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ON THE DETERMINATION OF THE COEFFICIENTS OF THE SECOND-ORDER HYPERBOLIC EQUATION WITH A NONLOCAL CONDITION

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Abstract. In this paper, we study the optimal control problem for the hyperbolic equation with a nonlocal boundary condition, when the desired control functions are present in the coefficients of the first-order derivatives. An existence theorem for optimal control is proved, and the necessary optimality condition is derived.

Keywords: nonlocal condition, inverse problem, method, hyperbolic equation, optimization.

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1 Introduction

Optimal control problems for the coefficients of the partial differential equations are of great applied importance, since these coefficients characterize the properties of the medium under consideration. Moreover, this type optimal control problems are closely related to the inverse problems of determining the coefficients of partial differential equations (Yuldashev, 2019; Guliyev et al., 2017; Kabanikhin & Iskakov, 2001; Tagiev & Habibov, 2016; Tagiev, 2012, 2010; Li & Lou, 2012). Note that such problems are nonlinear and, generally speaking, ill-posed problems. When the process under consideration is described by the boundary value problem with a nonlocal condition, then the study of the solvability of this problem and, accordingly, the problem of optimal control becomes much more complicated.

In this paper, the inverse problem of determining the coefficients of the equation is considered and this problem is reduced to the optimal control problem for the second-order hyperbolic equation with a nonlocal condition when the control functions are present in the coefficients of the first-order derivatives. An existence theorem for the optimal control is proved, the necessary optimality condition is derived and the gradient of the considered functional is calculated.

2 Formulation of the problem

Consider the problem of determining a pair of functions $(u(x,t),\vartheta(x))$ from the following relations

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+\sum_{i=1}^{3}\vartheta_i(x)\frac{\partial u}{\partial x_i}=f(x,t),(x,t)\in Q=\Omega\times(0,T), \tag{1}$$

$$u(x,0)=u_0(x),\frac{\partial u(x,0)}{\partial t}=u_1(x),\ x\in\Omega, \tag{2}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_S = \int_\Omega K(x,y)u(y,t)dy, (x,t) \in S, \tag{3}$$

$$\int_0^T R(x,t)u(x,t)dt = \varphi(x), x \in \Omega, \tag{4}$$

where T is a given positive number; $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$; $S = \partial\Omega \times (0,T)$ is a lateral surface of the cylinder Q ; ν is an outward normal to the boundary $\partial\Omega$.

If the functions $f(x,t), \vartheta(x) = (\vartheta_1(x), \vartheta_2(x), \vartheta_3(x)), u_0(x), u_1(x)$, $K(x,y)$ are given, then we obtain direct problem (1)-(3) on determination of the function $u(x,t)$.

If $\vartheta(x)$ is unknown function, then we set additional condition (4). By this way we come to inverse problem (1)-(4) on determination of the pair $(u(x,t), \vartheta(x))$.

Suppose that $f \in L_2(Q)$, $u_0 \in W_2^1(\Omega)$, $u_1 \in L_2(\Omega)$, $R \in L_\infty(Q)$, $\varphi \in L_2(\Omega)$, $K(x,y) \in L_\infty(\partial\Omega \times \Omega)$ are given functions.

Problem (1)-(4) is reduced to the following optimal control problem: find such a function $\vartheta(x)$ from the set

$$V = \Big\{\vartheta(x) : \vartheta(x) = (\vartheta_1(x), \vartheta_2(x), \vartheta_3(x)), \vartheta_i(x) \in C^1(\bar{\Omega}) : |\vartheta_i(x)| \le M_i,$$

$$\vartheta_i(x)|_{\partial\Omega} = 0, \left|\frac{\partial \vartheta_i(x)}{\partial x_k}\right| \le M_i^k,\ i,k = 1,2,3 \ on\ \Omega\Big\}, \tag{5}$$

which delivers a minimum of functionality

$$I(\vartheta) = \frac{1}{2}\int_\Omega \left[\int_0^T R(x,t)u(x,t;\vartheta)dt - \varphi(x)\right]^2 dx \tag{6}$$

subject to restrictions (1)-(3), where $u(x,t;\vartheta)$ is a solution to problem (1)-(3) at $\vartheta = \vartheta(x)$, $M_i, M_i^k,\ i,k = 1,2,3$ are given positive numbers. This problem we call problem (1)-(3),(5),(6).

The vector function $\vartheta(x)$ is called a control, and V - a class of admissible controls. There is a close connection between problems (1)-(3), (5), (6) and (1)-(4). Note that if $\min\limits_{\vartheta \in V} I(\vartheta) = 0$, then additional condition (4) is satisfied.

As a solution to boundary value problem (1)-(3) at each fixed control $\vartheta \in V$ we consider the function from $W_2^1(Q)$ equaling to $u_0(x)$ at $t = 0$ and satisfying the integral identity

$$\begin{gathered}\int_0^T \int_\Omega \left(-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \nabla u \nabla \eta + \sum_{i=1}^3 \vartheta_i \frac{\partial u}{\partial x_i}\eta\right) dxdt- \\ -\int_0^T \int_{\partial\Omega} \eta\,(x,t) \int_\Omega K\,(x,y)\,u(y,t)\ dydsdt- \\ -\int_\Omega u_1(x)\eta(x,0)dx = \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt\end{gathered} \tag{7}$$

at all $\eta = \eta(x,t)$ from $W_2^1(Q)$, equaling to zero at $t = T$.

It follows from (Kozhanov & Pulkina, 2010) that under the adopted conditions, boundary value problem (1)-(3) for each fixed control $\vartheta \in V$ has a unique generalized solution from $W_2^1(Q)$ and the estimate

$$\|u\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)}\right] \tag{8}$$

is valid, moreover, this solution has the properties (Lions & Magenes, 1971)

$$u \in C\left([0,T], W_2^1(\Omega)\right),\ \frac{\partial u}{\partial t} \in C\left([0,T], L_2(\Omega)\right).$$

Here and furthermore, c will denote various constants that do not depend on the values being estimated and on the admissible controls.

3 On a solvability of problem (1)-(3), (5), (6)

Theorem 1. *Let the conditions set in the formulation of problem* (1)-(3), (5),(6) *be fulfilled. Then* $V_* = \left\{ \vartheta_* \in V : \ I(\vartheta_*) = \min\limits_{\vartheta \in V} I(\vartheta) \right\}$ *is not empty, weekly compact in* $(W_2^1(\Omega))^3$ *and any minimizing sequence* $\{\vartheta_k\}$ *converges weekly to the set* V_* *in* $(W_2^1(\Omega))^3$, *where* $(W_2^1(\Omega))^3 = W_2^1(\Omega) \times W_2^1(\Omega) \times W_2^1(\Omega)$.

Proof. It is clear that the set V defined by relation (5) is weekly compact in the space $(W_2^1(\Omega))^3$. Let us show that functional (6) is weekly continuous in $(W_2^1(\Omega))^3$ on the set V. Let $\vartheta \in V$ be some element $\{\vartheta^k\} \in V$ be an arbitrary sequence such that $\vartheta^k \to \vartheta$ weekly in $(W_2^1(\Omega))^3$ at $k \to \infty$. Hence, from the compactness of the embedding $(W_2^1(\Omega))^3 \subset (L_2(\Omega))^3$ (Ladyzhenskaya, 1973), follows that

$$\vartheta^k \to \vartheta \textit{ strongly in } (L_2(\Omega))^3. \tag{9}$$

Moreover taking into account the definition of the set V we see that

$$\vartheta^k \to \vartheta \textit{ strongly in } (C^1(\bar{\Omega}))^3 \tag{9'}$$

Due to the unique solvability of boundary value problem (1)-(3) to each control $\vartheta^k \in V$ correspond the only solution $u_k = u(x,t;\vartheta^k)$ of problem (1)-(3) and the estimation $\|u_k\|_{W_2^1(Q)} \leq c\,, \ \forall\, k = 1, 2, ...$ is valid i.e. the sequence $\{u_k\}$ uniformly bounded in the norm of the space $W_2^1(Q)$. Then it follows from the embedding theorems (Ladyzhenskaya, 1973) that a subsequence $\{u_{k_l}\}$ can be selected from the sequence $\{u_k\}$ such that for $l \to \infty$

$$u_{k_l} \to u \textit{ strongly in } L_2(Q), \tag{10}$$

$$\frac{\partial u_{k_l}}{\partial t} \to \frac{\partial u}{\partial t}, \frac{\partial u_{k_l}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \textit{ weekly in } L_2(Q), \tag{11}$$

where $u = u(x,t) \in W_2^1(Q)$ is some element. Let us show that $u(x,t) = u(x,t;\vartheta)$ i.e. the function $u(x,t)$ is a generalized solution to problem (1)-(3) corresponding to the control $\vartheta \in V$. It is clear that the identities

$$\begin{aligned} &\int_0^T \int_\Omega \left(-\frac{\partial u_{k_l}}{\partial t}\frac{\partial \eta}{\partial t} + \nabla u_{k_l} \nabla \eta + \sum_{i=1}^3 \vartheta_i^{k_l} \frac{\partial u_{k_l}}{\partial x_i}\eta \right) dxdt - \\ &- \int_0^T \int_{\partial\Omega} \eta\,(x,t) \int_\Omega K\,(x,y)\, u_{k_l}(y,t)\; dydsdt - \int_\Omega u_1(x)\eta(x,0)dx = \\ &= \int_0^T \int_\Omega f(x,t)\eta(x,t)dxdt, l = 1, 2... \end{aligned} \tag{12}$$

are valid for all $\eta = \eta(x,t)$ from $W_2^1(\bar{Q})$ equaling to zero at $t = T$.

Passing to limit in (12) at $l \to \infty$ and using (9)-(11) we obtain that the function $u(x,t)$ is equal to $u_0(x)$ at $t = 0$ and satisfies to identity (7). From this and from the uniqueness of the solution to problem (1)-(3) corresponding to the control $\vartheta \in V$, it follows that $u(x,t) = u(x,t;\vartheta)$.

Using the uniqueness of the solution to problem (1)-(3) corresponding to the control $\vartheta \in V$ it is easy to check that relations (10), (11) are valid not only for the subsequence $\{u_{k_l}\}$, but also for the entire sequence $\{u_k\}$. Taking this into account, from (6) we obtain that $I(\vartheta^k) \to I(\vartheta)$ at $k \to \infty$, i.e. $I(\vartheta)$ is weekly continuous in $(W_2^1(\Omega))^3$ on the set V. Then, by virtue of Theorem 2 from Vasilyev (1981), we obtain that all the assertions of Theorem 1 are true. Theorem 1 is proved. □

To ensure the uniqueness of the solution to the optimal control problem, instead of functional (6), we can consider a functional of the form

$$I_\alpha(\vartheta) = I(\vartheta) + \alpha \sum_{i=1}^{3} \|\vartheta_i - \omega_i\|^2_{W_2^1(\Omega)}, \tag{13}$$

where $I(\vartheta)$ is defined by equality (6), $\alpha > 0$ is a given number, $\omega = (\omega_1, \omega_2, \omega_3) \in \left(W_2^1(\Omega)\right)^3$ is a given function.

Theorem 2. *Let the conditions of Theorem 1 and $\alpha > 0$ be satisfied. Then there exists a dense subset G of the space $(W_2^1(\Omega))^3$, such that for any $\omega \in G$ problem of minimizing functional* (13) *on the set V under conditions* (1)-(3) *has a unique solution.*

Proof. The functional $I(\vartheta)$ is bounded below and, by virtue of Theorem 1, is continuous on. Moreover, the set V is closed and bounded in a uniformly convex Banach space $(W_2^1(\Omega))^3$. Then the results of Ekland & Temam (1979) imply the assertion of Theorem 2. Theorem 2 is proved. □

4 Differentiability of functional (6)

Now let us investigate the differentiability of functional (6). Let $\psi(x,t) = \psi(x,t;\vartheta)$be a generalized solution from $W_2^1(Q)$ of the adjoint problem

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\vartheta_i \psi) = \int_{\partial\Omega} \psi(\xi,t) K(\xi,x) ds + R(x,t) \left(\int_0^T R(x,t) u(x,t) dt - \varphi(x) \right),$$

$$(x,t) \in Q, \tag{14}$$

$$\psi(x,T) = 0, \frac{\partial \psi(x,T)}{\partial t} = 0,\ x \in \Omega, \tag{15}$$

$$\left.\frac{\partial \psi}{\partial \nu}\right|_S = 0, (x,t) \in S. \tag{16}$$

As generalized solution of boundary value problem (14)-(16) for each fixed control $\vartheta \in V$we mean a function $\psi(x,t) = \psi(x,t;\vartheta)$ from $W_2^1(Q)$ equal to zero for $t = T$ and satisfying the integral identity

$$\begin{gathered}\int_0^T \int_\Omega \left(-\frac{\partial \psi}{\partial t}\frac{\partial \Phi}{\partial t} + \nabla\psi \nabla\Phi + \sum_{i=1}^{3} \vartheta_i \frac{\partial \Phi}{\partial x_i} \psi \right) dxdt - \\ - \int_0^T \int_{\partial\Omega} \psi(x,t) \int_\Omega K(x,y) \Phi(y,t)\, dydsdt - \\ - \int_0^T \int_\Omega \Phi(x,t) R(x,t) \left(\int_0^T R(x,t) u(x,t) dt - \varphi(x) \right) dxdt = 0\end{gathered} \tag{17}$$

for all $\Phi = \Phi(x,t)$ from $W_2^1(Q)$ equal to zero at $t = 0$.

Since (14)-(16) is linear with respect to $\psi(x,t)$, this problem has a unique generalized solution in the space $W_2^1(Q)$ and the

$$\|\psi\|_{W_2^1(Q)} \leq c\left[\|u\|_{W_2^1(Q)} + \|\varphi\|_{L_2(\Omega)}\right] \tag{18}$$

is valid (Kozhanov & Pulkina, 2010).

Then from (8) and (18) follows

$$\|\psi\|_{W_2^1(Q)} \leq c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\Omega)}\right]. \tag{18$'$}$$

Theorem 3. *Let the conditions imposed above on the data of problem* (1)-(3), (5), (6) *be satisfied. Then functional* (6) *is continuously Frechet differentiable on* V *and the differential at the point* $\vartheta \in V$ *with increment* $\delta\vartheta \in C^1(\bar{\Omega})$ *has the form*

$$\left\langle I'(\vartheta),\, \delta\vartheta\right\rangle = \int_0^T \int_\Omega \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\delta\vartheta_i \psi dxdt. \tag{19}$$

Proof. Consider the increment of functional (6)

$$\begin{aligned} &\delta I(\vartheta) = I(\vartheta + \delta\vartheta) - I(\vartheta) = \\ &= \int_\Omega \int_0^T R(x,t)\delta u dt \left(\int_0^T R(x,t)u(x,t)dt - \varphi(x)\right) dx + \\ &+ \frac{1}{2}\int_\Omega \left(\int_0^T R(x,t)\delta u dt\right)^2 dx, \end{aligned} \tag{20}$$

where $\delta u(x,t) = u(x,t;\vartheta + \delta\vartheta) - u(x,t;\vartheta)$; $u(x,t;\vartheta + \delta\vartheta)$ and $u(x,t;\vartheta)$ are solutions of problem (1)-(3) corresponding to the controls $\vartheta + \delta\vartheta,\ \vartheta \in V$. It is obvious that the function $\delta u(x,t)$ is a generalized solution from $W_2^1(Q)$ for the boundary value problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \Delta\delta u + \sum_{i=1}^{3}(\vartheta_i + \delta\vartheta_i)\frac{\partial \delta u}{\partial x_i} = -\sum_{i=1}^{3}\frac{\partial u}{\partial x_i}\delta\vartheta_i, (x,t) \in Q, \tag{21}$$

$$\delta u(x,0) = 0, \frac{\partial \delta u(x,0)}{\partial t} = 0,\, x \in \Omega, \tag{22}$$

$$\left.\frac{\partial \delta u}{\partial \nu}\right|_S = \int_\Omega K(x,y)\delta u(y,t)dy, (x,t) \in S. \tag{23}$$

The generalized solution from $W_2^1(Q)$ for problem (21)-(23) is equal to zero and for $t = 0$ satisfies the integral identity

$$\begin{aligned} &\int_0^T \int_\Omega \left(\frac{\partial \delta u}{\partial t}\frac{\partial \eta}{\partial t} - \nabla\delta u \nabla\eta - \sum_{i=1}^{3}(\vartheta_i + \delta\vartheta_i)\frac{\partial \delta u}{\partial x_i}\eta\right) dxdt + \\ &+ \int_0^T \int_{\partial\Omega} \eta\,(x,t) \int_\Omega K\,(x,y)\,\delta u(y,t)\; dydsdt = \int_0^T \int_\Omega \sum_{i=1}^{3}\frac{\partial u}{\partial x_i}\delta\vartheta_i \eta dxdt \end{aligned} \tag{24}$$

for all $\eta = \eta(x,t)$ from $W_2^1(Q)$ equal to zero at $t = T$. As for the solution of problem (1)-(3) for the solution of problem (21)-(23) the estimation

$$\|\delta u\|_{W_2^1(Q)} \leq c\sum_{i=1}^{3}\|\delta\vartheta_i\|_{C(\bar{\Omega})} \tag{25}$$

is valid. If to put $\Phi = \delta u(x,t)$ in (17) and $\eta = \psi(x,t;\vartheta)$ in (24) and then sum the obtained relation we get

$$\int_0^T \int_\Omega R(x,t)\delta u \left(\int_0^T R(x,t)u(x,t)dt - \varphi(x) \right) dxdt =$$

$$-\int_0^T \int_\Omega \psi \sum_{i=1}^3 \frac{\partial \delta u}{\partial x_i} \delta\vartheta_i dxdt-$$
$$-\int_0^T \int_\Omega \psi \sum_{i=1}^3 \frac{\partial u}{\partial x_i} \delta\vartheta_i dxdt.$$

Considering this equality in (20), we have

$$\delta I(\vartheta) = -\int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial u}{\partial x_i} \delta\vartheta_i \psi dxdt + R, \tag{26}$$

where

$$R = -\int_0^T \int_\Omega \psi \sum_{i=1}^3 \frac{\partial \delta u}{\partial x_i} \delta\vartheta_i dxdt + \frac{1}{2}\int_\Omega \left(\int_0^T R\delta u dt \right)^2 dx$$

is a reminder term.

It is obvious that the expression in the right hand side of (19) at the given $\vartheta \in V$ defines a linear functional of $\delta\vartheta$. Additionally

$$\left| \int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial u}{\partial x_i} \delta\vartheta_i \psi dxdt \right| \le c \sum_{i=1}^3 \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(Q)} \|\psi\|_{L_2(Q)} \sum_{i=1}^3 \|\delta\vartheta_i\|_{C(\bar\Omega)} .$$

Considering here estimations (8), (18') we obtain boundedness over $\delta\vartheta$ of the functional in the right hand side of (19).

Now we estimate the reminder term R included in (26). Using Caushy-Bunyakovski inequality, we get

$$|R| \le c \left(\sum_{i=1}^3 \left\| \frac{\partial \delta u}{\partial x_i} \right\|_{L_2(Q)} \|\psi\|_{L_2(Q)} \sum_{i=1}^3 \|\delta\vartheta_i\|_{C(\bar\Omega)} + \|\delta u\|^2_{W_2^1(Q)} \right).$$

Considering here estimate (25) we conclude that $R = o\left(\sum_{i=1}^3 \|\delta\vartheta_i\|_{C(\bar\Omega)} \right)$ or considering embedding $C^1(\bar\Omega) \subset C(\bar\Omega)$ we get $R = o\left(\sum_{i=1}^3 \|\delta\vartheta_i\|_{C^1(\bar\Omega)} \right)$.

Then it follows from (26) that functional (6) id Frechet differentiable on V and formula (19) is valid. Show that the mapping $\vartheta \to I'(\vartheta)$ generated by equality (19) acts continuously from V into adjoint to $C^1(\bar\Omega)$ space $(C^1(\bar\Omega))^*$.

Let $\delta\psi(x,t) = \psi(x,t;\vartheta+\delta\vartheta) - \psi(x,t;\vartheta)$. It follows from (14)-(16) that $\delta\psi(x,t)$ is a generalized solution from $W_2^1(Q)$ for the boundary value problem

$$\frac{\partial^2 \delta\psi}{\partial t^2} - \Delta\delta\psi - \sum_{i=1}^3 \frac{\partial}{\partial x_i}((\vartheta_i + \delta\vartheta_i)\delta\psi) = \sum_{i=1}^3 \frac{\partial}{\partial x_i}(\psi\delta\vartheta_i) + \int_{\partial\Omega} \delta\psi(\xi,t)K(\xi,x)ds+$$

$$+R(x,t)\int_0^T R(x,t)\delta u(x,t)dt(x,t) \in Q,$$

$$\delta\psi(x,T) = 0, \frac{\partial \delta\psi(x,T)}{\partial t} = 0,\ x \in \Omega,$$

$$\left. \frac{\partial \delta\psi}{\partial \nu} \right|_S = 0, (x,t) \in S.$$

Similarly to (18') for the solution of this problem the estimation

$$\|\delta\psi\|_{W_2^1(Q)} \leq c\left(\|\delta u\|_{W_2^1(Q)} + \sum_{i=1}^{3}\|\delta\vartheta_i\|_{C^1(\bar{\Omega})}\right). \tag{27}$$

is valid. Then (25), (27) and the embedding $C^1(\bar{\Omega}) \subset C(\bar{\Omega})$ implies the estimation

$$\|\delta\psi\|_{W_2^1(Q)} \leq c\sum_{i=1}^{3}\|\delta\vartheta_i\|_{C^1(\bar{\Omega})}. \tag{28}$$

Using (19) and Caushy-Bunyakovski inequality we get

$$\left\|I'(\vartheta+\delta\vartheta) - I'(\vartheta)\right\|_{\left(C'(\bar{\Omega})\right)^*} \leq c\sum_{i=1}^{3}\left(\left\|\frac{\partial\delta u}{\partial x_i}\right\|_{L_2(Q)}\|\psi\|_{L_2(Q)} + \right.$$

$$\left. + \left\|\frac{\partial u}{\partial x_i}\right\|_{L_2(Q)}\|\delta\psi\|_{L_2(Q)} + \left\|\frac{\partial\delta u}{\partial x_i}\right\|_{L_2(Q)}\|\delta\psi\|_{L_2(Q)}\right).$$

By virtue of (25) and (28) the last gives

$$\left\|I'(\vartheta+\delta\vartheta) - I'(\vartheta)\right\|_{\left(C'(\bar{\Omega})\right)^*} \leq c\sum_{i=1}^{3}\|\delta\vartheta_i\|_{C^1(\bar{\Omega})}, \tag{29}$$

where the right hand side tends to zero at $\|\delta\vartheta_i\|_{C^1(\bar{\Omega})} \to 0$, $i = 1,2,3$. It follows from this that $\vartheta \to I'(\vartheta)$ is a continuous mapping from V into $(C^1(\bar{\Omega}))^*$. Theorem 3 is proved. □

5 Necessary conditions for optimality and the formula for the gradient of functional (6)

Theorem 4. *Let the condition of Theorem 3 be fulfilled. Then for the optimality of the control $\vartheta_*(x) \in V$ in problem (1)-(3), (5), (6) it is necessary fulfilment of the inequality*

$$\int_0^T\int_\Omega\sum_{i=1}^{3}\frac{\partial u_*(x,t)}{\partial x_i}(\vartheta_i(x) - \vartheta_{i*}(x))\psi_*(x,t)dxdt \geq 0, \tag{30}$$

for any $\vartheta = \vartheta(x) \in V$, where $u_*(x,t) = u(x,t;\vartheta_*)$, $\psi_*(x,t) = \psi(x,t;\vartheta_*)$ are solutions of problems (1)-(3) and (14)-(16), correspondingly at $\vartheta = \vartheta_*(x)$.

Proof. The set V defined by relation (5) is convex in $(C^1(\bar{\Omega}))^3$. Moreover, according to Theorem 3, the functional $I(\vartheta)$ is continuously Frechet differentiable on V and its differential at a point $\vartheta \in V$ is determined by equality (19). Then, by virtue of Theorem 5 (Vasilyev, 1981; p.28), at the element $\vartheta_*(x) \in V$ it is necessary fulfilment of the inequality

$$\left\langle I'(\vartheta), \vartheta - \vartheta_*\right\rangle \geq 0,$$

for all $\vartheta \in V$. From this and from (19) follows the validity of inequality (30) for all $\vartheta \in V$. Theorem 4 is proved. □

Now we will show that it is possible to obtain a formula for the gradient of functional (6). We introduce the following boundary value problem on determination of the function $\psi_i = \psi_i(x;\vartheta)$

$$-\Delta\psi_i + \psi_i = f_i(x),\ x \in \Omega,\ i = 1,2,3, \tag{31}$$

$$\left.\frac{\partial \psi_i}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{32}$$

where $f_i(x) = \int_0^T \frac{\partial u}{\partial x_i}\psi dt$ (Tagiev, 2010).

As a solution of problem (31), (32) for a given $\vartheta \in V$ we mean a function $\psi_i = \psi_i(x;\vartheta)$ from $W_2^1(\Omega)$ satisfying the integral identity

$$\int_\Omega (\nabla\psi_i \nabla\eta + \psi_i\eta)\, dx = \int_\Omega f_i(x)\eta dx, \tag{33}$$

for arbitrary function $\eta = \eta(x)$ from $C^1\left(\bar{\Omega}\right)$.

The right hand side of equation (31) belongs to $L_1(\Omega)$ and boundary value problem (31), (32) is uniquely solvable in $\overset{\circ}{W}{}_2^1(\Omega)$ (Mikhailov, 1983).

Theorem 5. *Let the conditions of Theorem 3 be fulfilled. Then the gradient of functional* (6) *at an arbitrary point $\vartheta \in V$ is determined by the expression*

$$I'(\vartheta) = (\psi_1(x;\vartheta), \psi_2(x;\vartheta), \psi_3(x;\vartheta)). \tag{34}$$

Proof. Let ϑ, $\vartheta + \delta\vartheta \in V$ be arbitrary controls, where $\delta\vartheta \in C^1(\Omega)$ be an increment of the control on the point $\vartheta \in V$. Taking $\eta = \delta\vartheta_i$ in (33) we obtain

$$\int_\Omega \left(\sum_{k=1}^{3} \frac{\partial\psi_i}{\partial x_k}\frac{\partial\delta\vartheta_i}{\partial x_k} + \psi_i\delta\vartheta_i\right) dx = \int_\Omega f_i(x)\delta\vartheta_i dx = .$$

$$= \int_0^T \int_\Omega \frac{\partial u}{\partial x_i}\delta\vartheta_i\psi dx dt$$

Taking into account this equality in (19), we have

$$\left\langle I'(\vartheta), \delta\vartheta\right\rangle = \int_\Omega \sum_{i=1}^{3}(\sum_{k=1}^{3} \frac{\partial\psi_i}{\partial x_k}\frac{\partial\delta\vartheta_i}{\partial x_k} + \psi_i\delta\vartheta_i)dx.$$

Hence it follows that the gradient of functional (6) is determined by equality (34). Theorem 5 is proved. □

The following theorem gives the necessary optimality condition in problem (1)-(3), (5), (6) using the gradient of functional (6).

Theorem 6. *Let the conditions of Theorem 3 be fulfilled. Then for the optimality of the control $\vartheta_* = \vartheta_*(x) \in V$ in problem* (1)-(3), (5), (6) *it is necessary fulfilment of the inequality*

$$\int_\Omega \left(\sum_{i=1}^{3} \nabla\psi_{i*}(x)(\nabla\vartheta_i(x) - \nabla\vartheta_{i*}(x)) + \sum_{i=1}^{3}\psi_{i*}(x)(\vartheta_i(x) - \vartheta_{i*}(x))\right) dx \geq 0$$

for any $\vartheta = \vartheta(x) \in V$, where $\psi_{i*}(x) = \psi_i(x;\vartheta_*)$ is a solution of problem (31), (32) *at $\vartheta = \vartheta_*(x)$.*

The proof of Theorem 6 is quite similar to the proof of Theorem 4 using formula (34).

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