Advanced Mathematical Models & Applications
Vol.6, No.2, 2021, pp.128-161

AN ANALYTICAL ANALYSIS TO SOLVE THE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates some novel exact analytical solutions, including soliton wave, periodic wave, singular, and kink-singular wave solutions for the fractional Whitham-Broer-Kaup and generalized Hirota-Satsuma coupled KdV equations. Firstly, the related equations are well established by the space–time fractional models. Secondly, two types of wave solutions for the nonlinear fractional equations are obtained by utilizing the transformation wave method. A kind of new (G'/G)-expansion method is also generated, i.e. the so called the generalized (G'/G)-expansion method, which enriches the types of solutions of PDEs. It is demonstrated that our proposed method is further efficient, general, succinct, power full, straight forward and can be asserted to install the new exact solutions of different kinds of fractional equations in engineering and nonlinear dynamics.

Keywords: Generalized (G'/G)-expansion method, Fractional nonlinear differential equations, Soliton wave, periodic wave, singular, and kink-singular.
AMS Subject Classification: 34A34.
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Received: 02 March 2021; Revised: 11 April 2021; Accepted: 18 May 2021; Published: 24 August 2021.

1 Introduction

Up to now, scientists and researchers in general utilize the analytical methods for the purpose of solving nonlinear PDEs and obtain the exact solutions (Mammadov, 2014; Manafian & Heidari, 2019), fractional solutions (Dehghan et al., 2010a, 2010b), the $\tan(\phi/2)$-expansion method (Lakestani & Manafian, 2018; Ilhan et al., 2018), Sine-Gordon expansion method (Korkmaz et al., 2020; Rezazadeh et al., 2019), lump solutions (Manafian et al., 2020; Ilhan et al., 2019), and others He et al. (2012); Dehghan et al. (2011). The main idea here is that we transform the original variables into new ones, therefore simplify the analysis so that the soliton solutions in these new variables become easy to reach. For example, the (1+1)-dimensional fractional Whitham-Broer-Kaup (FWBK) equation is considered as

$$\mathrm{D_t^\alpha u + uD_x^\alpha u + D_x^\alpha v} + \beta \mathrm{D_x^{2\alpha} u} = 0,\ \ 0 < \alpha \le 1, \tag{1}$$

$$\mathrm{D_t^\alpha v + D_x^\alpha (uv)} - \beta \mathrm{D_x^{2\alpha} v} + \gamma \mathrm{D_x^{3\alpha} u} = 0,$$

$u(x,t)$ is the field of horizontal velocity, $v(x,t)$ is the height deviating from equilibrium position of liquid, β and γ are real constants that represent different diffusion powers. If $\alpha = 1$, Eqs. (1) is the generalization of the Whitham-Broer-Kaup equations, which can be used to describe the dispersive long wave in shallow water (Mohebbi et al., 2012; Wang et al., 2012, 2014). Also, in our work, we introduce the fractional generalized Hirota-Satsuma coupled KdV (FgHSCKdV) equations (Meng, 2013) as below

$$D_t^\alpha u - \frac{1}{2}D_x^{3\alpha}u + 3uD_x^\alpha u - 3D_x^\alpha(vw) = 0, \tag{2}$$

$$D_t^\alpha v + D_x^{3\alpha}v - 3uD_x^\alpha v = 0,$$

$$D_t^\alpha w + D_x^{3\alpha}w - 3uD_x^\alpha w = 0.$$

Fractional calculus has applications in many scientific disciplines based on mathematical modelling including signal and image processing, physics, aerodynamics, chemistry, economics, electrodynamics, polymer rheology, economics, biophysics, control theory (Kilbas et al., 2006; Miller & Ross, 1993; Podlubny, 1999).

To our knowledge, few studies have been done to solve nonlinear PDEs based on the generalized (G'/G)-expansion method (gG'/GEM) (Yokus et al., 2020; Foroutan et al., 2018). However, the new nonlinear Riccati equations nonlinear terms will still incur a lot of computation and are yet to be fixed. Therefore, we embark on the new research topic of constructing the analytic solutions of nonlinear fractional PDEs by exploring the generalized (G'/G)-expansion method. According to recent studies, we can obtain some of the new exact analytic solutions of nonlinear PDEs or fractional PDEs by way of constructing their corresponding nonlinear ordinary differential equations.

The rest of the paper is in the following organization: Section 2 gives the generalized (G'/G)-expansion method. The soliton wave, periodic wave, singular, and kink-singular wave solutions are constructed in section 3 for FWBK and FgHSCKdV equations. At last, we conclude our work in section 4.

2 The gG'/GEM

The gG'/GEM was summarized and improved for achieving the analytic solutions of NLPDEs.
Step 1. Assume a nonlinear fractional partial differential equation is given in general form as follows

$$\mathcal{K}(u, D_t^\alpha u, D_x^\alpha u, ...) = 0, \quad 0 < \alpha \leq 1, \tag{3}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are the fractional derivatives and $\mathcal{K}$ is a polynomial. After simple algebraic operations, this equation is transformed into an ordinary differential equation (ODE) with the below transformation

$$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} - \frac{ct^\alpha}{\Gamma(\alpha+1)}, \tag{4}$$

then one gets

$$\mathcal{O}(u, -cu', ku', ...) = 0. \tag{5}$$

Moreover, here we list some important properties for the modified Riemann-Liouville derivative as follows:
(1) $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t)$,

(2) $D^\alpha[f(g(t))] = f'_g(g(t))D^\alpha g(t)$,

(3) $D^\alpha[f(g(t))] = D_g^\alpha f(g(t))[g'(t)]^\alpha$,

(4) $D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha-\gamma)}t^{\gamma-\alpha},\ \gamma > 0$,

where Γ denotes the Gamma function.
Step 2. Then, assume that the searched wave solutions of Eq. (5) have the following representation

$$u(\xi) = \aleph(\theta(\xi)) = \sum_{i=0}^{\eta} A_i \left(p+\theta(\xi)\right)^i + \sum_{i=1}^{\eta} B_i \left(p+\theta(\xi)\right)^{-i}, \tag{6}$$

where, $A_i(0 \le i \le \eta)$ are constants to be determined, such that $A_\eta \neq 0$, $B_\eta \neq 0$ and $\theta(\xi) = G'(\xi)/G(\xi)$ is the solution of the following second order differential equation:

$$k_1 G G'' - k_2 G G' - k_3 (G')^2 - k_4 G^2 = 0. \tag{7}$$

If we try to find the solution of the (7), then we obtain special solutions that vary according to the state of the coefficients:
Family 1: **When** $k_2 \neq 0$**,** $r = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) > 0$**,** then $\theta(\xi) = \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}\frac{C_1 sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2 cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}{C_1 cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2 sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}$.
Family 2: **When** $k_2 \neq 0$**,** $r = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) < 0$**,** then $\theta(\xi) = \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}\frac{-C_1 sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2 cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}{C_1 cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2 sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}$.
Family 3: **When** $k_2 \neq 0$**,** $r = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) = 0$**,** then $\theta(\xi) = \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}$.
Family 4: **When** $k_2 = 0$**,** $r = k_1 - k_3$ **and** $q = rk_4 > 0$**,** then $\theta(\xi) = \frac{\sqrt{q}}{r}\frac{C_1 sinh\left(\frac{\sqrt{q}}{k_1}\xi\right)+C_2 cosh\left(\frac{\sqrt{q}}{k_1}\xi\right)}{C_1 cosh\left(\frac{\sqrt{q}}{k_1}\xi\right)+C_2 sinh\left(\frac{\sqrt{q}}{k_1}\xi\right)}$.
Family 5: **When** $k_2 = 0$**,** $r = k_1 - k_3$ **and** $q = rk_4 < 0$**,** then $\theta(\xi) = \frac{\sqrt{-q}}{r}\frac{-C_1 sin\left(\frac{\sqrt{-q}}{k_1}\xi\right)+C_2 cos\left(\frac{\sqrt{-q}}{k_1}\xi\right)}{C_1 cos\left(\frac{\sqrt{-q}}{k_1}\xi\right)+C_2 sin\left(\frac{\sqrt{-q}}{k_1}\xi\right)}$.
Family 6: **When** $k_4 = 0$ **and** $r = k_1 - k_3$, then $\theta(\xi) = \frac{C_1 k_2^2 \exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1 + C_1 k_1 k_2 \exp\left(\frac{-k_2}{k_1}\xi\right)}$.
Family 7: **When** $k_2 \neq 0$ **and** $r = k_1 - k_3 = 0$ then $\theta(\xi) = -\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right)$, where C_1, C_2 are the integration constants and $A_j(0 \le j \le \eta), B_j(1 \le j \le \eta)$, k_1, k_2, k_3 and k_4 are also the constants to be explored later. As usual, for determining η, the highest order derivative should be balanced with the highest order nonlinear terms in Eq. (6). However, the positive integer η can be determined in this way.
Step 3. Following these operations, according to the m value obtained above, let substitute (7) into Eq. (6). Therefore we obtain a set of algebraic equations that contains $\theta^s(\xi), (s = 0, 1, 2, ...)$. Then setting each coefficients of $\theta^s(\xi)$ to zero, we can get a set of over-determined equations for $A_0, A_1, B_1, ..., A_\eta, B_\eta, k_1, k_2, k_3$, and k_4. Since obtained algebraic equations system will be difficult to solve manually, symbolic computation as Maple can be used at this stage. Assume the estimation of the constants can be gotten by fathoming the mathematical conditions got in step 2. Substituting the estimations of the constants together with the arrangements of Eq. (7), we will acquire new and far reaching precise traveling wave arrangements of the nonlinear development Eq. (3).

3 Method Test

To continue, we test the method for two nonlinear fractional PDEs.

3.1 The FWBK equations

By utilizing the following transformation

$$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} - \frac{ct^\alpha}{\Gamma(\alpha+1)}, \tag{8}$$

then, Eq. (1) are transformed to

$$-cu+\frac{k}{2}u^2+kv+\beta k^2u'=0, \tag{9}$$
$$-cv+k(uv)-\beta k^2v'+\gamma k^3u''=0,$$

where Eq. (9) received by integrating respect to ξ and also the amount of integration is considered zero. The balance number will be obtained $\eta=1$ by using the balance principle between u' and u^2. Similarly, by considering the homogeneous balance between u'' and uv in Eq. (9) we obtain

$$\eta+2=\eta+\vartheta, \qquad \Rightarrow \vartheta=2. \tag{10}$$

Then, the exact solutions are given as

$$u=A_0+A_1\left(p+\theta\right)+\frac{B_1}{p+\theta},\ v(\xi)=E_0+E_1\left(p+\theta\right)+E_2\left(p+\theta\right)^2+\frac{D_1}{p+\theta}+\frac{D_2}{\left(p+\theta\right)^2}. \tag{11}$$

Substituting (11) into Eq. (9) and by utilizing the Maple 18, the below results will be reached as:

Option I:

$$k=\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}},\quad c=\frac{A_1^2k_1\sqrt{k_2^2-k_3k_4}}{4k_3^2\sqrt{\beta^2+\gamma}},\quad A_1=A_1,\quad B_1=0,\quad D_1=0,\quad D_2=0, \tag{12}$$

$$p=0,\quad z=1+\frac{\beta}{\sqrt{\beta^2+\gamma}},\quad E_0=-\frac{k_4A_1^2s}{2k_3},\quad E_1=-\frac{k_2A_1^2s}{2k_3},\quad E_2=-\frac{A_1^2s}{2}, \tag{13}$$

$$A_0=\frac{A_1}{2k_3}\left(k_2+\sqrt{k_2^2-4k_3k_4}\right),\qquad u(\xi)=A_0+A_1\Phi(\xi),\qquad v(\xi)=E_0+E_1\Phi(\xi)+E_2\Phi^2(\xi),$$

where k_1,k_2,k_3,γ are free amounts. According to **Family 1** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, will be get

$$u_{1_1}(\xi)=\frac{A_1}{2k_3}\left(k_2+\sqrt{k_2^2-4k_3k_4}\right)+A_1\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right], \tag{14}$$

$$v_{1_1}(\xi)=-\frac{k_4A_1^2z}{2k_3}-\frac{k_2A_1^2z}{2k_3}\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]-\frac{A_1^2z}{2}\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2,$$

$$u_{1_2}(\xi)=\frac{A_1}{2k_3}\left(k_2+\sqrt{k_2^2-4k_3k_4}\right)+A_1\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right], \tag{15}$$

$$v_{1_2}(\xi)=-\frac{k_4A_1^2z}{2k_3}-\frac{k_2A_1^2z}{2k_3}\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]-\frac{A_1^2z}{2}\left[\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2.$$

According to **Family 2** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, one get

$$u_{1_3}(\xi)=\frac{A_1}{2k_3}\left(k_2+\sqrt{k_2^2-4k_3k_4}\right)+A_1\left[\frac{k_2}{2r}+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right], \tag{16}$$

$$v_{1_3}(\xi)=-\frac{k_4A_1^2z}{2k_3}-\frac{k_2A_1^2z}{2k_3}\left[\frac{k_2}{2r}+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]-\frac{A_1^2z}{2}\left[\frac{k_2}{2r}+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2,$$

$$u_{1_4}(\xi)=\frac{A_1}{2k_3}\left(k_2+\sqrt{k_2^2-4k_3k_4}\right)+A_1\left[\frac{k_2}{2r}-\frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right], \tag{17}$$

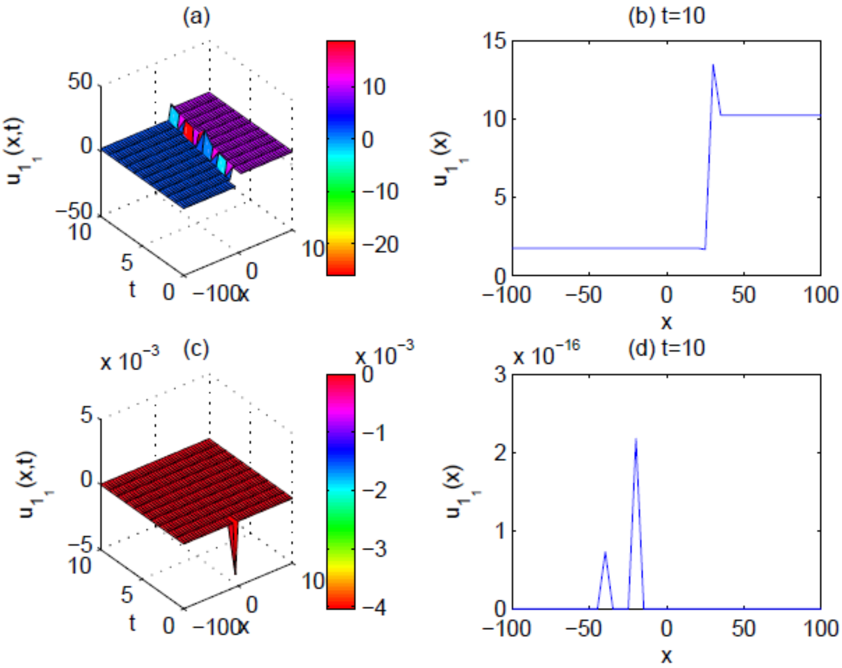

Figure 1: Plots of (a) and (b) real amounts and (c) and (d) imaginary amounts of Eq. (14) with providing amounts $A_1 = 3, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

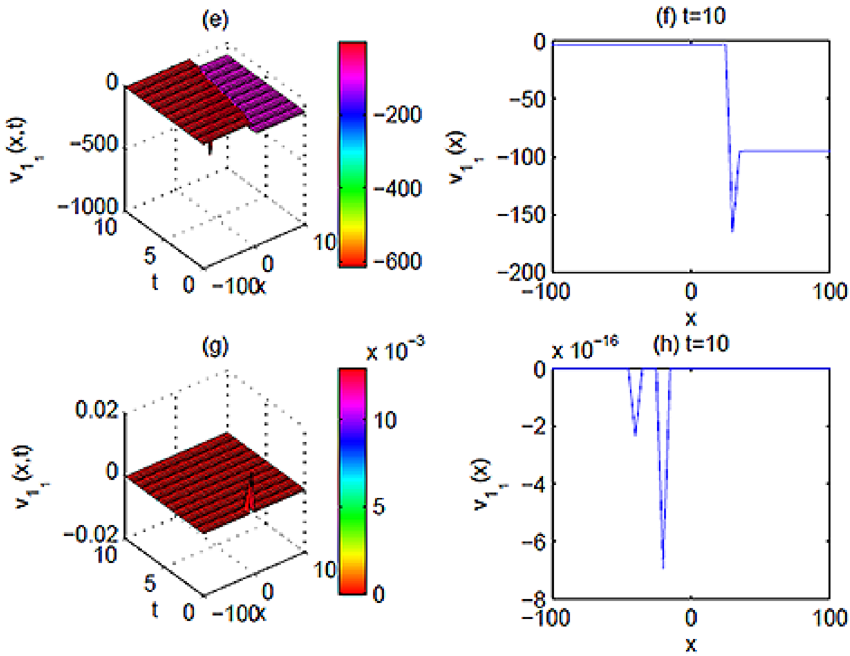

Figure 2: Plots of (e) and (f) real amounts and (g) and (h) imaginary amounts of Eq. (14) with providing amounts $A_1 = 3, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

$$v_{1_4}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} - \frac{k_2 A_1^2 z}{2k_3}\left[\frac{k_2}{2r} - \frac{\sqrt{-s}}{2r} tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right] - \frac{A_1^2 z}{2}\left[\frac{k_2}{2r} - \frac{\sqrt{-s}}{2r} tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2.$$

Based on the **Family 3**, become

$$u_{1_5}(\xi) = \frac{A_1}{2k_3}\left(k_2 + \sqrt{k_2^2 - 4k_3k_4}\right) + A_1\left[\frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right], \tag{18}$$

$$v_{1_5}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} - \frac{k_2 A_1^2 z}{2k_3}\left[\frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right] - \frac{A_1^2 z}{2}\left[\frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right]^2.$$

According to **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, one become

$$u_{1_6}(\xi) = \frac{A_1\sqrt{-k_3k_4}}{k_3} + \frac{A_1\sqrt{q}}{r} coth\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad v_{1_6}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} - \frac{A_1^2 q z}{2r^2} coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \tag{19}$$

$$u_{1_7}(\xi) = \frac{A_1\sqrt{-4k_3k_4}}{2k_3} + \frac{A_1\sqrt{q}}{r} tanh\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad v_{1_7}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} - \frac{A_1^2 q z}{2r^2} tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right). \tag{20}$$

Based on the **Family 5** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, will be reached as

$$u_{1_8}(\xi) = \frac{A_1\sqrt{-k_3k_4}}{k_3} + \frac{A_1\sqrt{-q}}{r} cot\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad v_{1_8}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} + \frac{A_1^2 q z}{2r^2} cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \tag{21}$$

$$u_{1_9}(\xi) = \frac{A_1\sqrt{-k_3k_4}}{k_3} - \frac{A_1\sqrt{-q}}{r} tan\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad v_{1_9}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} + \frac{A_1^2 q z}{2r^2} tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right). \tag{22}$$

By using of the (13) and **Family 6** we get

$$\begin{aligned} u_{1_{10}}(\xi) &= \frac{A_1 k_2}{k_3} + \frac{A_1 C_1 k_2^2 e^{\frac{-k_2}{k_1}\xi}}{r k_1 + C_1 k_1 k_2 e^{\frac{-k_2}{k_1}\xi}}, \\ v_{1_{10}}(\xi) &= -\frac{C_1 \frac{k_2^3 A_1^2 z}{2k_3} e^{\frac{-k_2}{k_1}\xi}}{r k_1 + C_1 k_1 k_2 e^{\frac{-k_2}{k_1}\xi}} - \frac{A_1^2 z}{2}\left[\frac{C_1 k_2^2 e^{\frac{-k_2}{k_1}\xi}}{r k_1 + C_1 k_1 k_2 e^{\frac{-k_2}{k_1}\xi}}\right]^2. \end{aligned} \tag{23}$$

According to **Family 7**, one get

$$u_{1_{11}}(\xi) = \frac{A_1}{2k_3}\left(k_2 + \sqrt{k_2^2 - 4k_3k_4}\right) + A_1\left[-\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right)\right], \tag{24}$$

$$v_{1_{11}}(\xi) = -\frac{k_4 A_1^2 z}{2k_3} - \frac{k_2 A_1^2 z}{2k_3}\left[-\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right)\right] - \frac{A_1^2 z}{2}\left[-\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right)\right]^2,$$

where $z = 1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}$ and $\xi = \frac{1}{\Gamma(\alpha+1)}\left(\frac{A_1 k_1}{2k_3\sqrt{\beta^2+\gamma}} x^\alpha - \frac{A_1^2 k_1 \sqrt{k_2^2 - k_3 k_4}}{4k_3^2\sqrt{\beta^2+\gamma}} t^\alpha\right)$.

Option II:

$$\begin{aligned} k &= \frac{A_1 k_1}{2k_3\sqrt{\beta^2+\gamma}}, \quad c = \frac{A_1^2 k_1 \sqrt{k_3 k_4 - k_2^2}}{2\sqrt{2} k_3^2 \sqrt{\beta^2+\gamma}}, \quad A_1 = A_1, \\ B_1 &= \frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2), \quad D_1 = 0, \quad p = \frac{k_2}{2k_3}, \end{aligned} \tag{25}$$

$$\begin{aligned} E_0 &= 0, \quad D_2 = \frac{A_1^2(k_2^4 + 16k_3^2k_4^2 - 8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right), \\ E_1 &= 0, \quad E_2 = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right), \end{aligned} \tag{26}$$

$$A_0 = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2}, \qquad u(\xi) = A_0 + A_1(p+\theta(\xi)) + \frac{B_1}{p+\theta(\xi)},$$
$$v(\xi) = E_2(p+\theta(\xi))^2 + \frac{D_2}{(p+\theta(\xi))^2},$$

where k_1, k_2, k_3, γ are free amounts. According to **Family 1** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, will be reached as

$$u_{2_1}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\Omega_1 + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\Omega_1 + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]}, \tag{27}$$

$$v_{2_1}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\Omega_1 + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 +$$
$$+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\Omega_1 + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2},$$

$$u_{2_2}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\Omega_1 + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\Omega_1 + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]}, \tag{28}$$

$$v_{2_2}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\Omega_1 + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 +$$
$$+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\Omega_1 + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2}.$$

where $\Omega_1 = \frac{k_2}{2k_3} + \frac{k_2}{2r}$. Via **Family 2** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, one get

$$u_{2_3}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\Omega_1 + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\Omega_1 + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]}, \tag{29}$$

$$v_{2_3}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\Omega_1 + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 +$$
$$+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\Omega_1 + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2},$$

$$u_{2_4}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]}, \tag{30}$$

$$v_{2_4}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 +$$
$$\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2}.$$

where $\Omega_1 = \frac{k_2}{2k_3} + \frac{k_2}{2r}$. According to **Family 3**, one get

$$u_{2_5}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right]}, \quad (31)$$

$$v_{2_5}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right]^2 + \frac{\frac{A_1^2(k_2^2-4k_3k_4)^2}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right]^2}.$$

Based on the **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, one become

$$u_{2_6}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4} + \frac{A_1\sqrt{q}}{r}\coth\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1 r}{4k_3^2\sqrt{q}}(4k_3k_4 - k_2^2)\tanh\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad (32)$$

$$v_{2_6}(\xi) = -\frac{A_1^2 q}{2r^2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1^2r^2k_4^2}{2qk_3^2}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)\tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$u_{2_7}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4} + \frac{A_1\sqrt{q}}{r}\tanh\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1 r}{4k_3^2\sqrt{q}}(4k_3k_4 - k_2^2)\coth\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad (33)$$

$$v_{2_7}(\xi) = -\frac{A_1^2 q}{2r^2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1^2r^2k_4^2}{2qk_3^2}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)\coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right).$$

According to **Family 5** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, one get

$$u_{2_8}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4} + \frac{A_1\sqrt{-q}}{r}\cot\left(\frac{\sqrt{-q}}{k_1}\xi\right) + \frac{A_1 r}{4k_3^2\sqrt{-q}}(4k_3k_4 - k_2^2)\tan\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad (34)$$

$$v_{2_8}(\xi) = \frac{A_1^2 q}{2r^2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1^2r^2k_4^2}{2qk_3^2}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)\tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$u_{2_9}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4} - \frac{A_1\sqrt{-q}}{r}\tan\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1 r}{4k_3^2\sqrt{-q}}(4k_3k_4 - k_2^2)\cot\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad (35)$$

$$v_{2_9}(\xi) = \frac{A_1^2 q}{2r^2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1^2r^2k_4^2}{2qk_3^2}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)\cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right).$$

Based on the **Family 6**, on become

$$u_{2_{10}}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{-k_2^2} + A_1\left[\frac{k_2}{2k_3} + \Xi_1\right] - \frac{\frac{A_1k_2^2}{4k_3^2}}{\left[\frac{k_2}{2k_3} + \Xi_1\right]}, \quad (36)$$

$$v_{2_{10}}(\xi) = -\frac{A_1^2}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\frac{k_2}{2k_3} + \Xi_1\right]^2 + + \frac{\frac{A_1^2k_2^4}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}} - 1\right)}{\left[\frac{k_2}{2k_3} + \Xi_1\right]^2},$$

where $\Xi_1 = \frac{A_1C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1 + C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}$. Via **Family 7** the below result will be reached as

$$u_{2_{11}}(\xi) = \frac{A_1}{\sqrt{2}k_3}\sqrt{4k_3k_4 - k_2^2} + A_1\left[\frac{k_2}{2k_3} - \frac{k_4}{k_2} + C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right] + \frac{\frac{A_1}{4k_3^2}(4k_3k_4 - k_2^2)}{\left[\frac{k_2}{2k_3} - \frac{k_4}{k_2} + C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]}, \quad (37)$$

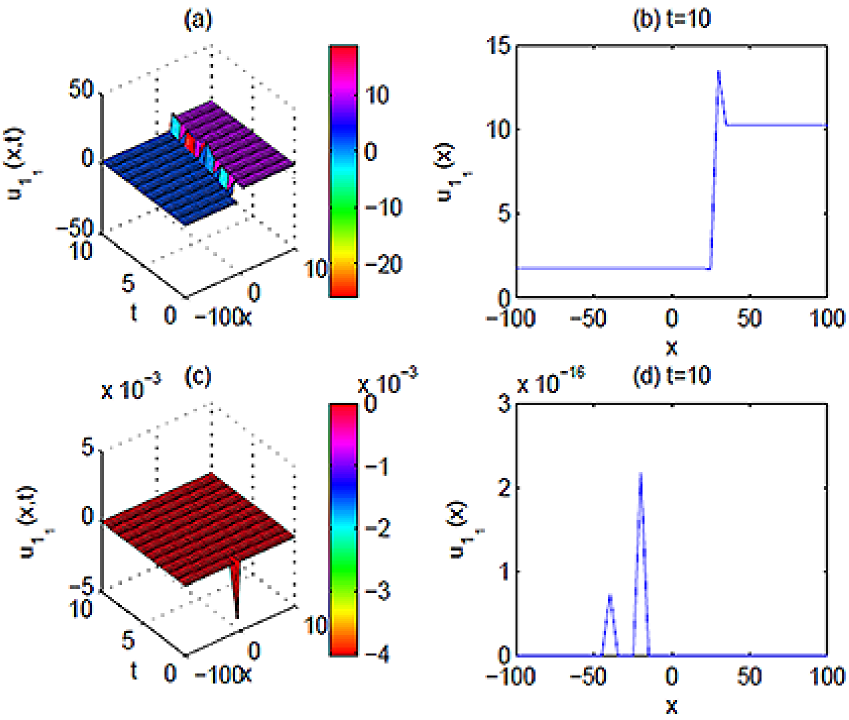

Figure 3: Plots of (i) and (j) real amounts and (k) and (l) imaginary amounts of Eq. (15) with providing amounts $A_1 = 3, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

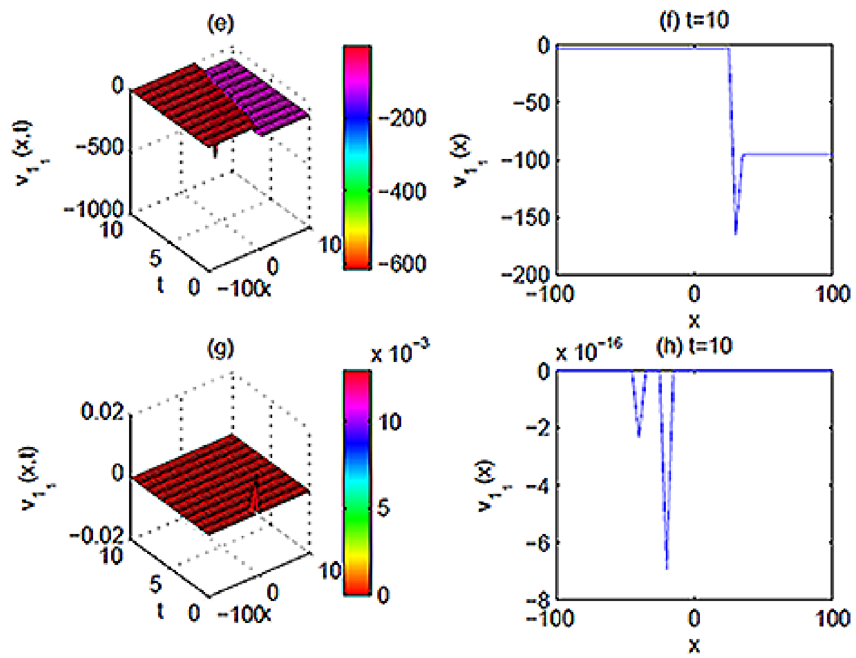

Figure 4: Plots of (m) and (n) real amounts and (o) and (p) imaginary amounts of Eq. (15) with providing amounts $A_1 = 3, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

$$v_{2_{11}}(\xi)=-\frac{A_1^2}{2}\left(1+\frac{\beta}{\sqrt{\beta^2+\gamma}}\right)\left[\frac{k_2}{2k_3}-\frac{k_4}{k_2}+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^2+$$
$$+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}\left(\frac{\beta}{\sqrt{\beta^2+\gamma}}-1\right)}{\left[\frac{k_2}{2k_3}-\frac{k_4}{k_2}+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^2},$$

where $\xi=\frac{1}{\Gamma(\alpha+1)}\left(\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}}x^\alpha-\frac{A_1^2k_1\sqrt{k_3k_4-k_2^2}}{2\sqrt{2}k_3^2\sqrt{\beta^2+\gamma}}t^\alpha\right)$.

Option III:

$$\begin{array}{l} k=\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}},\quad c=\frac{A_1^2k_1\sqrt{k_2^2-k_3k_4}}{2k_3^2\sqrt{\beta^2+\gamma}},\quad A_1=A_1,\\ B_1=-\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2),\quad D_1=0,\quad p=\frac{k_2}{2k_3}, \end{array}\tag{38}$$

$$z=1+\frac{\beta}{\sqrt{\beta^2+\gamma}},\quad E_0=-\frac{A_1(4k_3k_4-k_2^2)z}{4k_3^2},\quad D_2=\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)z}{32k_3^4},\quad E_1=0,$$
$$A_0=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2},\quad E_2=-\frac{A_1^2z}{2},\tag{39}$$

$$u(\xi)=A_0+A_1(p+\theta(\xi))+\frac{B_1}{p+\theta(\xi)},\quad v(\xi)=E_0+E_2(p+\theta(\xi))^2+\frac{D_2}{(p+\theta(\xi))^2},$$

where k_1,k_2,k_3,γ are free amounts. According to **Family 1** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, the solutions will be reached as below form

$$u_{3_1}(\xi)=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2}+A_1\left[\Omega_1+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]-\frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\Omega_1+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]},\tag{40}$$

$$v_{3_1}(\xi)=z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2}-\frac{A_1^2}{2}\left[\Omega_1+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\Omega_1+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2}\right],$$

$$u_{3_2}(\xi)=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2}+A_1\left[\Omega_1+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]-\frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\Omega_1+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]},\tag{41}$$

$$v_{3_2}(\xi)=z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2}-\frac{A_1^2}{2}\left[\Omega_1+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\Omega_1+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2}\right],$$

where $\Omega_1=\frac{k_2}{2k_3}+\frac{k_2}{2r}$. Via **Family 2** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, one get

$$u_{3_3}(\xi)=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2}+A_1\left[\Omega_1+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]-\frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\Omega_1+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]},\tag{42}$$

$$v_{3_3}(\xi)=z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2}-\frac{A_1^2}{2}\left[\Omega_1+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\Omega_1+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2}\right],$$

$$u_{3_4}(\xi)=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2}+A_1\left[\Omega_1-\frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]-\frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\Omega_1-\frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]},\tag{43}$$

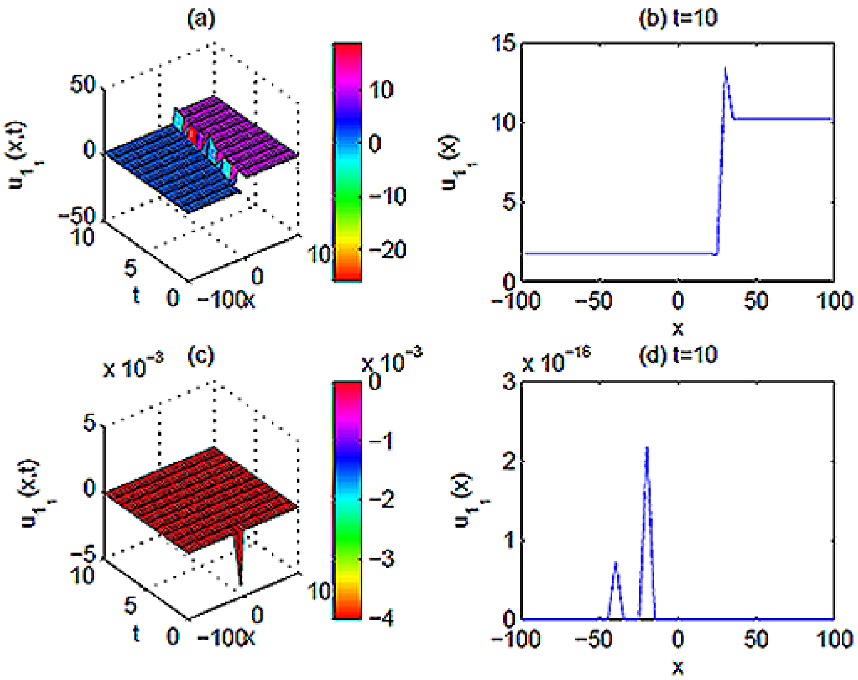

Figure 5: Plots of (q) and (r) real amounts and (s) and (t) imaginary amounts of Eq. (16) with providing amounts $A_1 = 3, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \beta = 2, \gamma = 2, \alpha = 0.9$.

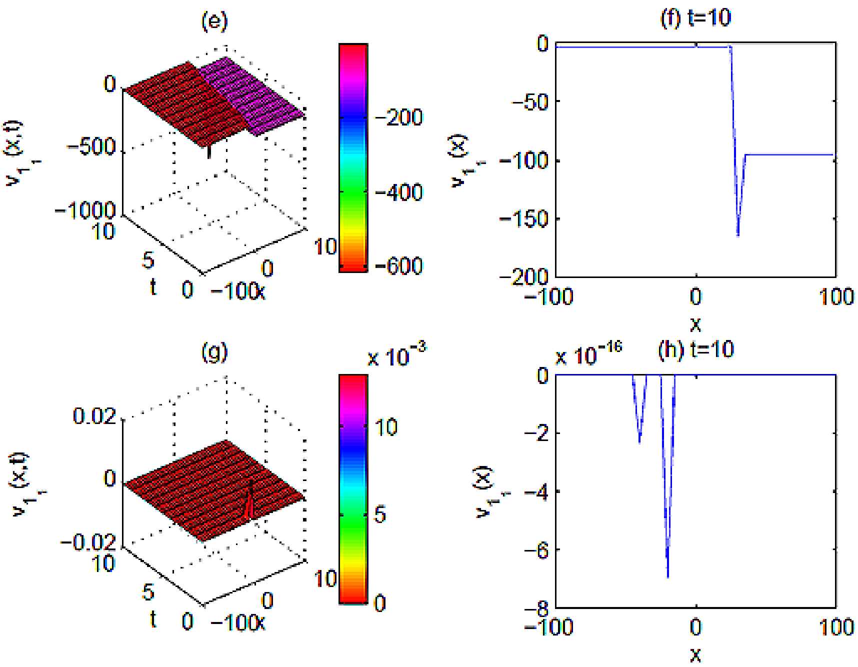

Figure 6: Plots of (u) and (v) real amounts and (w) and (x) imaginary amounts of Eq. (16) with providing amounts $A_1 = 3, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \beta = 2, \gamma = 2, \alpha = 0.9$.

$$v_{3_4}(\xi) = z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2} - \frac{A_1^2}{2}\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 + \frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\Omega_1 - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2}\right],$$

where $\Omega_1 = \frac{k_2}{2k_3} + \frac{k_2}{2r}$. According to **Family 3**, one become

$$u_{3_5}(\xi) = \frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2} + A_1\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right] - \frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right]}, \tag{44}$$

$$v_{3_5}(\xi) = z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2} - \frac{A_1^2}{2}\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right]^2 + \frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\frac{k_2}{2k_3} + \frac{k_2}{2r} + \frac{C_2}{C_1+C_2\xi}\right]^2}\right].$$

Based on the **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be got as below form

$$u_{3_6}(\xi) = \frac{2A_1}{k_3}\sqrt{k_3k_4} + \frac{A_1\sqrt{q}}{r}\coth\left(\frac{\sqrt{q}}{k_1}\xi\right) - \frac{A_1 r}{4k_3^2\sqrt{q}}(4k_3k_4-k_2^2)\tanh\left(\frac{\sqrt{q}}{k_1}\xi\right), \tag{45}$$

$$v_{3_6}(\xi) = z\left[-\frac{A_1k_4}{k_3} - \frac{A_1^2q}{2r^2}\coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1^2k_4^2r^2}{2k_3^2q}\tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right)\right],$$

$$u_{3_7}(\xi) = \frac{A_1}{k_3}\sqrt{4k_3k_4} + \frac{A_1\sqrt{q}}{r}\tanh\left(\frac{\sqrt{q}}{k_1}\xi\right) - \frac{A_1 r}{4k_3^2\sqrt{q}}(4k_3k_4-k_2^2)\coth\left(\frac{\sqrt{q}}{k_1}\xi\right), \tag{46}$$

$$v_{3_7}(\xi) = z\left[-\frac{A_1k_4}{k_3} - \frac{A_1^2q}{2r^2}\tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{A_1^2k_4^2r^2}{2k_3^2q}\coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right)\right].$$

According to **Family 5** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions become

$$u_{3_8}(\xi) = \frac{A_1}{k_3}\sqrt{4k_3k_4} + \frac{A_1\sqrt{-q}}{r}\cot\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1 r}{4k_3^2\sqrt{-q}}(4k_3k_4-k_2^2)\tan\left(\frac{\sqrt{-q}}{k_1}\xi\right), \tag{47}$$

$$v_{3_8}(\xi) = z\left[-\frac{A_1k_4}{k_3} + \frac{A_1^2q}{2r^2}\cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1^2k_4^2r^2}{2k_3^2q}\tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)\right].$$

$$u_{3_9}(\xi) = \frac{A_1}{k_3}\sqrt{4k_3k_4} - \frac{A_1\sqrt{-q}}{r}\tan\left(\frac{\sqrt{-q}}{k_1}\xi\right) + \frac{A_1 r}{4k_3^2\sqrt{-q}}(4k_3k_4-k_2^2)\cot\left(\frac{\sqrt{-q}}{k_1}\xi\right), \tag{48}$$

$$v_{3_9}(\xi) = z\left[-\frac{A_1k_4}{k_3} + \frac{A_1^2q}{2r^2}\tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{A_1^2k_4^2r^2}{2k_3^2q}\cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)\right].$$

Via **Family 6**, the solutions become as

$$u_{3_{10}}(\xi) = \frac{A_1\sqrt{-k_2^2}}{k_3} + A_1\left[\frac{k_2}{2k_3} + \frac{A_1C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1 + C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right] + \frac{\frac{A_1k_2^2}{4k_3^2}}{\left[\frac{k_2}{2k_3} + \frac{A_1C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1+C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right]}, \tag{49}$$

$$v_{3_{10}}(\xi) = z\left[\frac{A_1k_2^2}{4k_3^2} - \frac{A_1^2}{2}\left[\frac{k_2}{2k_3} + \frac{A_1C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1 + C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right]^2 + \frac{\frac{A_1^2k_2^4}{32k_3^4}}{\left[\frac{k_2}{2k_3} + \frac{A_1C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{rk_1+C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right]^2}\right].$$

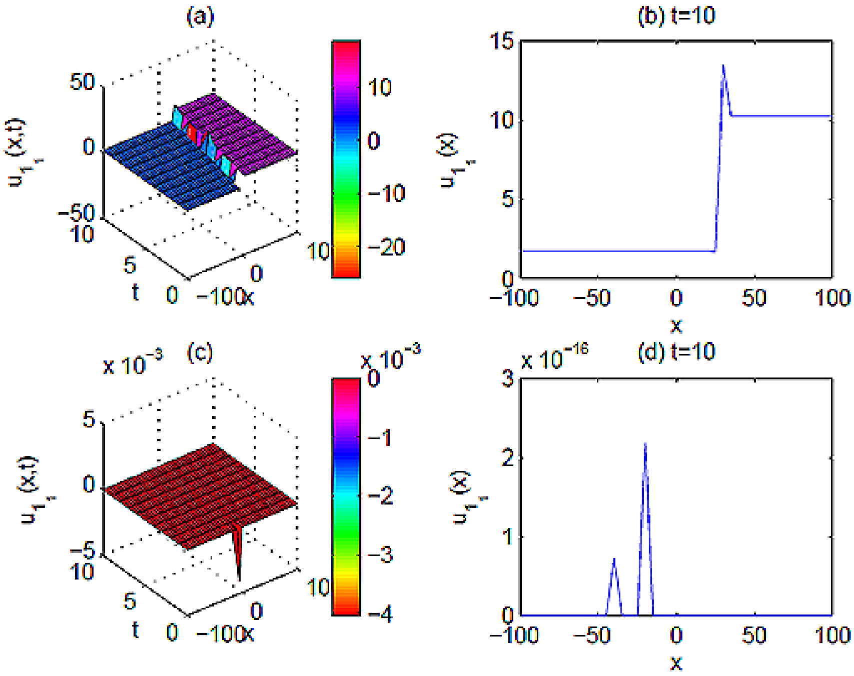

Figure 7: Plots of (a1) and (b1) real amounts and (c1) and (d1) imaginary amounts of Eq. (17) with providing amounts $A_1 = 3, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \beta = 2, \gamma = 2, \alpha = 0.9$.

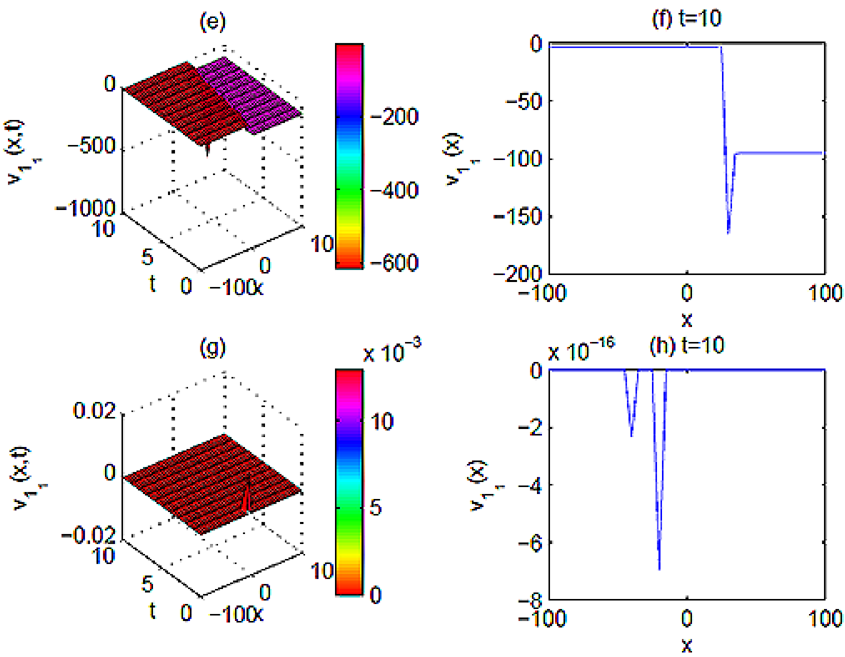

Figure 8: Plots of (e1) and (f1) real amounts and (g1) and (h1) imaginary amounts of Eq. (17) with providing amounts $A_1 = 3, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \beta = 2, \gamma = 2, \alpha = 0.9$.

Based on the **Family 7**, the solutions will be reached as below form

$$u_{3_{11}}(\xi)=\frac{A_1}{k_3}\sqrt{4k_3k_4-k_2^2}+A_1\left[\Omega_1+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]-\frac{\frac{A_1}{4k_3^2}(4k_3k_4-k_2^2)}{\left[\Omega_1+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]},\quad (50)$$

$$v_{3_{11}}(\xi)=z\left[-\frac{A_1(4k_3k_4-k_2^2)}{4k_3^2}-\frac{A_1^2}{2}\left[\Omega_1+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^2+\frac{\frac{A_1^2(k_2^4+16k_3^2k_4^2-8k_3k_2^2k_4)}{32k_3^4}}{\left[\Omega_1+C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^2}\right],$$

where $\Omega_1=\frac{k_2}{2k_3}-\frac{k_4}{k_2}$ and $z=1+\frac{\beta}{\sqrt{\beta^2+\gamma}}$ and $\xi=\frac{1}{\Gamma(\alpha+1)}\left(\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}}x^\alpha-\frac{A_1^2k_1\sqrt{k_2^2-k_3k_4}}{2k_3^2\sqrt{\beta^2+\gamma}}t^\alpha\right)$.

Option IV:

$$k=\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}},\quad c=\frac{A_1^2k_1\sqrt{k_2^2-k_3k_4}}{4k_3^2\sqrt{\beta^2+\gamma}},\quad A_1=A_1,\quad B_1=0,\quad D_1=D_2=0,\quad p=\frac{k_2}{2k_3},\quad (51)$$

$$z=1+\frac{\beta}{\sqrt{\beta^2+\gamma}},\quad E_0=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2},\quad E_1=0,\quad E_2=-\frac{A_1^2z}{2},\quad (52)$$

$$A_0=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4},\quad u(\xi)=A_0+A_1(p+\theta(\xi)),\quad v(\xi)=E_0+E_2(p+\theta(\xi))^2,$$

where k_1,k_2,k_3,γ are free amounts. According to **Family 1** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, the solutions will be received as below form

$$u_{4_1}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right],\quad (53)$$

$$v_{4_1}(\xi)=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2}-\frac{A_1^2z}{2}\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2,$$

$$u_{4_2}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right],\quad (54)$$

$$v_{4_2}(\xi)=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2}-\frac{A_1^2z}{2}\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2.$$

Via **Family 2** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, the solutions become

$$u_{4_3}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right],\quad (55)$$

$$v_{4_3}(\xi)=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2}-\frac{A_1^2z}{2}\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2,$$

$$u_{4_4}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}-\frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right],\quad (56)$$

$$v_{4_4}(\xi)=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2}-\frac{A_1^2z}{2}\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}-\frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2.$$

According to **Family 3**, one get

$$u_{4_5}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{C_2}{C_1+C_2\xi}\right],\quad (57)$$

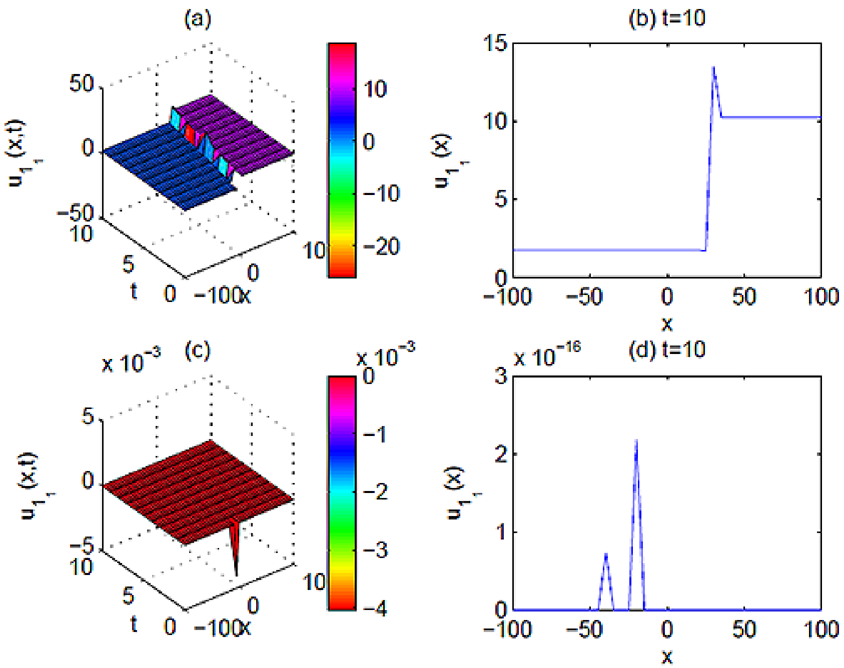

Figure 9: Plots of (i1) and (j1) real amounts and (k1) and (l1) imaginary amounts of Eq. (18) with providing amounts $A_1 = 3, C_1 = 1, C_2 = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

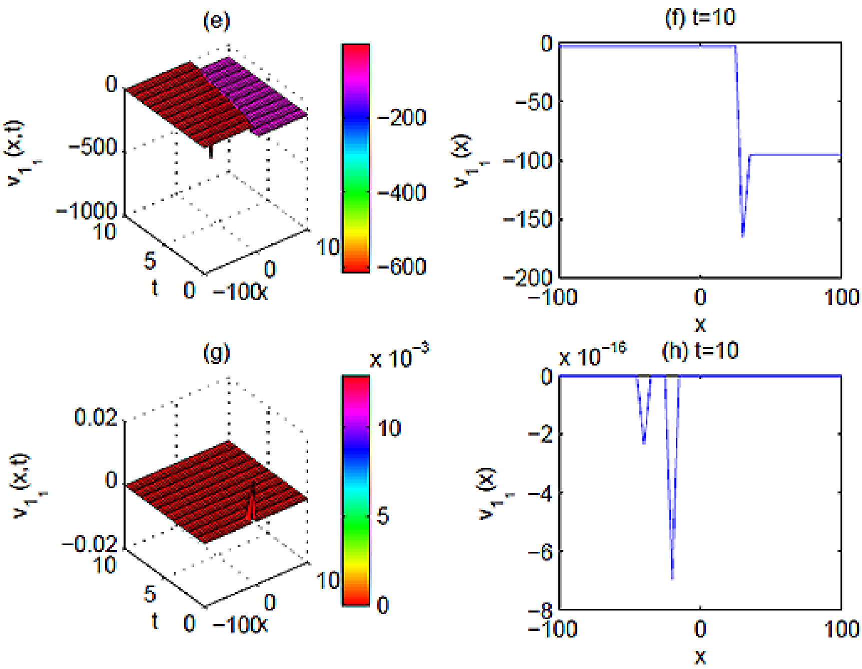

Figure 10: Plots of (m1) and (n1) real amounts and (o1) and (p1) imaginary amounts of Eq. (18) with providing amounts $A_1 = 3, C_1 = 1, C_2 = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 1, \beta = 2, \gamma = 2, \alpha = 0.9$.

$$v_{4_5}(\xi)=\frac{A_1^2(k_2^2-4k_3k_4)z}{8k_3^2}-\frac{A_1^2z}{2}\left[\frac{k_2}{2k_3}+\frac{k_2}{2r}+\frac{C_2}{C_1+C_2\xi}\right]^2.$$

Based on the **Family 4** (if $C_1=0$ but $C_2\neq 0$; $C_1\neq 0$ but $C_2=0$), respectively, the solutions will be reached as below form

$$u_{4_6}(\xi)=\frac{A_1\sqrt{-k_3k_4}}{k_3}+\frac{A_1\sqrt{q}}{r}coth\left(\frac{\sqrt{q}}{k_1}\xi\right),\quad v_{4_6}(\xi)=-\frac{k_4A_1^2z}{2k_3}-\frac{A_1^2qz}{2r^2}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right),\quad (58)$$

$$u_{4_7}(\xi)=\frac{A_1\sqrt{-k_3k_4}}{k_3}+\frac{A_1\sqrt{q}}{r}tanh\left(\frac{\sqrt{q}}{k_1}\xi\right),\quad v_{4_7}(\xi)=-\frac{k_4A_1^2z}{2k_3}-\frac{A_1^2qz}{2r^2}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right).\quad (59)$$

Based on the **Family 5**, the solutions will be received as below form

$$u_{4_8}(\xi)=\frac{A_1\sqrt{-k_3k_4}}{k_3}+\frac{A_1\sqrt{-q}}{r}cot\left(\frac{\sqrt{-q}}{k_1}\xi\right),\quad v_{4_8}(\xi)=-\frac{k_4A_1^2z}{2k_3}+\frac{A_1^2qz}{2r^2}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),\quad (60)$$

$$u_{4_9}(\xi)=\frac{A_1\sqrt{-k_3k_4}}{k_3}-\frac{A_1\sqrt{-q}}{r}tan\left(\frac{\sqrt{-q}}{k_1}\xi\right),\quad v_{4_9}(\xi)=-\frac{k_4A_1^2z}{2k_3}+\frac{A_1^2qz}{2r^2}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right).\quad (61)$$

Via **Family 6** we get

$$u_{4_{10}}(\xi)=\frac{A_1k_2}{2k_3}+A_1\left[\frac{k_2}{2k_3}+\frac{C_1k_2^2e^{\frac{-k_2}{k_1}\xi}}{rk_1+C_1k_1k_2e^{\frac{-k_2}{k_1}\xi}}\right],\quad (62)$$

$$v_{4_{10}}(\xi)=z\left\{\frac{k_2^2A_1^2}{8k_3^2}-\frac{A_1^2}{2}\left[\frac{k_2}{2k_3}+\frac{C_1k_2^2e^{\frac{-k_2}{k_1}\xi}}{rk_1+C_1k_1k_2e^{\frac{-k_2}{k_1}\xi}}\right]^2\right\}.$$

According to **Family 7** we have the below result

$$u_{4_{11}}(\xi)=\frac{A_1}{2k_3}\sqrt{k_2^2-4k_3k_4}+A_1\left[\frac{k_2}{2k_3}-\frac{k_4}{k_2}+C_1e^{\frac{k_2}{k_1}\xi}\right],\quad (63)$$

$$v_{4_{11}}(\xi)=z\left\{\frac{A_1^2(k_2^2-4k_3k_4)}{8k_3^2}-\frac{A_1^2}{2}\left[\frac{k_2}{2k_3}-\frac{k_4}{k_2}+C_1e^{\frac{k_2}{k_1}\xi}\right]^2\right\},$$

where $z=1+\frac{\beta}{\sqrt{\beta^2+\gamma}}$ and $\xi=\frac{1}{\Gamma(\alpha+1)}\left(\frac{A_1k_1}{2k_3\sqrt{\beta^2+\gamma}}x^\alpha-\frac{A_1^2k_1\sqrt{k_2^2-k_3k_4}}{4k_3^2\sqrt{\beta^2+\gamma}}t^\alpha\right)$.

3.2 The gHSCKdV equations

By utilizing the following transformation

$$\xi=\frac{kx^\alpha}{\Gamma(\alpha+1)}-\frac{ct^\alpha}{\Gamma(\alpha+1)},\quad (64)$$

then, Eq. (2) are transformed to

$$-cu'+\frac{k^3}{2}u'''+3kuu'-3k(vw)'=0,\quad -cv'+k^3v'''-3kuv'=0,\quad (65)$$
$$-cw'+k^3w'''-3kuw'=0.$$

The balance number between (u''' and $(vw)'$), (v''' and uv'), and also (w''' and uw') in Eq. (65), respectively, one become

$$\eta+3=\vartheta+\kappa+1,\qquad \vartheta+3=\eta+\vartheta+1,\quad \kappa+3=\theta+\kappa+1,\quad (66)$$

then, the below amounts will be reached as

$$\eta = \vartheta = \kappa = 2, \tag{67}$$

where

$$u = \sum_{i=0}^{\eta}(.), \qquad v = \sum_{i=0}^{\vartheta}(.), \qquad w = \sum_{i=0}^{\kappa}(.). \tag{68}$$

Then the exact solutions will be received as below forms

$$u(\xi) = A_0 + A_2\,(p+\theta(\xi))^2 + \frac{B_2}{(p+\theta(\xi))^2}, \quad v(\xi) = C_0 + C_2\,(p+\theta(\xi))^2 + \frac{D_2}{(p+\theta(\xi))^2}, \tag{69}$$

$$w(\xi) = E_0 + E_2\,(p+\theta(\xi))^2 + \frac{F_2}{(p+\theta(\xi))^2}.$$

Appending (69) into Eq. (65) and by utilizing the Maple 18, the below results will be concluded **Option I**:

$$k = k, \quad c = -3kA_0, \quad A_0 = A_0, \quad B_2 = \frac{4k^2k_4^2}{k_1^2}, \quad E_0 = E_0, \quad D_2 = D_2, \quad p = p, \quad k_2 = k_3 = 0, \tag{70}$$

$$C_0 = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right), \quad E_2 = 0, \quad F_2 = \frac{4k^4k_4^4}{D_2k_1^4}, \quad A_2 = 0, \quad C_2 = 0, \quad k_4 = k_4,$$

$$u(\xi) = A_0 + B_2(p+\theta(\xi))^{-2}, \quad v(\xi) = C_0 + D_2(p+\theta(\xi))^{-2}, \quad w(\xi) = E_0 + F_2(p+\theta(\xi))^{-2},$$

where k_4, k are free amounts. By utilizing **Family 1** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, will be reached as below form

$$u_{1_1}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_1\right]^{-2}, \quad w_{1_1}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi\right]^{-2}, \tag{71}$$

$$v_{1_1}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_1\right]^{-2}, \;\; \Pi_1 = p + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right),$$

$$u_{1_2}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_2\right]^{-2}, \quad w_{1_2}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_2\right]^{-2}, \tag{72}$$

$$v_{1_2}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_2\right]^{-2}, \;\; \Pi_2 = p + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right).$$

Via **Family 2** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be received as below forms

$$u_{1_3}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_3\right]^{-2}, \quad w_{1_3}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_3\right]^{-2}, \tag{73}$$

$$v_{1_3}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_3\right]^{-2}, \;\; \Pi_3 = p + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right),$$

$$u_{1_4}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_4\right]^{-2}, \quad w_{1_4}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_4\right]^{-2}, \tag{74}$$

$$v_{1_4}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_4\right]^{-2}, \;\; \Pi_4 = p - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right).$$

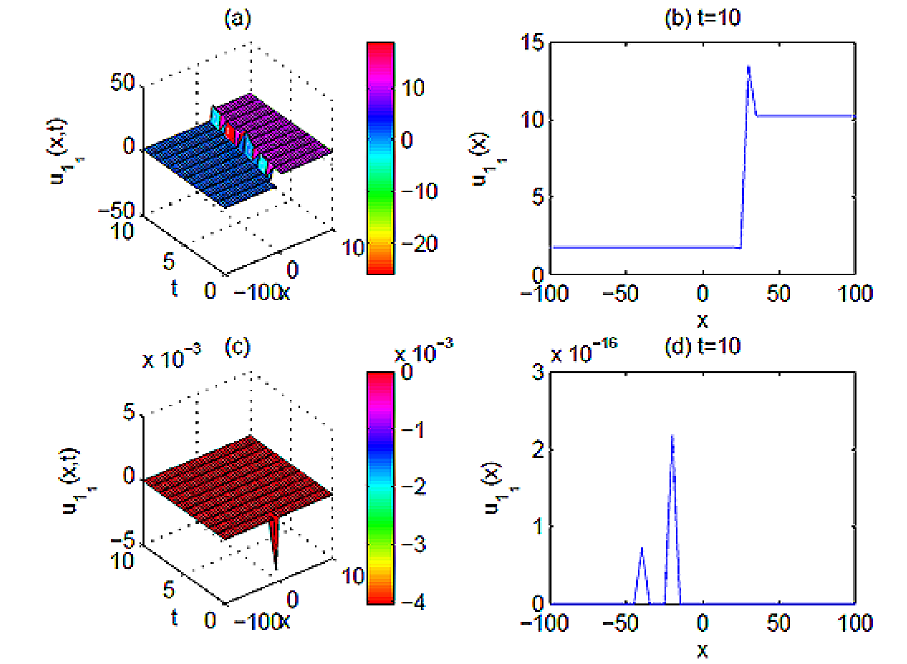

Figure 11: Plots of (a) and (b) real amounts and (c) and (d) imaginary amounts of Eq. (71) with providing amounts $p = 2, A_0 = 3, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

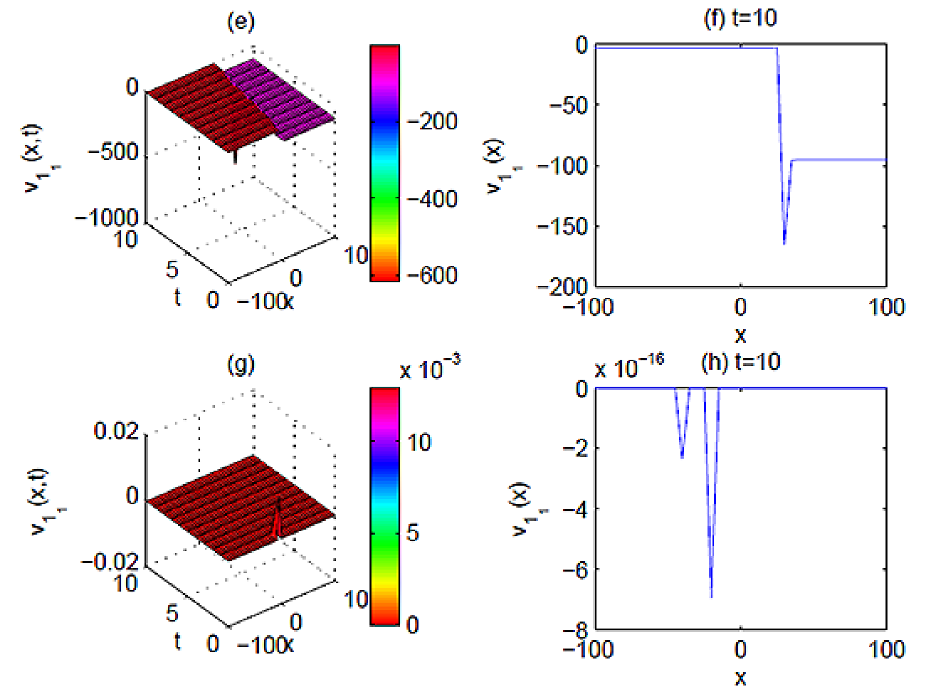

Figure 12: Plots of (e) and (f) real amounts and (g) and (h) imaginary amounts of Eq. (71) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

Based on the **Family 3** we get

$$u_{1_5}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}, \quad w_{1_5}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}, \tag{75}$$

$$v_{1_5}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}.$$

According to **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be reached as below forms

$$u_{1_6}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_5\right]^{-2}, \; w_{1_6}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_5\right]^{-2}, \tag{76}$$

$$v_{1_6}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_5\right]^{-2}, \;\; \Pi_5 = p + \frac{\sqrt{q}}{r}coth\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$u_{1_7}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_6\right]^{-2}, \quad w_{1_7}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_6\right]^{-2}, \tag{77}$$

$$v_{1_7}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_6\right]^{-2}, \;\; \Pi_6 = p + \frac{\sqrt{q}}{r}tanh\left(\frac{\sqrt{q}}{k_1}\xi\right).$$

Based on the **Family 5** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be received as below forms

$$u_{1_8}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_7\right]^{-2}, \quad w_{1_8}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_7\right]^{-2}, \tag{78}$$

$$v_{1_8}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_7\right]^{-2}, \;\; \Pi_7 = p + \frac{\sqrt{-q}}{r}cot\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$u_{1_9}(\xi) = A_0 + \frac{4k^2k_4^2}{k_1^2}\left[\Pi_8\right]^{-2}, \quad w_{1_9}(\xi) = E_0 + \frac{4k^4k_4^4}{D_2k_1^4}\left[\Pi_8\right]^{-2}, \tag{79}$$

$$v_{1_9}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[\Pi_8\right]^{-2}, \;\; \Pi_8 = p - \frac{\sqrt{-q}}{r}tan\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

where $\xi = \frac{1}{\Gamma(\alpha+1)}\left(kx^\alpha - 3kA_0t^\alpha\right)$.

Option II:

$$k = k, \quad c = 8k^3\left(\frac{k_4}{k_1} - p^2\right) - 3kA_0, \quad B_2 = \frac{4k^2(p^2k_1 - k_4)^2}{k_1^2}, \quad D_2 = D_2, \quad p = \frac{k_2}{2k_1}, \tag{80}$$

$$k_1 = k_3, \;\; C_0 = C_0, \;\; F_2 = \frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right), \;\; A_2 = C_2 = E_2 = 0,$$

$$k_4 = k_4, \quad E_0 = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2) + 2A_0(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right], \tag{81}$$

$$u(\xi) = A_0 + B_2(p + \theta(\xi))^{-2}, \quad v(\xi) = C_0 + D_2(p + \theta(\xi))^{-2}, \quad w(\xi) = E_0 + F_2(p + \theta(\xi))^{-2},$$

where k_4, k are free amounts. According to **Family 1** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be reached as below forms

$$u_{2_1}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}, \tag{82}$$

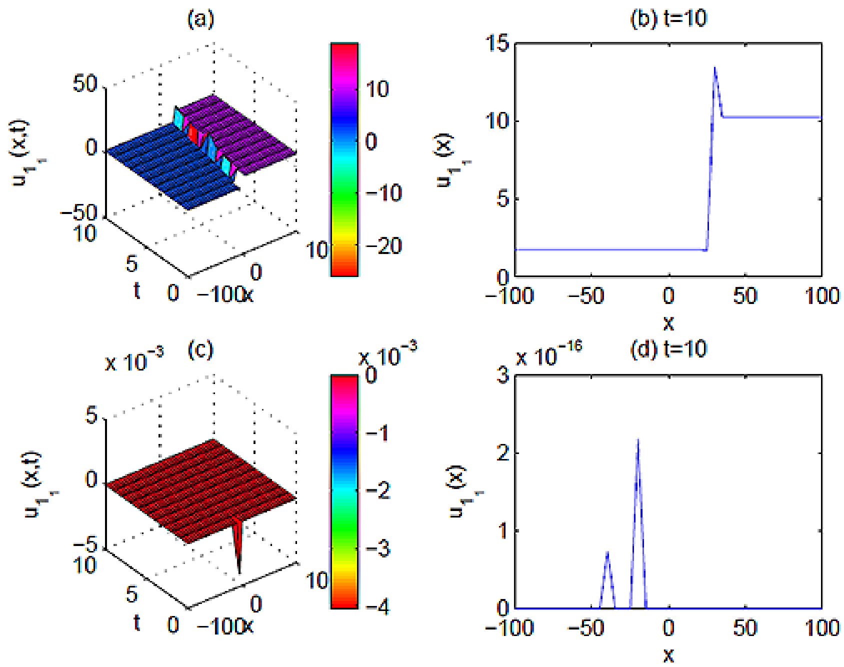

Figure 13: Plots of (i) and (j) real amounts and (k) and (l) imaginary amounts of Eq. (71) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

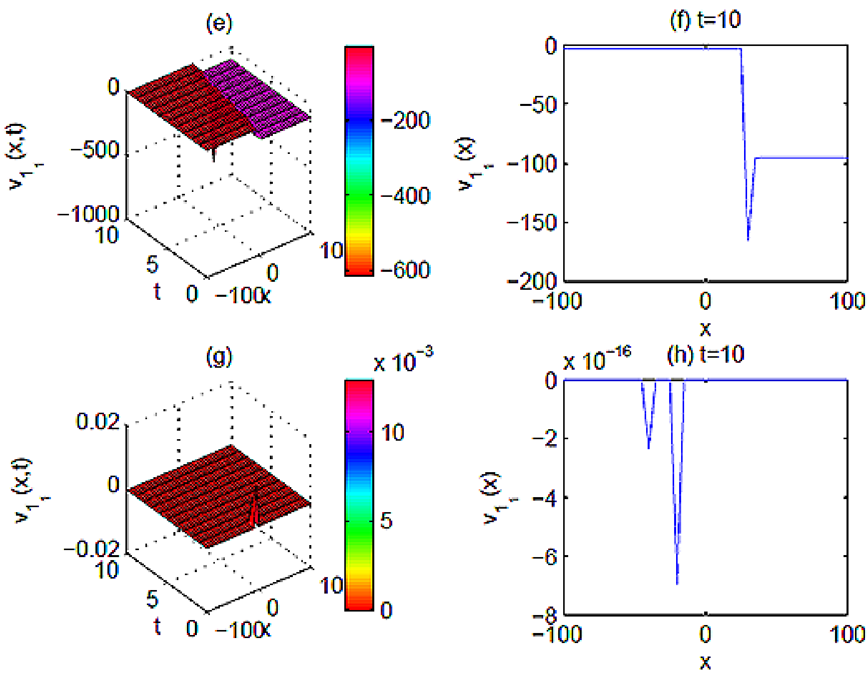

Figure 14: Plots of (m) and (n) real amounts and (o) and (p) imaginary amounts of Eq. (72) with providing amounts $p = 2, A_0 = 3, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

$$v_{2_1}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$

$$w_{2_1}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$

$$u_{2_2}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}, \qquad (83)$$

$$v_{2_2}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$

$$w_{2_2}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}.$$

Based on the **Family 2** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be got as below forms

$$u_{2_3}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2}, \qquad (84)$$

$$v_{2_3}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$

$$w_{2_3}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$

$$u_{2_4}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2}, \qquad (85)$$

$$v_{2_4}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$

$$w_{2_4}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2}.$$

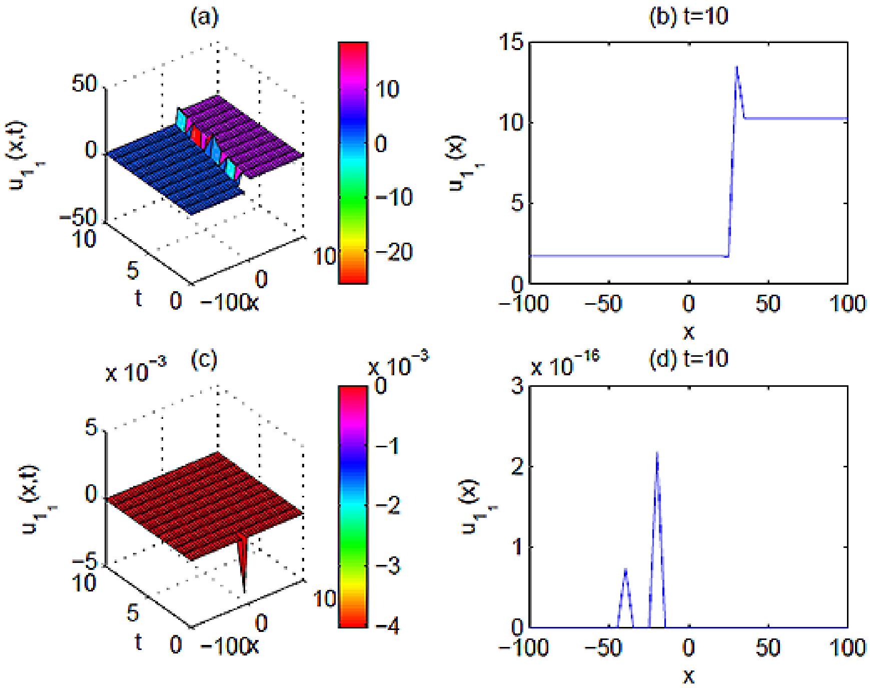

Figure 15: Plots of (q) and (r) real amounts and (s) and (t) imaginary amounts of Eq. (71) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

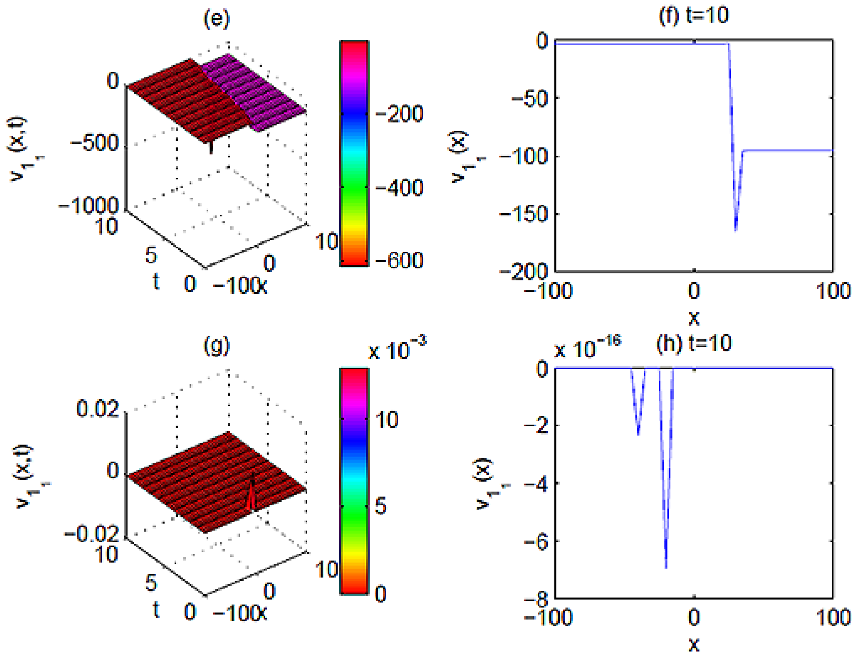

Figure 16: Plots of (u) and (v) real amounts and (w) and (x) imaginary amounts of Eq. (72) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 2, k_2 = 2, k_3 = 1, k_4 = 1, \alpha = 0.9$.

According to **Family 3** we have

$$u_{2_5}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}, \tag{86}$$

$$v_{2_5}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right]^{-2},$$

$$w_{2_5}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}.$$

Based on the **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be reached as below forms

$$u_{2_6}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad v_{2_6}(\xi) = C_0 + D_2\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \tag{87}$$

$$w_{2_6}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$u_{2_7}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad v_{2_7}(\xi) = C_0 + D_2\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \tag{88}$$

$$w_{2_7}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] +$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right).$$

According to **Family 5**, we get the below solutions

$$u_{2_8}(\xi) = A_0 - \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad v_{2_8}(\xi) = C_0 - D_2\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \tag{89}$$

$$w_{2_8}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + \right.$$

$$\left. D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] -$$

$$\frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$u_{2_9}(\xi) = A_0 - \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad v_{2_9}(\xi) = C_0 - D_2\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \tag{90}$$

$$w_{2_9}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] - \frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right).$$

Via **Family 7** we get

$$u_{2_{10}}(\xi) = A_0 + \frac{4k^2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} - \frac{k_4}{k_2} + C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^{-2}, \tag{91}$$

$$v_{2_{10}}(\xi) = C_0 + D_2\left[\frac{k_2}{2k_1} - \frac{k_4}{k_2} + C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^{-2},$$

$$w_{2_{10}}(\xi) = -\frac{4k^2}{D_2^2}\left[C_0k^2\frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3 + p^8k_1^4}{k_1^4} + D_2\frac{4k^2(k_4^3 - 3p^2k_4^2k_1 + 3p^4k_4k_1^2)}{k_1^3} + \frac{2A_0D_2(2p^2k_4k_1^2 - p^4 - k_4^2k_1)}{k_1^3}\right] + \frac{4k^4}{D_2}\left(p^8 + \frac{k_4^4 - 4p^2k_4^3k_1 + 6p^4k_4^2k_1^2 - 4p^6k_4k_1^3}{k_1^4}\right)\left[\frac{k_2}{2k_1} - \frac{k_4}{k_2} + C_1\exp\left(\frac{k_2}{k_1}\xi\right)\right]^{-2},$$

where $\xi = \frac{1}{\Gamma(\alpha+1)}\left[kx^\alpha\left(8k^3\left(\frac{k_4}{k_1} - p^2\right) - 3kA_0\right)t^\alpha\right]$.

Option III:

$$c = k\left(8k^2\frac{k_4}{k_1} - 8k^2p^2 - 3A_0\right), \quad B_2 = \frac{4k^4(p^2k_1 - k_4)^2}{k_1^2}, \quad p = \frac{k_2}{2k_1}, \quad A_2 = 4k^2, \tag{92}$$

$$D_2 = \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}, \quad C_0 = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2}, \tag{93}$$

$$C_2 = \frac{4k^4}{E_2}, \quad F_2 = \frac{E_2(p^2k_1 - k_4)^2}{k_1^2}, \quad E_0 = E_0, \quad u(\xi) = A_0 + A_2(p + \theta(\xi))^2 + B_2(p + \theta(\xi))^{-2},$$

$$v(\xi) = C_0 + C_2(p + \theta(\xi))^2 + D_2(p + \theta(\xi))^{-2}, \qquad w(\xi) = E_0 + E_2(p + \theta(\xi))^2 + F_2(p + \theta(\xi))^{-2},$$

where k_4, k are free amounts. According to **Family 1** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, one get

$$u_{3_1}(\xi) = A_0 + 4k^2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 + \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}, \tag{94}$$

$$v_{3_1}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 + \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$

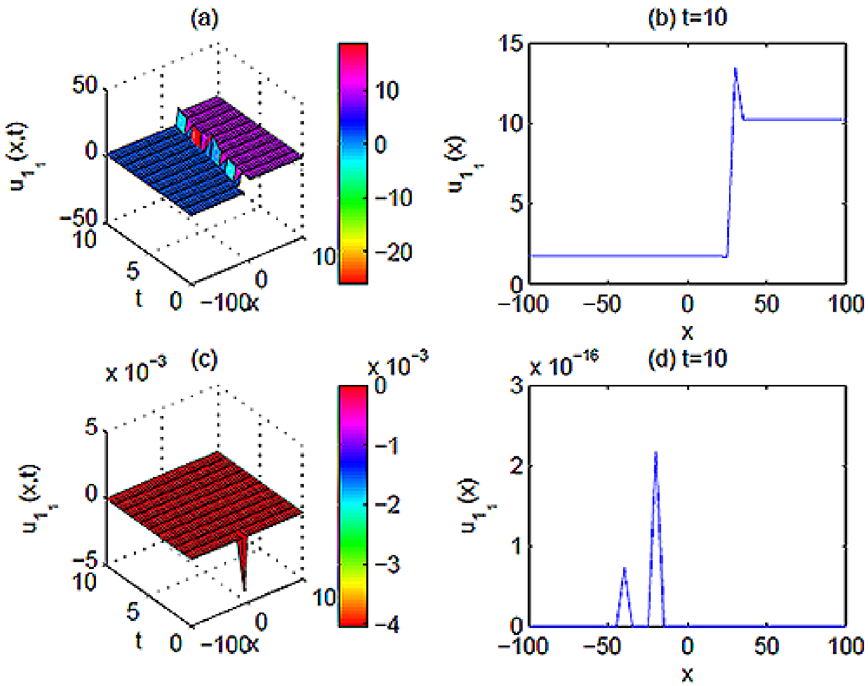

Figure 17: Plots of (a1) and (b1) real amounts and (c1) and (d1) imaginary amounts of Eq. (73) with providing amounts $p = 2, A_0 = 3, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

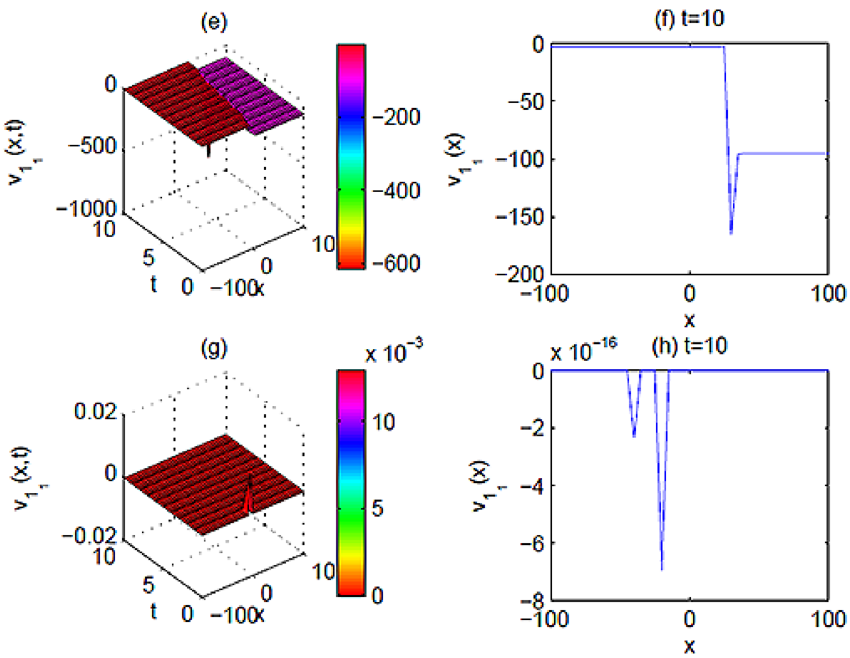

Figure 18: Plots of (e1) and (f1) real amounts and (g1) and (h1) imaginary amounts of Eq. (73) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

$$w_{3_1}(\xi) = E_0 + E_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 +$$
$$\frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}coth\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$

$$u_{3_2}(\xi) = A_0 + 4k^2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 + \tag{95}$$
$$\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$
$$v_{3_2}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2$$
$$+\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2},$$
$$w_{3_2}(\xi) = E_0 + E_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^2 +$$
$$\frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{s}}{2r}tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}.$$

Based on the **Family 2** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be reached as below forms

$$u_{3_3}(\xi) = A_0 + 4k^2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 + \tag{96}$$
$$\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$
$$v_{3_3}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2$$
$$+\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$
$$w_{3_3}(\xi) = E_0 + E_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 +$$
$$\frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r}cot\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$

$$u_{3_4}(\xi) = A_0 + 4k^2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 + \tag{97}$$
$$\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$
$$v_{3_4}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2$$
$$\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2},$$

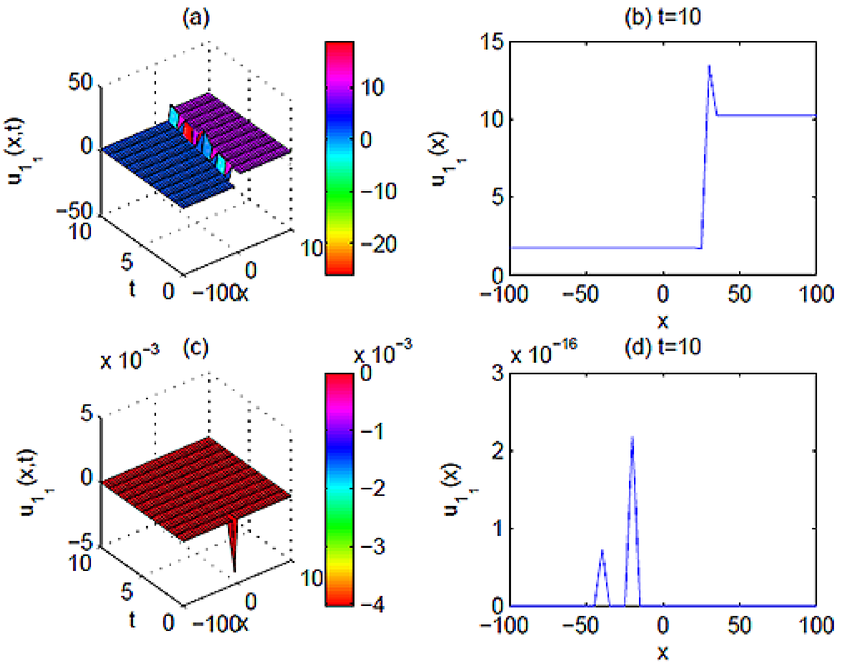

Figure 19: Plots of (i1) and (j1) real amounts and (k1) and (l1) imaginary amounts of Eq. (73) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

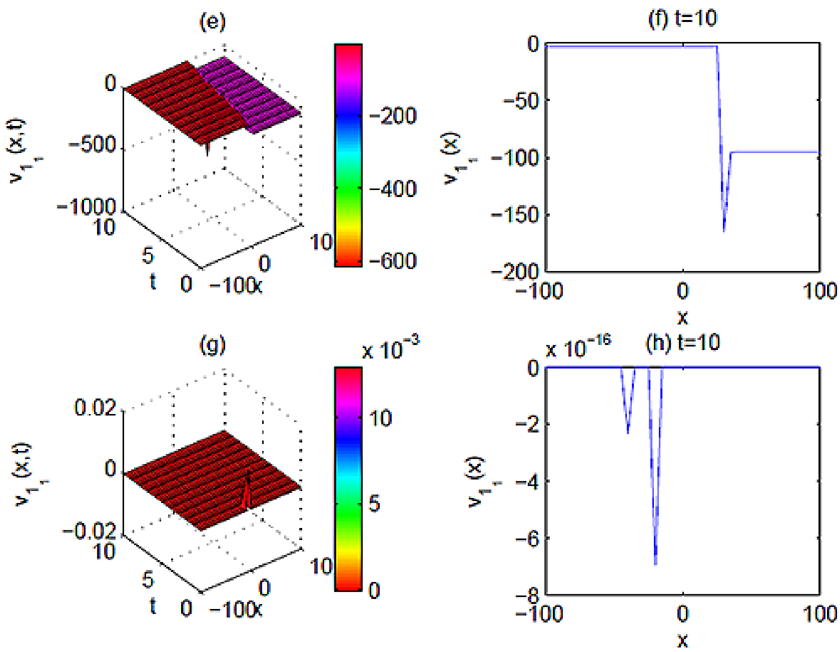

Figure 20: Plots of (m1) and (n1) real amounts and (o1) and (p1) imaginary amounts of Eq. (73) with providing amounts $p = 2, A_0 = 3, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

$$w_{3_4}(\xi) = E_0 + E_2\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^2 +$$
$$\frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\frac{k_2}{2k_1} + \frac{k_2}{2r} - \frac{\sqrt{-s}}{2r}tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2}.$$

According to **Family 3** we have

$$u_{3_5}(\xi) = A_0 + 4k^2\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^2 + \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}, \quad (98)$$

$$v_{3_5}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^2$$
$$+\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^{-2},$$

$$w_{3_5}(\xi) = E_0 + E_2\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^2 + \frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\left[\Omega_1 + \frac{C_2}{C_1 + C_2\xi}\right]^{-2},$$

where $\Omega_1 = \frac{k_2}{2k_1} + \frac{k_2}{2r}$. Via **Family 4** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, we get the below results

$$u_{3_6}(\xi) = A_0 + 4k^2\frac{q}{r^2}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad (99)$$

$$v_{3_6}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\frac{q}{r^2}qcoth^2\left(\frac{\sqrt{q}}{k_1}\xi\right)$$
$$+\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$w_{3_6}(\xi) = E_0 + E_2\frac{q}{r^2}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$u_{3_7}(\xi) = A_0 + 4k^2\frac{q}{r^2}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right), \quad (100)$$

$$v_{3_7}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} + \frac{4k^4}{E_2}\frac{q}{r^2}qtanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right)$$
$$+\frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{E_2k_1^2}\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right),$$

$$w_{3_7}(\xi) = E_0 + E_2\frac{q}{r^2}tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right) + \frac{E_2(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right).$$

According to **Family 5** (if $C_1 = 0$ but $C_2 \neq 0$; $C_1 \neq 0$ but $C_2 = 0$), respectively, the solutions will be concluded as below forms

$$u_{3_8}(\xi) = A_0 - 4k^2\frac{q}{r^2}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right) - \frac{4k^4(p^4k_1^2 - 2p^2k_4k_1 + k_4^2)}{k_1^2}\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad (101)$$

$$v_{3_8}(\xi) = \frac{4k^2(4k^2p^2E_2 - 4k^2E_2\frac{k_4}{k_1} - k^2E_0 + 2A_0E_2)}{E_2^2} - \frac{4k^4}{E_2}\frac{q}{r^2}qcot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)$$

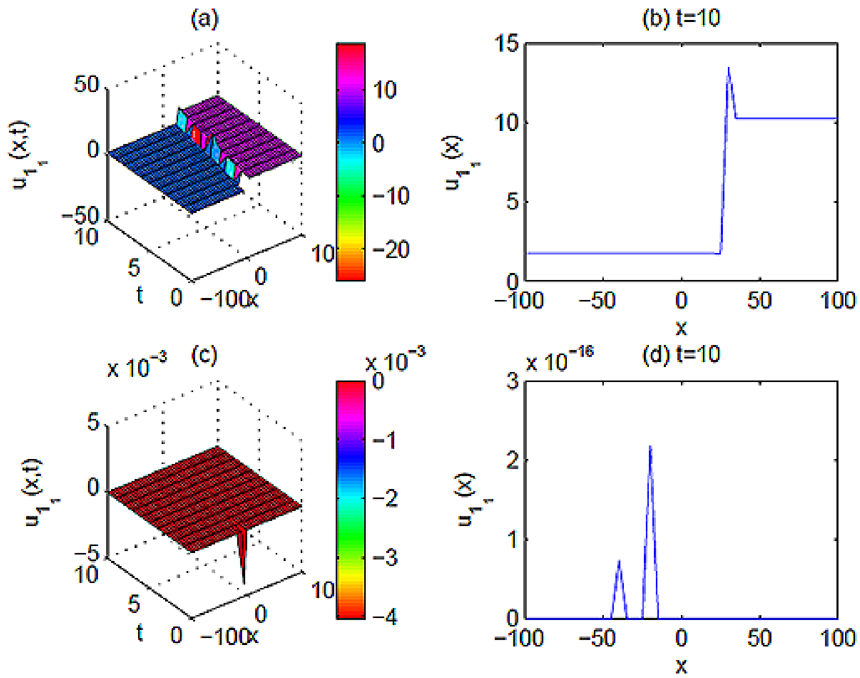

Figure 21: Plots of (q1) and (r1) real amounts and (s1) and (t1) imaginary amounts of Eq. (74) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

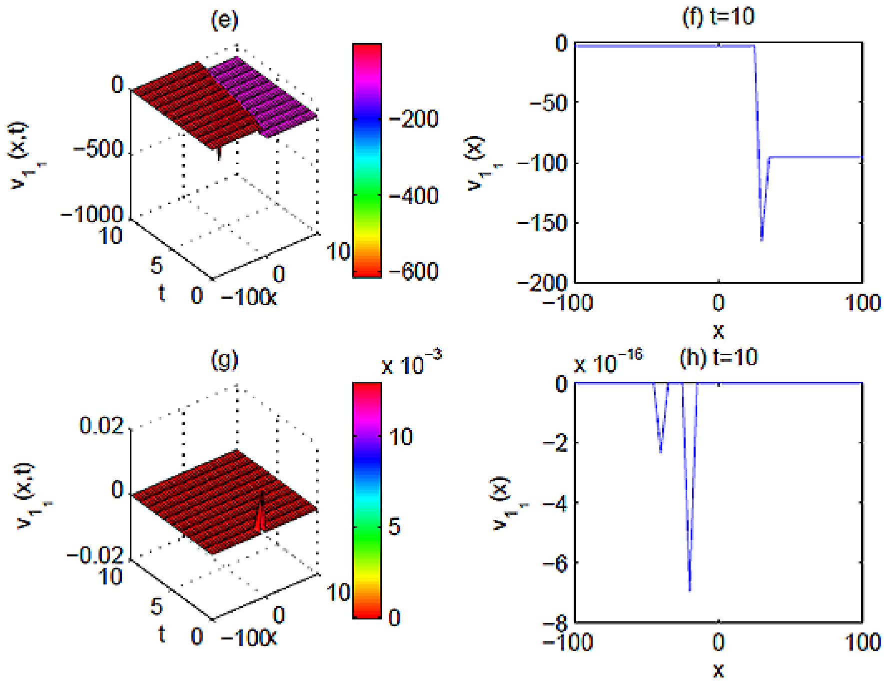

Figure 22: Plots of (u1) and (v1) real amounts and (w1) and (x1) imaginary amounts of Eq. (74) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 2, \alpha = 0.9$.

$$-\frac{4k^4(p^4k_1^2-2p^2k_4k_1+k_4^2)}{E_2k_1^2}\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$w_{3_8}(\xi)=E_0-E_2\frac{q}{r^2}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)-\frac{E_2(p^4k_1^2-2p^2k_4k_1+k_4^2)}{k_1^2}\frac{r^2}{q}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$u_{3_9}(\xi)=A_0-4k^2\frac{q}{r^2}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)-\frac{4k^4(p^4k_1^2-2p^2k_4k_1+k_4^2)}{k_1^2}\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right), \quad (102)$$

$$v_{3_9}(\xi)=\frac{4k^2(4k^2p^2E_2-4k^2E_2\frac{k_4}{k_1}-k^2E_0+2A_0E_2)}{E_2^2}-\frac{4k^4}{E_2}\frac{q}{r^2}qtan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)$$

$$-\frac{4k^4(p^4k_1^2-2p^2k_4k_1+k_4^2)}{E_2k_1^2}\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right),$$

$$w_{3_9}(\xi)=E_0-E_2\frac{q}{r^2}tan^2\left(\frac{\sqrt{-q}}{k_1}\xi\right)-\frac{E_2(p^4k_1^2-2p^2k_4k_1+k_4^2)}{k_1^2}\frac{r^2}{q}cot^2\left(\frac{\sqrt{-q}}{k_1}\xi\right).$$

Via **Family 6** we get

$$u_{3_{10}}(\xi)=A_0+4k^2p^4\left[\Lambda_1\right]^2+4k^2p^4\left[\Lambda_1\right]^{-2}, \quad (103)$$

$$v_{3_{10}}(\xi)=\frac{4k^2(4k^2p^2E_2-k^2E_0+2A_0E_2)}{E_2^2}+\frac{4k^4}{E_2}\left[\Lambda_1\right]^2+\frac{4k^4p^4}{E_2}\left[\Lambda_1\right]^{-2},$$

$$w_{3_{10}}(\xi)=E_0+E_2\left[\Lambda_1\right]^2+E_2p^4\left[\Lambda_1\right]^{-2}, \quad \Lambda_1=\frac{k_2}{2k_1}+\frac{C_1k_2^2e^{\frac{-k_2}{k_1}\xi}}{rk_1+C_1k_1k_2e^{\frac{-k_2}{k_1}\xi}},$$

where $\xi=\frac{1}{\Gamma(\alpha+1)}\left[kx^\alpha-k\left(8k^2\frac{k_4}{k_1}-8k^2p^2-3A_0\right)t^\alpha\right]$.

4 Results and Discussion

In this section we will discuss the physical explanation of the found exact solutions to a nonlinear fractional model. We expose the graphical representation of these solutions and accomplish about the different kinds of solutions with two fractional order. To sketch plots we have utilized the Maple software package. Every exact solutions are offered in Maple 3D plot and 2D plot view for proper understanding. Some appropriate values are given in Figs. 2-24 to analyze the dynamics properties briefly. These figures are depended on the family conditions which are of important physically. It has been investigated that all figures including 3D plot and 2D plot with two fractional-order $\delta=1$ designed for the nonlinear fractional models involving the modified Riemann-Liouville derivative. To the best of our knowledge, these complex exponential function solutions have not been submitted to literature in advance. The analytical solutions and figures obtained in this paper give us a different physical interpretation for the nonlinear FWBK and FgHSCKdV equations.

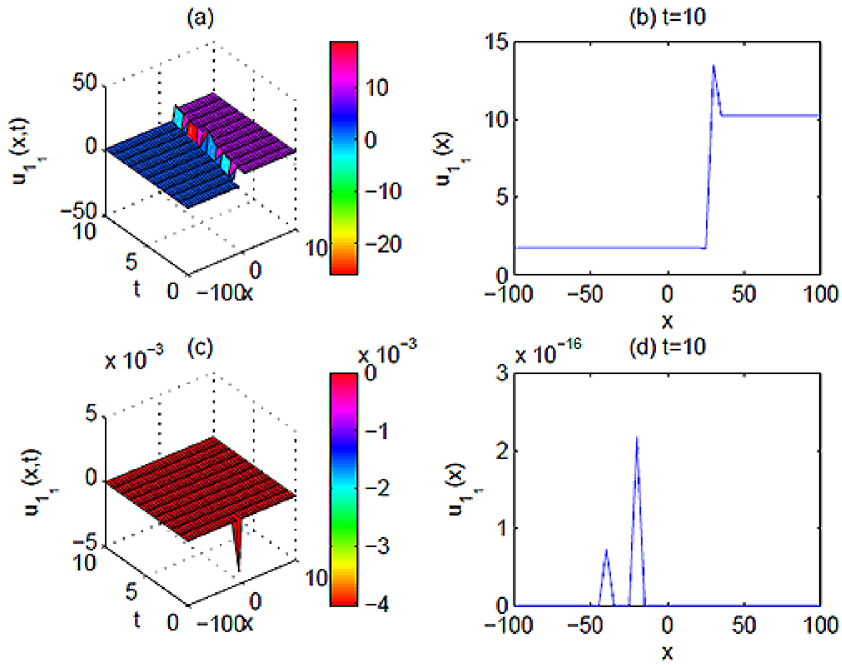

Figure 23: Plots of (a2) and (b2) real amounts and (c2) and (d2) imaginary amounts of Eq. (75) with providing amounts $p = 2, A_0 = 3, C_1 = 2, C_2 = 3, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 1, \alpha = 0.9$.

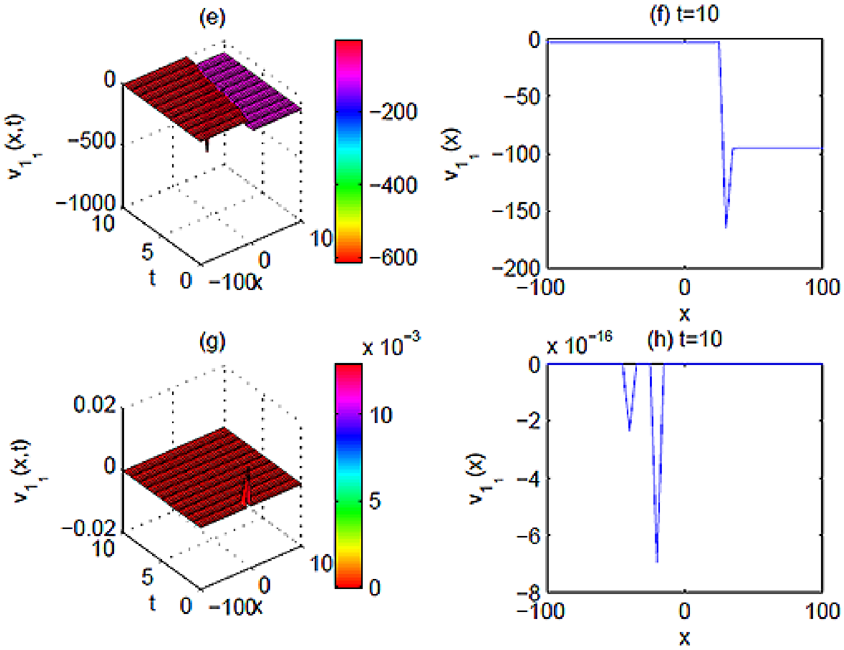

Figure 24: Plots of (e2) and (f2) real amounts and (g2) and (h2) imaginary amounts of Eq. (75) with providing amounts $p = 2, A_0 = 3, E_0 = 3, D_2 = 2, C_1 = 2, C_2 = 3, k = 2, k_1 = 1, k_2 = 2, k_3 = 2, k_4 = 1, \alpha = 0.9$.

5 Conclusion

In this work, on the basis of the generalized (G'/G)-expansion method, we have successfully obtained the nonlinear FWBK and FgHSCKdV equations. And then, several sets of soliton wave solutions, several sets of periodic wave solutions and other types of singular and kink-singular wave solutions were successfully constructed by utilizing the gG'/GEM. Moreover, we have obtained the other solutions by utilizing a nonlinear Riccati equation based on the ODE. On the other hand, the density, 3-D and 2-D graphs of those presented wave solutions were given from the perspective of dynamics. Hope these novel findings can promote the understanding of the (1+1)-dimensional the nonlinear FWBK and FgHSCKdV equations.

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