Advanced Mathematical Models & Applications
Vol.5, No.3, 2020, pp.307-312

LEFT REGULAR ORDERED AG-GROUPOIDS

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Abstract. In this article, we extend the notions of left(right) ideal, left(right) ideal element to the left(right) simple ordered AG-groupoids, left(right) regular ordered AG-groupoids, and regular ordered AG-groupoid. We present the main results which characterize this class of non-associative and non-commutative ordered semigroups through these objects.

Keywords: Ordered AG-groupoid, left(right)ideal, left(right) ideal element, left(right) simple ordered AG-groupoid, left(right)regular ordered AG-groupoid.
AMS Subject Classification: 06F99, 20N99.
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Received: 5 October 2020; Revised: 29 October 2020; Accepted: 3 November 2020;
Published: 29 December 2020.

1 Introduction

In 1972, a generalization of commutative semigroups was initiated by Kazim & Naseeruddin (1977). In triple commutative law: $abc = cba$, the authors arranged braces on the left side of the law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. They have called it the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law : $(ab)c = (cb)a$. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid). An AG-groupoid is a midway structure between an abelian semigroup and a groupoid.

A groupoid S is to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp.$(ab)(cd) = (db)(ca)$). In Kazim & Naseeruddin (1977), an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like Yousafzai & Khalaf (2019). Although semigroups concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science and information science. Algebraic structures especially ordered semigroups play a prominent role in mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages (See Khan et al. (2019)).

In Kehayopulu (1990), if $(S, \cdot, \leq)$ is an ordered semigroup, then for $\emptyset \neq A \subseteq S$, the authors defined a subset $(A]$ of S such that : $(A] = \{s \in S : s \leq a$ for some $a \in A$. A non-empty subset A of S is a subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of S is a left (resp. right) ideal of S if following hold : (1) $SA \subseteq A$ (resp.$AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ **?**. Equivalently, A is a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).

In Kehayopulu (1990) an ordered semigroup S is to be regular, if for every $a \in S$, there exists an element $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for every $A \subseteq S$. (2) $a \in (aSa]$ for every $a \in S$.

Kehayopulu & Tsingelis (2002) studied the fuzzy sets in ordered groupoids. S. Lee, in his articles Lee (2005a) and Lee (2005b), respectively studied the prime left(right) ideals of ordered groupoids, and the prime bi-ideals of groupoids. Shah & Kausar (2014) initiated fuzzy bi-ideals on ordered AG-groupoids and also characterized the ordered AG-groupoids by using fuzzy bi-ideals. Shah & Kausar (2014) again presented the characterizations of non-associative ordered semigroups through the properties of their intuitionistic fuzzy bi-ideals. N. Kausar characterized the ordered AG-groupoids by using the fuzzy ideals with thresholds $(\alpha, \beta]$ in Kausar (2019), and the LA-rings in Kausar et al. (2019). Kausar et al. (2020b) presented the idea of fuzzy ideals on AG-Groupoid in Kausar et al (2020d). Kausar et al characterized the ordered AG-groupoid through the interior fuzzy ideal, the LA-Ring through the intuitionistic ideals in Kausar et al. (2020c).

Kausar et al. (2020a) established the anti fuzzy interior ideals on ordered AG-groupoids, and discussed their essential properties. In their articles, Yousafzai & Khalaf (2019) and Yousafzai et al. (2019) extended the concept of the AG-groupoid and non-associative ordered semigroups to the soft set theory. Recently, Shah & Kausar (2014) studied the characterizations of non-associative ordered semigroups in terms of intuitionistic fuzzy bi-ideals. A comprehnsive survey of the development of the non-associative rings is given in Shah et al. (2019).

In this note, following the concepts of the left regular ordered semigroups given by Lee et. al Kwon & Lee (1998), we are going to initiate the concept of left (right) ideal, left (right) ideal element, left (right) simple ordered AG-groupoid, left (right) regular ordered AG-groupoid, regular ordered AG-groupoid. The main result of this note is that if S is an ordered AG-groupoid with the left identity e such that $x^2 (x \in S)$ is a left ideal element, then S is a left regular if and only if there exists a family $\{S_\alpha \mid \alpha \in Y\}$ of left simple AG-subgroupoids of S such that $S = \cup\{S_\alpha \mid \alpha \in Y\}$.

2 Left regular ordered AG-groupoids

Definition 1. *Referring to Shah & Kausar (2014), an ordered AG-groupoid S, is an AG-groupoid, at the same time partially ordered set, such that $a \leq b$, implies $ac \leq bc$ and $ca \leq cb$ for all, $a, b, c \in S$.*

Example 1. *Consider a set $S = \{a, b, c, d, e\}$ with the following multiplication $''.''$ and order relation $'' \leq''$.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

$\leq := \{(a, a), (a, b), (b, a), (e, e)\}$. *Then $(S, \cdot, \leq)$is an ordered AG-groupoid with left identity e.*

For $\emptyset \neq A \subseteq S$. Then A is an AG-subgroupoid of S if $A^2 \subseteq A$ and $A$is a left (resp. right) ideal of Sif the following hold (1) $SA \subseteq A$(resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. A is an ideal of S if A is both a left and a right ideal of S.

An element s of S is a left (right) ideal element of S if $xs \leq s$ $(sx \leq s)$ for all, $x \in S$. An element t of S is semiprime if $a^2 \leq t$ for some $a \in S$implies $a \leq t$. A non-empty subset $P$of S is semiprime ideal if $A^2 \subseteq P$ implies $A \subseteq P$, for any ideal A of S.

An ordered AG-groupoid S is a left (resp. right) simple if and only if $(Sa] = S$ (resp.$(aS] = S)$ for every $a \in S$. Equivalently, we say that : an ordered AG-groupoid S is a left (resp. right) simple if for every left (resp. right) ideal A of S, we have $A = S$.

An AG-subgroupoid A of S is called a left (resp. right) simple if for every left (resp. right) ideal L of A we have $L = A$.

An ordered AG-groupoid S is called left (resp.right) regular if for every $a \in S$ there exists an element $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$). Equivalent definitions are as follows: (1) $a \in (Sa^2]$ (resp. $(a^2S]$) for every $a \in S$. (2) $A \subseteq (SA^2]$ (resp. $(A^2S]$) for every $A \subseteq S$.

An ordered AG-groupoid S is called regular if for every $a \in S$, there exists an element $x \in S$ such that $a \leq (ax)a$. Equivalent definitions are as follows: (1) $a \in ((aS)a]$ for every $a \in S$. (2) $A \subseteq ((AS)A]$ for every $A \subseteq S$.

If an ordered AG-groupoid S is a left simple and a right simple, then it is regular. Indeed, let $a \in S$. Since $(Sa] = S$and $(aS] = S$. Now $a \in (Sa] = ((aS]a] = ((aS)a]$ implies $a \in ((aS)a]$.

In other words, let x and $a \in S$, then by right simple and left simple we can write $x \leq ax$ and $a \leq xa$. Now $a \leq xa \leq (ax)a$, i.e., S is regular.

We denote by $L(a), R(a), I(a)$ the left ideal, the right ideal and the ideal of S, respectively, generated by a. We have $L(a) = \{s \in S : s \leq a$ or $s \leq xa$ for some $x \in S\} = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$.

We define the relations £ and $\mathcal{L}$ and σ on S as follows:

1. a £ b if and only if L(a) = L(b),

2. $a\mathcal{L}b$ if and only if $R(a) = R(b)$,

3. $a\sigma b$ if and only if $I(a) = I(b)$.

An equivalence relation is a relation on a set, generally denoted by $'' \sim''$ that is reflexive, symmetric, and transitive Bhuniya & Hansda (2020)

1. Reflexivity means $a \sim a$.
2. Symmetry, if $a \sim b$ then $b \sim a$.
3. Transitivity, if $a \sim b$ and $b \sim c$ then $a \sim c$.

A right congruence on a group (set) is an equivalence relation on the group (set) with the property that the equivalence relation is preserved on right multiplication by any element of the group (set).

Lemma 1. *Let A and B be two non empty subsets of an ordered AG-groupoid S. Then the following properties hold:*

(1) $((A]] = (A]$.
(2) $(A](B] \subseteq (AB]$
(3) $((A](B]] = (AB]$.
(4) *If $A \subseteq B$, then $(A] \subseteq (B]$.*
(5)$(A \cap B] \neq (A] \cap (B]$, *in general.*

Proof. Obvious. ☐

Lemma 2. *Let S be an ordered AG-groupoid. Then £ there is a right congruence on S i.e. there is an equivalence relation on S such that $a£b \Rightarrow ac£bc$ for all $c \in S$.*

Proof. Obvious. ☐

Theorem 1. *Let S be an ordered AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is a left regular.*

(2) *$L(a) \subseteq L(a^2)$ for all $a \in S$.*

(3) *$a \pounds a^2$ for all $a \in S$.*

Proof. Suppose that S is a left regular and $a \in S$. Let $t \in L(a)$, then $t \leq a$ or $t \leq xa$ for some $x \in S$. This implies that $a \leq ya^2$, Sbeing a left regular. If $t \leq a$, then $t \leq a \leq ya^2$ for some $y \in S$. If $t \leq xa$, then $t \leq xa \leq x(ya^2) = ((ee)x)(ya^2) = ((xe)e)(ya^2) = ((xe)y)(ea^2) = ((xe)y)a^2$ for some $y \in S$. Hence $L(a) \subseteq L(a^2)$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and $t \in L(a^2)$. Then $t \leq a^2$ or $t \leq xa^2$ for some $x \in S$. This means that $t \leq aa$ or $t \leq xa^2 = ((ee)x)(aa) = ((xe)e)(aa) = ((xe)a)(ea) = ((xe)a)a$ for some $x \in S$. In any case $t \leq za \in Sa \subseteq (a \cup Sa] = L(a)$for some $z \in S$, i.e., $L(a^2) \subseteq L(a)$. Thus $L(a) = L(a^2)$, i.e., $a \pounds a^2$ for all $a \in S$, therefore $(2) \Rightarrow (3)$. Suppose that (3) holds. This implies that $a \in L(a) = L(a^2) = (a^2 \cup Sa^2]$. Thus $a \leq a^2$ or $a \leq xa^2$ for some $x \in S$, i.e., $a \leq a^2 = ea^2$ or $a \leq xa^2$. In any case, we can get $a \leq ta^2$ for some $t \in S$, i.e., a is a left regular. Hence S is a left regular, i.e., $(3) \Rightarrow (1)$. □

Theorem 2. *Let S be an ordered AG-groupoid with left identity e such that x^2(for all$x \in S$) is a left ideal element, then the following conditions are equivalent.*

(1) *S is a left regular.*

(2) *Every left ideal element of S is semiprime.*

(3) *Every left ideal of S is semiprime.*

Proof. Suppose that (1) holds and t be a left ideal element of S and $a^2 \leq t$. Since $a \leq xa^2$ for some $x \in S$, S being a left regular, thus $a \leq xa^2 \leq xt \leq t$. Hence t is semiprime, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and L be a left ideal of $S$and $a^2 \in L$. Since $a \leq a^2 = aa$, i.e., $a \in L$, by the definition of left ideal. Hence L is a semiprime, i.e., $(2) \Rightarrow (3)$. Suppose that (3) holds and $L(a^2)$ is a left ideal. Since $a^2 \in L(a^2)$, i.e., $a \in L(a^2)$, $L(a^2)$ being a semiprime. Thus $L(a) \subseteq L(a^2)$. Hence S is a left regular by the Theorem 1, i.e., $(3) \Rightarrow (1)$. □

Theorem 3. *Let S be an ordered AG-groupoid with left identity $e$such that x^2(for all$x \in S$) is a left ideal element. Then S is a left regular if and only if there exists a family $\{S_\alpha \mid \alpha \in Y\}$ of left simple AG-subgroupoid of S such that $S = \cup\{S_\alpha \mid \alpha \in Y\}$.*

Proof. Let S be a left regular. We denote by $(x)_\pounds$ the $\pounds$-class of S containing x(for all$x \in S$). We have to show $(x)_\pounds$ is a left simple AG-subgroupoid of S for all $x \in S$. Since $(x)_\pounds$ is not empty because $x \in (x)_\pounds$ and let $a, b \in (x)_\pounds$. This implies that $a \pounds x$ and $b \pounds x$. Then $ab \pounds xb$ and $b^2 \pounds xb$, $\pounds$ being a right congruence by the Lemma 2. This means that $ab \pounds xb$ and $xb \pounds b^2$, $\pounds$ being symmetric and $ab \pounds b^2$, $\pounds$ being transitive. Since S is a left regular, i.e., $b^2 \pounds b$ by Theorem 1. Thus $ab \pounds b$, i.e., $ab \in (b)_\pounds = (x)_\pounds$. Hence $(x)_\pounds$ is an AG-subgroupoid of S. Let L be a left ideal of $(x)_\pounds$ and $z \in L$. Let $y \in (x)_\pounds$ and $z \in L \subseteq (x)_\pounds = (y)_\pounds$. Since $y \in L(y) = L(z) = L(z^2)$, by the Theorem 1. Then $y \leq z^2$ or $y \leq tz^2$ for some $t \in S$. If $y \leq z^2$, then $y \leq z^2 = zz \in (x)_\pounds L \subseteq L$. If $y \leq tz^2$, then $y \leq tz^2 \leq z^2$, $x^2 (x \in S)$ being a left ideal element. This implies $y \in L$ and $L = (x)_\pounds$. Thus every $(x)_\pounds$ is a left simple AG-subgroupoid of S. Hence $S = \cup\{S_\alpha \mid \alpha \in Y\}$.

Conversely, suppose that there is a family $\{S_\alpha \mid \alpha \in Y\}$ such that $S = \cup\{S_\alpha \mid \alpha \in Y\}$, where each S_α is a left simple AG-subgroupoid of S. Let L be a left ideal of S and $a^2 \in L$, $a \in S$, then $a \in S_\alpha$ for some $\alpha \in Y$. Consider a subset $L \cap S_\alpha$ of S. Since S_α is an AG-subgroupoid of S, so $a^2 \in S_\alpha$, i.e., $L \cap S_\alpha \neq \emptyset$. Now $S_\alpha(L \cap S_\alpha) \subseteq S_\alpha L \cap S_\alpha^2 \subseteq SL \cap S_\alpha \subseteq L \cap S_\alpha$.

Let $x \in L \cap S_\alpha$ and $x \geq y \in S_\alpha$. Since $x \in L$ and $y \leq x$, i.e., $y \in L$, because L is a left ideal of S. Thus $y \in L \cap S_\alpha$, i.e., $L \cap S_\alpha$ is a left ideal of S_α. Since S_α is left simple, i.e., $L \cap S_\alpha = S_\alpha$. Thus $a \in L$, i.e., L is a semiprime. Hence S is a left regular by the Theorem 2. □

3 Conclusion

In this note, we have presented the concepts of left (right) ideal, left(right) ideal element, left(right) simple ordered AG-groupoid, left (right) regular ordered AG-groupoid and regular ordered AG-groupoid. We have also presented the main result that for an ordered AG-groupoid S with the left identity e, $x^2(x \in S)$ is a left regular if and only if there exists a family $\{S_\alpha \mid \alpha \in Y\}$ of left simple AG-subgroupoids of S such that $S = \cup\{S_\alpha \mid \alpha \in Y\}$.

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