THE EXTENDED LENGTH BIASED TWO PARAMETERS MIRRA DISTRIBUTION WITH AN APPLICATION TO ENGINEERING DATA

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Abstract. In this article, we introduce an extension of the two parameters Mirra distribution (MD) called length biased two parameters Mirra distribution (MD), which offers more flexibility in modeling some life time data. Some statistical properties of the suggested model are explicitly derived. These properties include the moments, moment generating function, the coefficient of variation, coefficient of skewness and coefficient of kurtosis, the distributions of order statistics, reliability analysis and the maximum likelihood estimators of the model parameters. Also, we obtain the Rényi entropy, stochastic ordering, mean and median deviations, Lorenz and Bonferroni curves and the Gini index. An application of the model to a real data set consists of the failure times of 84 Aircraft Windshield is evaluated and compared with some other well-known existing distributions.

Keywords: Mirra distribution, Maximum likelihood estimators, Reliability analysis, Rényi entropy, Gini index, Stochastic ordering, Lorenz curve, Bonferroni curve.

AMS Subject Classification: 60E05, 62E15.

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1 Introduction

Sen et al. (2017) suggested the two parameters Mirra distribution (MD) with probability density function (pdf) given by

$$f_{MD}(x; \alpha, \theta) = \frac{\theta^3}{\alpha + \theta^2} \left(1 + \frac{\alpha}{2} x^2\right) e^{-\theta x}, \quad x > 0, \quad \alpha > 0, \quad \theta > 0,$$

and cumulative distribution function (cdf) defined as

$$F_{MD}(x; \alpha, \theta) = 1 - \frac{\theta^2 \left(1 + \frac{\alpha}{2} x + \frac{\alpha}{2} x^2\right) e^{-\theta x}}{\alpha + \theta^2}. \quad (2)$$

The rth mean of the MD is

$$E(X^r) = \frac{\theta^{-r} \Gamma(r + 1) \left(2\theta^2 + \alpha(r + 1)(r + 2)\right)}{2(\alpha + \theta^2)}, \quad r = 1, 2, 3, \ldots. \quad (3)$$

and for $r = 1$ we obtain the mean of the distribution as $\mu = \frac{3\alpha + \theta^2}{\alpha + \theta^2}$. The mode of the MD is $M(X) = \frac{1 + \sqrt{1 - 2\theta^2/\alpha}}{\theta}, \; \text{if} \; \theta^2 < \alpha/2,$ and zero otherwise.

In 1934, Fisher introduced the concept of weighted distributions for modeling ascertainment bias. Let $X$ be a random variable with pdf $f(x)$, then the corresponding weighted distribution function is defined by

$$f^w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad (4)$$
where $w(x)$ is a non-negative weight function such that $E(w(x))$ exists. Rao (1965) defined the size-biased distributions which is a special case of the weighted distributions when the weighted function is given by $w(x) = x^a$ with pdf

$$f_a(x) = \frac{x^a f(x)}{E_a(X)}.$$  \hfill (5)

For $a = 1, 2$ we can get the length-biased and size-biased distributions, respectively. Recently, the idea of weighted distribution attracts the researchers to introduce several weighted distributions for fitting real data. For instance, see Gupta and Kundu (2009) for a new class of weighted exponential distributions; Ahmed et al. (2013) for size biased Gamma distribution; Mudasir and Ahmad (2015) for length biased Nakagami distribution; Reyad et al. (2017) for length-biased weighted Erlang distribution; Al-Kadim and Fadhil (2018) for double weighted Lomax distribution; Al-Omari et al. (2019) for size-biased Ishita distribution; Al-Omari and Alsmairan (2019) for length-biased Suja distribution; Alsmairan and Al-Omari (2020) for weighted Suja distribution; Sherina and Oluyede (2014) proposed weighted inverse Weibull distribution; Modi and Gill (2017) proposed exponentiated generalized Lindley distribution. Beghriche and Beghriche (2019) proposed a size biased gamma Lindley distribution. Zeghdoudi and Nedjar (2016) proposed gamma Lindley distribution.

The paper is organized as follows: In Section 2, we define the length-biased Mirra distribution and shapes of the pdf and cdf are presented. Some distributional properties are given in Section 3. The applicability of the suggested model is accomplished in Section 4. Finally, the paper is concluded in Section 5.

### 2 The extended length biased two parameters Mirra distribution

The pdf of the length biased two parameters Mirra distribution is defined as

$$f_{LBMD}(x; \alpha, \theta) = \frac{\theta^4}{3\alpha + \theta^2 x} \left(1 + \frac{x^2 \alpha}{2}\right) e^{-x\theta}, \; 0 < x < \infty, \; \alpha > 0.$$ \hfill (6)

It is clear that $f_{LBMD}(x; \alpha, \theta)$ satisfies the conditions of being a pdf, that is $f(x) \geq 0$ for all $x$ and $\int_0^\infty f_{LBMD}(x; \alpha, \theta) dx = 1$. The corresponding cumulative distribution function of the LBMD is

$$F_{LBMD}(x; \alpha, \theta) = 1 - \frac{\alpha \left[6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6\right] + 2\theta^2(\theta x + 1)}{2(3\alpha + \theta^2)} e^{-x\theta}, \; x > 0, \; \alpha > 0, \; \theta > 0.$$ \hfill (7)

In Figure 1 we plot the pdf and cdf of the LBMD for some values of $\theta$ and $\alpha$.

![Figure 1](image_url)

**Figure 1:** Plots of pdf and cdf of the LBMD for $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5$ and $\theta = 0.9$. 
Figure 1 indicates that the pdf of the LBMD is asymmetric and positively skewed depending on the distribution parameters values.

3 Distributional Properties

This section is devoted in finding important distributional properties of the LBMD

3.1 Raw and central moments

The $r$th moment of the LBMD is

$$E(X_{LBMD}^r) = \frac{(2 + r)(3 + r)\alpha + 2\theta^2}{2\theta(3\alpha + \theta^2)} \Gamma[2 + r], \quad \alpha, \theta > 0, \quad r = 1, 2, 3, \ldots \quad (8)$$

For $r = 1, 2, 3, 4$ in Equation (8), the first four moments of the LBMD distribution, respectively, are given as

$$E(X_{LBMD}^1) = \frac{2(6\alpha + \theta^2)}{\theta(3\alpha + \theta^2)} = \mu, \quad E(X_{LBMD}^2) = \frac{6(10\alpha + \theta^2)}{\theta^2(3\alpha + \theta^2)},$$

$$E(X_{LBMD}^3) = \frac{12(30\alpha + 2\theta^2)}{\theta^3(3\alpha + \theta^2)}, \quad E(X_{LBMD}^4) = \frac{120(21\alpha + \theta^2)}{\theta^4(3\alpha + \theta^2)}, \quad \theta > 0.$$ 

Hence, the variance of the distribution is

$$\text{Var}(X_{LBMD}) = E(X_{LBMD}^2) - (E(X_{LBMD}^1))^2 = \frac{2(18\alpha^2 + 15\alpha\theta^2 + \theta^4)}{(3\alpha\theta + \theta^3)^2}. \quad (9)$$

3.2 Moment generating function

The moment generating function, $M_X(t)$ of the LBMD is

$$M_{LBMD}(t) = \frac{\theta^4}{3\alpha + \theta^2} \frac{3\alpha + (\theta - t)^2}{(\theta - t)^4}, \quad \theta > 0, \quad \alpha > 0. \quad (10)$$

The characteristic function (CF) of LBMD is

$$\varphi_{LBMD}(t) = \frac{\theta^4}{3\alpha + \theta^2} \frac{3\alpha + (\theta - it)^2}{(\theta - it)^4}, \quad t \in R, \quad i = \sqrt{-1}. \quad (11)$$

3.3 Coefficient of skewness

$$Sk_{LBMD} = \frac{\sqrt{2}(54\alpha^3 + 54\alpha^2\theta^2 + 27\alpha\theta^4 + \theta^6)}{(18\alpha^2 + 15\alpha\theta^2 + \theta^4)^{3/2}} \quad (12)$$

3.4 Coefficient of kurtosis

$$Ku_{LBMD} = \frac{6(243\alpha^4 + 378\alpha^3\theta^2 + 180\alpha^2\theta^4 + 36\alpha\theta^6 + \theta^8)}{(18\alpha^2 + 15\alpha\theta^2 + \theta^4)^2}. \quad (13)$$
### 3.5 Coefficient of variation

The coefficient of variation is

$$C_{V_{LBMD}} = \frac{\sqrt{18\alpha^2 + 15\theta^2 + \theta^4}}{\sqrt{2}(6\alpha + \theta^2)}.$$  \hspace{1cm} (14)

Tables (1) and (2) contain the mean, standard deviation (SD), the coefficient of variation, the coefficient of skewness and coefficient of kurtosis for some selected values of the distribution parameters.

**Table 1:** The mean, SD, CV, SK and CK for $\alpha = 0.6, 2$ and some selected values of $\theta$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Mean</th>
<th>SD</th>
<th>CV</th>
<th>SK</th>
<th>CK</th>
</tr>
</thead>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3.285710</td>
<td>2.05039</td>
<td>0.62857</td>
<td>1.02109</td>
<td>4.43037</td>
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<tr>
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<td>0.932311</td>
<td>0.711500</td>
<td>1.30484</td>
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<tr>
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<td>1.44070</td>
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<tr>
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<td>0.400456</td>
<td>0.727359</td>
<td>1.47819</td>
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<tr>
<td>↑</td>
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<td>0.426866</td>
<td>0.308865</td>
<td>0.723565</td>
<td>1.48091</td>
</tr>
<tr>
<td>↑</td>
<td>6</td>
<td>0.349206</td>
<td>0.251476</td>
<td>0.720135</td>
<td>1.47388</td>
</tr>
<tr>
<td>↑</td>
<td>7</td>
<td>0.295838</td>
<td>0.212256</td>
<td>0.717473</td>
<td>1.46522</td>
</tr>
<tr>
<td>↑</td>
<td>8</td>
<td>0.256839</td>
<td>0.183760</td>
<td>0.715469</td>
<td>1.45730</td>
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<tr>
<td>↑</td>
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<td>0.227053</td>
<td>0.162106</td>
<td>0.713956</td>
<td>1.45064</td>
</tr>
<tr>
<td>↑</td>
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<td>0.203536</td>
<td>0.145081</td>
<td>0.712800</td>
<td>1.44519</td>
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<tr>
<td>↑</td>
<td>11</td>
<td>0.184483</td>
<td>0.131334</td>
<td>0.711903</td>
<td>1.44075</td>
</tr>
<tr>
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<td>1.43713</td>
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<tr>
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<tr>
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<td>0.134392</td>
<td>0.085391</td>
<td>0.709798</td>
<td>1.42967</td>
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<td></td>
</tr>
<tr>
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<td>0.95107</td>
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</tr>
<tr>
<td>↑</td>
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<td>1.600000</td>
<td>1.019800</td>
<td>0.637377</td>
<td>1.04847</td>
</tr>
<tr>
<td>↑</td>
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<td>0.646357</td>
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<tr>
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<td>4</td>
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<td>0.456813</td>
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<td>1.34281</td>
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<tr>
<td>↑</td>
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<td>0.477419</td>
<td>0.347070</td>
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<td>1.41842</td>
</tr>
<tr>
<td>↑</td>
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<td>0.380952</td>
<td>0.277664</td>
<td>0.728869</td>
<td>1.45774</td>
</tr>
<tr>
<td>↑</td>
<td>7</td>
<td>0.316883</td>
<td>0.230658</td>
<td>0.727895</td>
<td>1.47547</td>
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<tr>
<td>↑</td>
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<td>0.271429</td>
<td>0.197044</td>
<td>0.725953</td>
<td>1.48139</td>
</tr>
<tr>
<td>↑</td>
<td>9</td>
<td>0.237548</td>
<td>0.171945</td>
<td>0.723832</td>
<td>1.48120</td>
</tr>
<tr>
<td>↑</td>
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<td>0.211321</td>
<td>0.152539</td>
<td>0.721835</td>
<td>1.47806</td>
</tr>
<tr>
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<td>0.190408</td>
<td>0.137105</td>
<td>0.720060</td>
<td>1.47366</td>
</tr>
<tr>
<td>↑</td>
<td>12</td>
<td>0.173333</td>
<td>0.124544</td>
<td>0.718521</td>
<td>1.46890</td>
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<tr>
<td>↑</td>
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<td>0.159121</td>
<td>0.114122</td>
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<td>1.46420</td>
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<tr>
<td>↑</td>
<td>14</td>
<td>0.147100</td>
<td>0.105334</td>
<td>0.716072</td>
<td>1.45980</td>
</tr>
<tr>
<td>↑</td>
<td>15</td>
<td>0.136797</td>
<td>0.097824</td>
<td>0.715106</td>
<td>1.45575</td>
</tr>
</tbody>
</table>

Based on Table 1, the distribution mean, standard deviation and coefficient of variation values are decreasing as theta values are increasing for fixed values of alpha. But the coefficients of skewness and kurtosis values are decreasing up to some values then start increasing. Table 2 shows that the distribution kurtosis average value is about 4.4 for all cases in the table.
Table 2: The mean, SD, CV, SK and CK for $\theta = 0.6, 2$ and some selected values of $\alpha$.

\[
\begin{array}{cccccc}
\alpha & \text{Mean} & \text{SD} & \text{CV} & \text{SK} & \text{CK} \\
\hline
\theta = 0.6 & & & & & \\
1 & 6.30952 & 3.40276 & 0.539306 & 0.953903 & 4.37941 \\
\uparrow & 2 & 6.47799 & 3.37490 & 0.520980 & 0.967782 & 4.42614 \\
\uparrow & 3 & 6.53846 & 3.36279 & 0.514309 & 0.976002 & 4.44738 \\
\uparrow & 4 & 6.56958 & 3.35611 & 0.510857 & 0.980973 & 4.45921 \\
\uparrow & 5 & 6.58854 & 3.35190 & 0.508746 & 0.984262 & 4.46672 \\
\uparrow & 6 & 6.60131 & 3.34900 & 0.507323 & 0.986589 & 4.47190 \\
\uparrow & 7 & 6.61049 & 3.34688 & 0.506298 & 0.988320 & 4.47568 \\
\uparrow & 8 & 6.61741 & 3.34526 & 0.505525 & 0.989657 & 4.47857 \\
\uparrow & 9 & 6.62281 & 3.34399 & 0.504921 & 0.990720 & 4.48085 \\
\uparrow & 10 & 6.62714 & 3.34297 & 0.504436 & 0.991586 & 4.48269 \\
\uparrow & 11 & 6.63070 & 3.34212 & 0.504038 & 0.992304 & 4.48421 \\
\uparrow & 12 & 6.63366 & 3.34141 & 0.503705 & 0.992909 & 4.48548 \\
\uparrow & 13 & 6.63618 & 3.34081 & 0.503423 & 0.993426 & 4.48656 \\
\uparrow & 14 & 6.63834 & 3.34029 & 0.503181 & 0.993873 & 4.48750 \\
\uparrow & 15 & 6.64021 & 3.33984 & 0.502971 & 0.994264 & 4.48831 \\
\theta = 2 & & & & & \\
1 & 1.42857 & 0.97938 & 0.685565 & 1.188640 & 4.88569 \\
\uparrow & 2 & 1.60000 & 1.01980 & 0.637377 & 1.048470 & 4.88911 \\
\uparrow & 3 & 1.69231 & 1.02916 & 0.608140 & 0.994214 & 4.38165 \\
\uparrow & 4 & 1.75000 & 1.03078 & 0.589015 & 0.970143 & 4.34948 \\
\uparrow & 5 & 1.78947 & 1.03002 & 0.575600 & 0.958847 & 4.34236 \\
\uparrow & 6 & 1.81818 & 1.02852 & 0.565685 & 0.953627 & 4.34464 \\
\uparrow & 7 & 1.84000 & 1.02684 & 0.558065 & 0.951518 & 4.35051 \\
\uparrow & 8 & 1.85714 & 1.02519 & 0.552027 & 0.951074 & 4.35762 \\
\uparrow & 9 & 1.87097 & 1.02365 & 0.547125 & 0.951545 & 4.36494 \\
\uparrow & 10 & 1.88235 & 1.02224 & 0.543067 & 0.952517 & 4.37203 \\
\uparrow & 11 & 1.89189 & 1.02096 & 0.539652 & 0.953755 & 4.37870 \\
\uparrow & 12 & 1.90000 & 1.01980 & 0.536739 & 0.955123 & 4.38489 \\
\uparrow & 13 & 1.90698 & 1.01875 & 0.534224 & 0.956540 & 4.39061 \\
\uparrow & 14 & 1.91304 & 1.01780 & 0.532032 & 0.957956 & 4.39586 \\
\uparrow & 15 & 1.91837 & 1.01693 & 0.530103 & 0.959345 & 4.40069 \\
\end{array}
\]
Table 3: The MLE and MSE of the estimated LBMD parameters for some selected values.

<table>
<thead>
<tr>
<th>N</th>
<th>$\hat{\alpha}$</th>
<th>$MSE(\hat{\alpha})$</th>
<th>$\theta$</th>
<th>$MSE(\theta)$</th>
<th>$\alpha = 0.5$</th>
<th>$MSE(\hat{\alpha})$</th>
<th>$\theta$</th>
<th>$MSE(\theta)$</th>
</tr>
</thead>
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<tr>
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</tr>
<tr>
<td>300</td>
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<td>0.5075</td>
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<td>0.5077</td>
<td>0.0043</td>
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<td>0.0002</td>
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<td>0.0002</td>
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</tr>
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<td>0.5061</td>
<td>0.0036</td>
<td>0.8997</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

3.6 Parameters estimation

Let $x_1, x_2, ..., x_n$ be the sample observations of the sample $X_1, X_2, ..., X_n$ chosen from the LBMD. The likelihood function for the LBMD is defined as

$$L_{LBMD}(\alpha, \theta | x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{\theta^4}{3\alpha + \theta^2} x_i \left(1 + \frac{x_i^2 \alpha}{2}\right) e^{-\theta x_i}$$

The log-likelihood function is given by

$$\log [L_{LBMD}(\alpha, \theta | x_1, x_2, ..., x_n)] = n \log \left(\frac{\theta^4}{3\alpha + \theta^2}\right) + \log \left(\prod_{i=1}^{n} x_i\right) + n \log \left(1 + \frac{x_i^2 \alpha}{2}\right) - \theta \sum_{i=1}^{n} x_i$$

$$= n \log \left(\frac{\theta^4}{3\alpha + \theta^2}\right) + \sum_{i=1}^{n} \log (x_i) + \sum_{i=1}^{n} \log \left(1 + \frac{x_i^2 \alpha}{2}\right) - \theta \sum_{i=1}^{n} x_i.$$

Differentiating (15) partially with respect to $\alpha$ and $\theta$, respectively and setting the result equals to zero, we have the log likelihood equations as

$$\frac{\partial \log [L_{LBMD}(\alpha, \theta | x_1, x_2, ..., x_n)]}{\partial \alpha} = \frac{3n \theta^2}{3\alpha + \theta^2} + \sum_{i=1}^{n} \frac{x_i^2}{2 \left(1 + \frac{\alpha x_i^2}{2}\right)}$$

$$\frac{\partial \log [L_{LBMD}(\alpha, \theta | x_1, x_2, ..., x_n)]}{\partial \theta} = - \frac{2 \theta^2 + 4 \left(3\alpha + \theta^2\right)}{3\alpha + \theta^2} - n \bar{x}.$$

By solving this nonlinear system of Equations (16) and (17) numerically such as Newton-Raphson we get the ML estimators for $\alpha$ and $\theta$. Table 3 lists the estimated distribution parameters using the MLE for some parameters selection. It turns out that the MSE values are decreasing as the sample size are increasing for all cases given in Table 3.

3.7 Reliability analysis

This section introduces the reliability analysis of the suggested LBMD. The reliability and hazard rate functions of the LBMD distribution are, respectively given by

$$R_{LBMD}(x, \alpha, \theta) = 1 - F_{LBMD}(x, \alpha, \theta) = \frac{\alpha \left[6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6\right] + 2\theta^2(\theta x + 1)}{2 \left(3\alpha + \theta^2\right)} e^{-x\theta},$$
and

\[ H_{LBMD}(x, \alpha, \theta) = \frac{f_{LBMD}(x, \alpha, \theta)}{1 - F_{LBMD}(x, \alpha, \theta)} = \frac{\theta^4 x (2 + x^2 \alpha)}{\alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1)} \]  

(19)

Figure (2) explores the plots of the \( R_{LBMD}(x; \alpha, \theta) \) and \( H_{LBMD}(x; \theta, \alpha) \) for \( \alpha = 0.1, 0.2, 0.3, 0.4, 0.5 \) and \( \theta = 0.9 \).

\[ O_{LBMD}(x; \alpha, \theta) = \frac{F_{LBMD}(x; \alpha, \theta)}{1 - F_{LBMD}(x; \alpha, \theta)} = \frac{2 \left( 3\alpha + \theta^2 \right) - \left\{ \alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1) \right\} e^{-x\theta}}{\alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1)e^{-x\theta}}. \]  

(21)

The possible odds and the reversed hazard rate functions shapes of LBMD distribution are displayed in Figure (3) for some selected parameters values.

The odds and the reversed hazard rate functions of the LBMD respectively, are defined as

\[ H_{LBMD}(x, \alpha, \theta) = \frac{f_{LBMD}(x, \alpha, \theta)}{1 - F_{LBMD}(x, \alpha, \theta)} = \frac{\theta^4 x (2 + x^2 \alpha)}{\alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1)} \]  

(20)

The possible odds and the reversed hazard rate functions shapes of LBMD distribution are displayed in Figure (3) for some selected parameters values.

\[ O_{LBMD}(x; \alpha, \theta) = \frac{F_{LBMD}(x; \alpha, \theta)}{1 - F_{LBMD}(x; \alpha, \theta)} = \frac{2 \left( 3\alpha + \theta^2 \right) - \left\{ \alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1) \right\} e^{-x\theta}}{\alpha \left[ 6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6 \right] + 2\theta^2(\theta x + 1)e^{-x\theta}}. \]  

(21)

The possible odds and the reversed hazard rate functions shapes of LBMD distribution are displayed in Figure (3) for some selected parameters values.

3.8 Order statistics

Let \( X_{(1:m)}, X_{(2:m)}, \ldots, X_{(m:m)} \) be the order statistics of a random sample of size \( m \), \( X_1, X_2, \ldots, X_m \) selected from \( f_{LBMD}(x; \alpha, \theta) \) with \( F_{LBMD}(x; \alpha, \theta) \), respectively. The pdf of the \( i \)th order statis-
tics say $X_{(i:m)}$ is

$$f_{(j:m)}(x; \alpha, \theta) = \frac{m!}{(j-1)!(m-j)!}[F(x; \alpha, \theta)]^{j-1}[1 - F(x; \alpha, \theta)]^{m-j}f(x; \alpha, \theta)$$

$$= \frac{\theta^j x^j e^{-\theta x}}{2^m(3\alpha + \theta^2)^m} \Gamma(j) \Gamma(m+1-j) \left\{ \alpha \left\{ \theta x(\theta x + 3) + 6 \right\} + 2\theta^2(\theta x + 1) \right\}^{m-j} \times \left( 2(3\alpha + \theta^2) - \left( \frac{\alpha}{\theta^2} - \frac{\alpha}{\theta x + 1} \right) \right)^{j-1}$$

(22)

Hence, the pdf of minimum and maximum order statistics say $X_{(1:m)}$ and $X_{(m:m)}$, are respectively, defined by

$$f_{(1:m)}(x; \alpha, \theta) = \frac{\theta^j x^j e^{-\theta x} (\alpha x^2 + 2)}{2^m(3\alpha + \theta^2)^m} \left\{ \alpha \left\{ \theta x(\theta x + 3) + 6 \right\} + 2\theta^2(\theta x + 1) \right\}^{m-1}$$

(23)

$$f_{(m:m)}(x; \alpha, \theta) = \frac{\theta^j x^j e^{-\theta x} (\alpha x^2 + 2)}{2^m(3\alpha + \theta^2)^m} \left\{ \alpha \left\{ \theta x(\theta x + 3) + 6 \right\} + 2\theta^2(\theta x + 1) \right\}^{m-1}$$

(24)

### 3.9 Stochastic ordering

The stochastic ordering can be used to compare two positive continuous distributions. A random variable $X$ is smaller than a random variable $Y$ in

- mean residual life order $X \leq Y$ if $m_X(x) \leq m_Y(x)$ for all $x$.
- likelihood ratio order $X \leq Y$ if $f_X(x) \leq Y$ decreases for all $x$.
- likelihood ratio order $X \leq Y$ if $h_X(x) \leq Y$ for all $x$.
- likelihood ratio order $X \leq Y$ if $F_X(x) \leq Y$ for all $x$.

It is shown by Shaked and Shanthikumar (1994) that

$$X \leq Y \Rightarrow X \leq Y \Rightarrow X \leq Y \Rightarrow X \leq Y$$

(25)

**Theorem 1.** Let $X_{LBMD} \sim f_X(x; \alpha, \theta)$ and $Y_{LBMD} \sim f_Y(x; \delta, \lambda)$, then if $\lambda < \theta$, we have $X \leq Y$, and hence $X \leq Y$ and $X \leq Y$, and $X \leq Y$.

**Proof.** Let $X_{LBMD} \sim f_X(x; \alpha, \theta)$ and $Y_{LBMD} \sim f_Y(x; \delta, \lambda)$. Hence,

$$f_{LBMD}(x; \alpha, \theta) = \frac{\theta^j x^j e^{-\theta x}}{3\alpha + \theta^2} \left( 1 + \frac{x^2 \alpha}{2} \right) e^{-x\theta}$$

and

$$f_{LBMD}(x; \delta, \lambda) = \frac{\lambda^j x^j e^{-\lambda x}}{3\delta + \lambda^2} \left( 1 + \frac{x^2 \delta}{2} \right) e^{-\lambda x}$$

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Therefore,

\[ f_X(x; \alpha, \theta) = \frac{\theta^4}{3\alpha + 2 \delta} x \left(1 + \frac{x^2 \alpha}{2}\right) e^{-\theta x} \]

and its logarithm is

\[ \ln f_X(x; \alpha, \theta) = \ln \left( \frac{\theta^4}{3\alpha + 2 \delta} x \left(1 + \frac{x^2 \alpha}{2}\right) e^{-\theta x} \right) \]

The derivative of the last equation with respect to \( x \) is

\[ \frac{\partial}{\partial x} \ln f_X(x; \alpha, \theta) = -\theta + \lambda + \frac{(\delta x^2 + 2) \left(\frac{2\alpha x}{\delta x^2 + 2} - \frac{2\delta x(\alpha x^2 + 2)}{(\delta x^2 + 2)^2}\right)}{\alpha x^2 + 2} \]

Now, if \( \lambda < \theta \), then \( \frac{\partial}{\partial x} \ln \left( f_X(x; \alpha, \theta) \right) < 0 \), which implies that \( X \leq Y \), and hence \( X \leq Y \), \( X \leq Y \) \( \text{LRO} \), \( X \leq Y \) \( \text{HRO} \), \( X \leq Y \) \( \text{MRLO} \) and \( X \leq Y \). The proof is completed.

**3.10 Mean and median deviations**

The mean deviations can be used as a measure of spread in a population. The mean deviations about the mean \( D(\mu) \) and about the median \( D(m) \) are given by the following:

\[ D(\mu) = \int_0^\infty |x - \mu| f(x) dx = 2 \mu F(\mu) - 2 \int_0^\mu x f(x) dx \]

and

\[ D(m) = \int_0^\infty |x - m| f(x) dx = 2 \mu - 2 \int_0^m x f(x) dx \]

where \( \mu = E(X) \) and \( m = median(X) \). For the LBMD we have

\[ D(\mu) = \frac{8(4\alpha + \theta^2)(6\alpha + \theta^2)^3}{\theta(3\alpha + \theta^2)^4} e^{-\frac{6\alpha}{3\alpha + \theta^2} - 2} \]

and

\[ D(m) = e^{-m \theta} \alpha \left[ \theta m (\theta m (\theta m + 4) + 12) - 12 e^{\theta m} + 24 \right] + 2 \theta^2 \left[ \theta m (\theta m + 2) - e^{\theta m} + 2 \right] \]

(26)
3.11 Lorenz and Bonferroni curves and Gini Index

Let $X$ be a non-negative random variable with a continuous and twice differentiable cumulative distribution function $F(x)$. The Lorenz curve is commonly used in economics in order to represent the distribution of income and the inequality of wealth distribution. Lorenz (1905) defined this curve as

$$L(t) = \frac{\int_0^t \int_0^x f(x)dx}{\int_0^{\infty} x f(x)dx}.$$

For the LBMD it is defined as

$$L_{LBMD}(t) = 1 + e^{-\theta t} \left[-\alpha \left[\theta t (\theta t + 4) + 12 + 24 - 2\theta^2 (\theta t + 2)\right]\right] \frac{4(6\alpha + \theta^2)}{4(6\alpha + \theta^2)} (27)$$

The Bonferroni curve of $X$ is defined as

$$B(p) = \frac{1}{p\mu} \int_0^p F^{-1}(t)dt, \ p \in [0, 1]$$

For the LBMD it is given by

$$B_{LBMD}(t) = \alpha \left[-\theta t (\theta t + 4) + 12 + 24 e^{\theta t} - 24 - 2\theta^2 (\theta t + 2) - 2e^{\theta t} + 2\right] e^{-\theta t} \frac{4p(6\alpha + \theta^2)}{4p(6\alpha + \theta^2)} (28)$$

The Gini index, $G$, is given by

$$G = 1 - \frac{1}{\mu} \int_0^\infty (1 - F(x))^2 dx = \frac{1}{\mu} \int_0^\infty F(x)(1 - F(x))dx.$$  

The Gini index for the LBMD is

$$G_{LBMD} = \frac{3}{64} \left( -\frac{21\alpha}{3\alpha + \theta^2} + \frac{29\alpha}{6\alpha + \theta^2} + 8 \right). (29)$$

Table (4) reveals the Gini index values of the LBMD for $\theta = 1, 2, ..., 14$ and $\alpha = 1, 2, 3, 4, 5$. According to Table (4), the Gini index values are all less than 1 and are running from 0.2 to 0.4 for all values of the distribution parameters given in the table.

3.12 Stress-strength reliability

Let the random variables $X$ and $Y$ are independent and observed from the pdf $f(x)$. The stress-strength reliability explain the life of a component that has a random strength $Y$ that is subjected to a random stress $X$, and it is defined as

$$R = P(Y < X) = \int_0^\infty P(Y < X|X = x) f(x)dx = \int_0^\infty f(x; \alpha)F(x; \delta)dx$$

**Theorem 2.** Let $X$ and $Y$ be two independent random variables follow the LBMD. Then, the stress-strength reliability is given by

$$R_{LBMD} = \frac{\beta^4}{(3\alpha + \theta^2)(3\alpha + \theta^2)(\beta + \theta)^7} \left[ 3\alpha \left( \frac{\beta^5 + 3\beta^3\delta + 35\theta^3 (\beta^2 + 3\delta)}{+21\beta^2 (\beta^2 + 3\delta)} \right) + 7\beta^2 (\beta^2 + 3\delta) + 30\beta^4 + 10\theta^5 
+ \theta^2(\beta + \theta)^2 \left( \frac{\beta^3 + 5\beta^2\delta + 3\beta\delta}{+7\beta^2 + 15\delta\theta + 3\theta^3} \right) \right]. (30)$$
Table 4: Gini index values of the LBMD for some values of $\theta$ and $\alpha$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha = 3$</th>
<th>$\alpha = 4$</th>
<th>$\alpha = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.370313</td>
<td>0.302885</td>
<td>0.294326</td>
<td>0.289615</td>
<td>0.286637</td>
</tr>
<tr>
<td>2</td>
<td>0.383594</td>
<td>0.348047</td>
<td>0.33206</td>
<td>0.323103</td>
<td>0.315862</td>
</tr>
<tr>
<td>3</td>
<td>0.384981</td>
<td>0.373214</td>
<td>0.361979</td>
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<td>0.344201</td>
</tr>
<tr>
<td>4</td>
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<td>0.382610</td>
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<td>0.363988</td>
</tr>
<tr>
<td>5</td>
<td>0.382126</td>
<td>0.384972</td>
<td>0.382983</td>
<td>0.384937</td>
<td>0.384964</td>
</tr>
<tr>
<td>6</td>
<td>0.380786</td>
<td>0.384766</td>
<td>0.384896</td>
<td>0.383594</td>
<td>0.381476</td>
</tr>
<tr>
<td>7</td>
<td>0.379727</td>
<td>0.383774</td>
<td>0.384952</td>
<td>0.384937</td>
<td>0.384132</td>
</tr>
<tr>
<td>8</td>
<td>0.378906</td>
<td>0.382648</td>
<td>0.384279</td>
<td>0.384981</td>
<td>0.385005</td>
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<tr>
<td>9</td>
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<td>0.381605</td>
<td>0.383381</td>
<td>0.384447</td>
<td>0.384964</td>
</tr>
</tbody>
</table>

Table 5, revealed that the Rényi entropy values are all positive and depend on the parameters values, where it is increasing for some values and decreasing for other values.

The harmonic mean of length biased LBMD distribution is given in the following theorem.

3.13 Rényi entropy

The Rényi entropy is defined as

$$RE(\eta) = \frac{1}{1 - \eta} \log \left( \int_0^{\infty} f(x)^\eta dx \right), \eta > 0, \eta \neq 1$$

Theorem 3. Let $X \sim f_{LBMD}(x; \alpha, \theta)$, then the and Rényi entropy of $X$ is defined as

$$RE(\eta) = \left\{ \frac{1}{1 - \eta} \log \left( \frac{\eta^\gamma}{(n+\gamma)^\eta} \right) \right\}_{\eta = \frac{\gamma}{\gamma+1}}^{\eta = \frac{\gamma+1}{\gamma}}$$

Table 5, revealed that the Rényi entropy values are all positive and depend on the parameters values, where it is increasing for some values and decreasing for other values.

The harmonic mean of length biased LBMD distribution is given in the following theorem.
Table 5: Renyi entropy values for some selected LBMD distribution parameters.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\eta = 0.7$</th>
<th>$\eta = 0.7$</th>
<th>$\eta = 0.4$</th>
<th>$\eta = 0.4$</th>
<th>$\theta$</th>
<th>$\alpha = 0.4$</th>
<th>$\alpha = 0.4$</th>
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<tbody>
<tr>
<td></td>
<td>$\theta = 4$</td>
<td>$\theta = 2$</td>
<td>$\theta = 4$</td>
<td>$\theta = 2$</td>
<td>$\eta = 0.7$</td>
<td>$\eta = 0.7$</td>
<td>$\eta = 0.3$</td>
<td>$\eta = 0.7$</td>
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</tr>
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<td>2</td>
<td>0.56527</td>
<td>1.41525</td>
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<td>4.43615</td>
<td>4.43404</td>
<td>4.75079</td>
<td>4.74915</td>
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<tr>
<td>3</td>
<td>0.61847</td>
<td>1.43983</td>
<td>0.83124</td>
<td>1.62579</td>
<td>0.4</td>
<td>3.06927</td>
<td>3.06318</td>
<td>3.37040</td>
<td>3.37075</td>
</tr>
<tr>
<td>4</td>
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<td>1.45067</td>
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<td>0.48030</td>
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</tbody>
</table>

Theorem 4. The harmonic mean of LBMD distribution is given by

$$H_{LBMD}(x) = \frac{\theta}{3\alpha + \theta^2} (\alpha + \theta^2)$$

(32)

Table 6 shows that the LBMD harmonic mean values are increasing as the parameter theta increasing for fixed $\alpha = 3, 7, 0.5, 0.1$. But for increasing alpha values the distribution harmonic mean values are decreasing for all $\theta = 3, 7, 0.5, 0.1$.

4 Applications of real data

In this section, the LBMD distribution is compared with some competitive models to illustrate its performance in real data modeling. We compare the LBMD with several competitive models such as

- Darna distribution (DD) with pdf
  $$f(x) = \frac{\theta}{2\alpha^2 + \theta^2} \left(2\alpha + \frac{\theta^2 x^2}{2\alpha^3}\right) e^{-\frac{\theta x}{\alpha}}; x > 0, \alpha > 0, \theta > 0.$$  

- Janardan distribution (JD) with pdf
  $$f(x) = \frac{\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta x}{\alpha}}; x > 0, \theta > 0, \alpha > 0.$$  

- Two parameters Sujatha distribution (TSPD) with pdf
  $$f(x) = \frac{\theta^3}{\alpha\theta^2 + \theta + 2} \left(\alpha + x + x^2\right) e^{-\frac{\theta x}{\alpha}}; x > 0, \theta > 0.$$  

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Table 6: Harmonic mean values of LBMD for some selected parameters.

<table>
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The data represent the failure times for a particular model 84 Aircraft Windshield given Table 16.11 of Murthy et al. (2004). The data set is the strength data of glass of the aircraft window reported by Fuller et al (1994). The data are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.505, 2.194, 3.030, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.524, 3.376, 4.663. The same data is studied by Ramos et al. (2013).

The choice of the best model for these data can be done with model selection criteria. In this section, the following goodness-of-fit tests are used, namely: Kolmogorov-Smirnov (KS), Cramer-von-Mises (W), and Anderson-Darling (A). Additionally, four information criteria are considered including Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) where these measures are defined as

\[
AIC = -2 \cdot MLL + 2t, \quad CAIC = -2 \cdot MLL + \frac{2t \cdot n}{n - t - 1}, \quad BIC = -2 \cdot MLL + t \cdot \log(n)
\]

and

\[
HQIC = 2 \cdot \log \left\{ \log(n) \left| t - 2 \cdot MLL \right| \right\}
\]

where \( t \) is the number of parameters and \( n \) is the sample size. The distribution with the lowest values of the AIC and BIC statistics is the best model for the data. Table 7 contains the results for the fitted models.

According to the results in Table 7, the LBMD distribution has the lowest values of these statistics, followed by MD, TSPD, JD and DD. Therefore, the suggested distribution is the best choice for the Aircraft Windshield data. The fitted pdfs and empirical cdfs plots of the five
models are sketched in Figure 4. Therefore, we assert that the LBMD fitting successfully the empirical plots of the data set.

![Figure 4: Estimated densities and cdfs of the models based on the real data.](image)

## 5 Conclusions

In this paper, the LBMD is defined. Several structural properties of the LBMD are studied in details include the moments, moment generating function, reliability analysis, the coefficients of variation, skewness and kurtosis, distributions of order statistics and harmonic mean. Also, the maximum likelihood estimation is used to estimate the distribution parameters, and the Rényi entropy, stochastic ordering, mean and median deviations, and stress-strength are derived. Finally, an application to a real data set is considered to demonstrate the applicability of the suggested distribution in practical situations. The empirical results assert that the LBMD distribution could be considered as a competitive model and can offer better modeling ability than its counterparts considered in this study.

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References


