $f(tx) \leq tf(x)$, with $t \in [0,1]$.

Geometrically a function $f : [0,b] \longrightarrow \mathbb{R}$ is m-convex if for any $x, y \in [0,b]$, say $x \leq y$, the segment between the points $(x, f(x))$ and $(my, mf(y))$ is above the graph of f in $[x, my]$.

Definition 24. Let $f(x)$ be a positive function on $[0,b]$ and $m \in (0,1]$. If

$$f(x^t y^{m(1-t)}) \leq [f(x)]^t [f(y)]^{m(1-t)} \tag{27}$$

holds for all $x, y \in [0,b]$and $t \in [0,1]$ then we say that the function $f(x)$ is m-geometrically convex on $[0,b]$. It is clear that when $m = 1$, **m-geometrically convex** functions become geometrically convex functions.

4.11 (α, m) - Convex Function

In Mihesana (1993) the author introduced the class of (α, m)-convex functions in the following way:

Definition 25. The function $f : [0,b] \to \mathbb{R}$ is said to be (α, m)**-convex**, where $(\alpha, m) \in [0,1]^2$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y). \tag{28}$$

This class is usually denoted by$K_m^A(b)$. This concept was generalized with the notion of generalized (α,m)-convex in Noor et al. (2017b) and Sun & Liu (2017).

Definition 26. Let $f(x)$ be a positive function on $[0,b]$ and $(\alpha, m) \in (0,1] \times (0,1]$. If

$$f(x^t y^{m(1-t)}) \leq [f(x)]^{t\alpha} [f(y)]^{m(1-t^\alpha)} \tag{29}$$

holds for all $x, y \in [0,b]$and $t \in [0,1]$ then we say that the function $f(x)$ is (α, m)**-geometrically convex** on $[0,b]$.

If $\alpha = m = 1$, (α, m)-geometrically convex functions becomes a geometrically convex function on $[0,b]$.

4.12 (m, h_1, h_2)-Convex Function

Definition 27. Let $h_1, h_2 : [0,1] \longrightarrow \mathbb{R}_+$ and $m \in [0,1]$. A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is said to be (m, h_1, h_2)**-convex** function if

$$f(tx + (1-t)y) \leq h_1(t) f(x) + m h_2(t) f(y) \tag{30}$$

holds for all $x.y \in I$ and $t \in [0,1]$. If the inequality is reversed is said to be (m, h_1, h_2)-concave function.

Definition 28. Sea $h_i : [0,1] \longrightarrow \mathbb{R}_0, m : [0,1] \longrightarrow (0,1]$ such that $h_i = 0$ for $i_1, 2$, and $f : (0,b] \longrightarrow \mathbb{R}_0$. If

$$f(x^t y^{(1-t)m(t)}) \leq h_1(t) f(x) + m(t) h_2(1-t) f(y) \tag{31}$$

for $x, y \in [0,b)$ and $t \in [0,1]$, then f is said to be an (m, h_1, h_2)**- geometric arithmetically convex** function or, simply speaking an (m, h_1, h_2)-GA-convex function.

4.13 Quasi-Convex Function

Now we consider that X will be a non-empty convex subset of a real vector space and Y will be a real vector space (see Seto et al. (2018)).

Definition 29. A function $f : X \to \mathbb{R}$ is said to be **quasi-convex** if

$$f((1-t)x + ty) \leq \max(f(x), f(y)), \tag{32}$$

for any $x, y \in X$ and $t \in (0, 1)$.

An interesting characterization of convexity in terms of quasi-convexity has been obtained by Crouzeix (see Crouzeix (1977) and Crouzeix (1980)). It states that a real-valued function $f : X \to \mathbb{R}$ is convex if and only if $f + g$ is quasi-convex for any linear/affine function $g : X \to \mathbb{R}$.

Consider a convex cone $C \subset Y$, i.e. $0 \in C = tC = C + C$ for all $t \in [0, \infty)$. Then C induces on Y a partial ordering (i.e. a reflexive and transitive binary relation, which is compatible with the linear structure of Y, cf. Jahn (2011)), defined for any $y_0, y_1 \in Y$ by

$$y_0 \leq_C y_1 \Leftrightarrow y_1 \in Y_0 + C.$$

(32) can be extended for vector-valued functions, by means of the partial ordering $\leq_C$.

Definition 30. A vector-valued function $f : X \to Y$ is said to be:

- **C-convex** (convex in the sense of Luenberger (1969)) if for any $x_0, x_10 \in X$ and $t \in (0, 1)$, we have
$$f((1-t)x_0 + tx_1) \leq_C (1-t)f(x_0) + tf(x_1), \tag{33}$$
- **C-quasi-convex** (strongly quasiconvex w.r.t. C in the sense of Borwein (1974)) if for any $y \in Y$, the level set
$$f^{-1}(y - C) := (x \in X | f(x) \leq_C y) = (x \in X | f(x) + (-y) \leq_C 0) \tag{34}$$
is convex; in other words (see, e.g. Luc (1989)), f is C-quasi-convex if and only if for any $x_0, x_1 \in X$ and $y \in Y$
$$f(x_0) \leq_C y, f(x_1) \leq_C y \to f((1-t)x_0 + tx_1) \leq_C y \tag{35}$$
for all $t \in (0, 1)$, for any $t \in (0, 1)$;
- **quasi-convex** (in the sense of Jahn (1986, 2011)) if for any $x_0, x_1 \in X$,
$$f(x_0) \leq_C f(x_1) \to f((1-t)x_0 + tx_1) \leq_C f(x_1) \tag{36}$$
for any $t \in (0, 1)$.

Definition 31. A function f is called **η-quasi-convex**, if

$$f(tx + (1-t)y) \leq \max(f(y), f(y) + \eta(f(x), f(y))\mathbb{R}) \tag{37}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In the above definition if we set $\eta(x, y) = x - y$, then we approach to the quasi-convex. Note that by taking $x = y$ we get $t\eta(f(x), f(x) \geq 0$ for any $x \in I$ and $t \in [0, 1]$ which implies that

$$\eta(f(x), f(x)) \geq 0$$

for any $x \in I$. Also if we take $t = 1$ we get

$$f(x) - f(y) \leq \eta(f(x), f(y)), \tag{38}$$

for any $x, y \in I$. The second condition obviously implies the first. If $f : I \to \mathbb{R}$ is a convex function and $eta : I \times I \to \mathbb{R}$ is an arbitrary bi-function that satisfies

$$\eta(x, y) \geq x - y \tag{39}$$

for any $x, y \in I$ then

$$f(tx + (1-t)y) \leq f(y) + t[f(x) - f(y)] \leq f(y) + t\eta(f(x), f(y)), \tag{40}$$

showing that f is $\eta - convex$.

5 Diagram

As we have said previously, the definition of convex function has been extended and generalized depending on the problem and applications in question.

In summary we now present the following diagram where the relationships between the classical convex function definition and their different generalizations are represented.

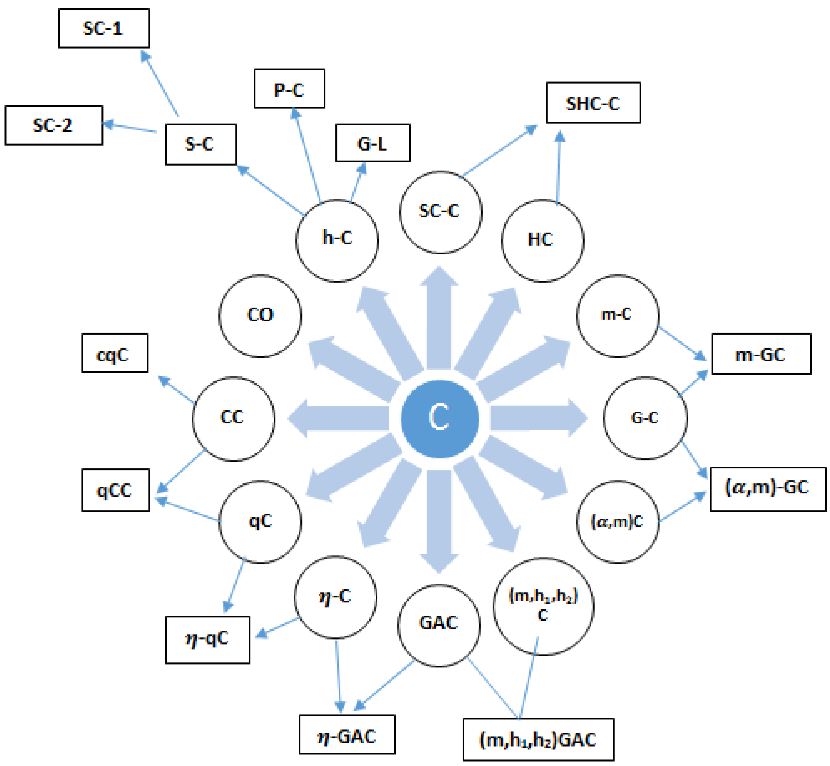

Figure 2: Diagram

References:

- **C** Classical convex function.
- **CO** Convexity with respect to another function.
- **SC-C** Strongly convex function with modulus C.
- **HC** Harmonically convex function.
- **SHC-C** Strongly harmonically convex function with modulus C.

- **h-C** h-convex function.
- **P-C** P- convex function.
- **G-L** Godunova-Levin function.
- **S-C** s-convex function.
- **SC-1** s-convex function in the first sense.
- **SC-2** s-convex function in the second sense.
- **m-C** m-convex function.
- **G-C** Geometrically convex function.
- **m-GC** m-geometrically convex function.
- **(α,m)-C** (α,m) convex function.
- **(α,m)-GC** (α,m) geometrically convex function.
- **GAC** Geometeric arithmetically convex function.
- **(m,h_1, h_2)-C** (m,h_1, h_2) convex function.
- **(m,h_1, h_2)-GAC** (m,h_1, h_2) geometric arithmetically convex function.
- η**-C** η convex function.
- η**-GAC** Generalized geometric arithmetically convex function with respect to η.
- **qC** Quasi-convex function.
- **CC** C- convex function.
- **cqC** C- quasi-convex function.
- **qCC** Quasi-convex function with respect to C.
- η**qC** η-quasi-convex function.

6 A new notion of convexity as an Epilogue

We have seen a great variety of Definitions of convexity, coming from different areas of Mathematics, we now want to present a new definition that includes some of the previously presented.

Definition 32. (Partially h-convex) Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is partially h-convex function, or that f belongs to the class $PX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + (1 - h(\alpha))f(y). \tag{41}$$

Remark 4. It is easy to see that for some h choices we can get some of the known definitions. For example:

- $h(t) = t$, in this case we have the classic convex function.
- $s = 1$, this notion can be considered a particular case of s-convexity (see Pinheiro (2007)) in the second sense (or h-convex), with h an additive function such that $h(1) = 1$.

- $h : [0, 1] \to [0, 1]$, in this case, we come to the notion of class $\mathbb{F}$, see Ng (1987).

Obviously, using this definition to obtain the Hermite-Hadamard Inequality (2) will give us more general results than those reported in the literature for the notion of convexity and consistent with those obtained for s-conevity. All of which indicates the breadth of this new notion of convexity.

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