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CERTAIN FINITE INTEGRALS INVOLVING GENERALIZED WRIGHT FUNCTION

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Abstract. The primary goal of this article is to infer new integral formulas for the generalized Wright function and for its auxiliary functions, communicated in terms of generalized Wright hypergeometric function (GWHF) and also in terms of generalized hypergeometric function (HGF). Certain integrals are obtained as a special cases from the main results by specializing the suitable values of the parameters involved.

Keywords: Fox-Wright function, generalized Wright function, generalized hypergeometric function, MacRobert integral formula and Edward double integral formula.

AMS Subject Classification: [2010] 33C20, 33C60, 33C99.

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1 Introduction and Preliminaries

Integral transform techniques are very useful in solving and studying mathematical physics differential and integral equations. These techniques consist of integrating an equation with a certain weight function of two arguments which often lead to the simplification of a given initial issue. The key condition for applying an integral transformation is the validity of the inversion theorem that enables one to find an unknown function that knows its image. Based on a weight function and an integration domain are used the Fourier, Laplace, Mellin, Hankel, Meyer, Hilbert and other transforms. Through these, several transforms can be solved in the theory of oscillation, heat conductivity, neutron diffusion and slow-down, hydrodynamics, elasticity theory, and physical kinetics. From the time of Laplace up to the present time, different theories of integral transforms have been proposed by the integrals with different kernels and ranges of integration. These integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The most useful significance of integral transforms lies in the fact that they transform a class of differential equations into a class of algebraic equations, so that solution of those differential equations can be obtained by easily algebraic methods and by use of results of integral transforms.

Integral transformations appear as an important tool with special functions which deal with the areas of applied science that are most intensively growing. The special function called Wright function and its generalizations has gained considerable prominence and significance in integral and differential equations solutions because of its applications appear naturally.

The Wright generalized hypergeometric function (WGHF) ${}_r\Psi_s[x]$, also called Fox-Wright function (Wright, 1906, 1935, 1940) is defined as:

$$ {}_r\Psi_s[x] = {}_r\Psi_s \left[\begin{array}{c} (\gamma_1, \acute{\gamma_1}), ..., (\gamma_r, \acute{\gamma_r}); \\ (l_1, \acute{l_1}), ..., (l_s, \acute{l_s}); \end{array} x \right] \tag{1} $$

$$ = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + \acute{\gamma_1}k), ..., \Gamma(\gamma_r + \acute{\gamma_r}k)}{\Gamma(l_1 + \acute{l_1}k), ..., \Gamma(l_s + \acute{l_s}k)} \frac{x^k}{k!}, \tag{2} $$

$$ = H^{1,r}_{r,s+1} \left[-x \middle| \begin{array}{c} (1-\gamma_1, \acute{\gamma_1}), ..., (1-\gamma_r, \acute{\gamma_r}) \\ (0,1), (1-l_1, \acute{l_1}), ..., (1-l_s, \acute{l_s}) \end{array} \right], \tag{3} $$

where $H^{1,r}_{r,s+1}[x]$ denotes the Fox-H function (Fox, 1928), coefficients $\gamma'_1, \cdots, \gamma'_r$, $l'_1, \cdots, l'_s \in \mathbb{R}^+$ and the series absolutely converges for all $x \in \mathbb{C}$ when $1 + \sum_{j=1}^{s} l'_j - \sum_{m=1}^{r} \gamma'_m > 0$.

When $\acute{\gamma_1} =, ..., = \acute{\gamma_r} = 1$, $\acute{l_1} =, ..., = \acute{l_s} = 1$ in (1), Fox-Wright function reduces to simpler generalized hypergeometric function ${}_rF_s$[x] (Wright, 1940)

$$ {}_r\Psi_s \left[\begin{array}{c} (\gamma_1, \acute{\gamma_1}), ..., (\gamma_r, \acute{\gamma_r}); \\ (l_1, \acute{l_1}), ..., (l_s, \acute{l_s}); \end{array} x \right] = \frac{\Gamma(\gamma)_1, ..., \Gamma(\gamma)_r}{\Gamma(l)_1, ..., \Gamma(l)_s} \, {}_rF_s(\gamma_1, ..., \gamma_r; l_1, ..., l_s; x). \tag{4} $$

where $(\nu)_n$ is the known Pochhammer's symbol (Rainville, 1960).

Due to its precious position in many fields of science such as mathematical, physical, and engineering sciences, the classical Wright function is considered as an important special function. Also, other special functions related to the Wright function have become essential resource for the scientists and engineers in many areas of applied mathematics. Consequently, several extensions of special functions (such as Wright function, Wright hypergeometric function, Fox-H function, Meijer-G function, etc) are studied during the recent decades.

We begin to recall the classical Wright function $W_{\nu,\mu}(r)$ better-known as Bessel Maitland function (Podlubny, 1999; Kiryakova, 2008)

$$ W_{\nu,\mu}(r) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu n + \mu)} \frac{r^n}{n!}, \qquad \mu \in \mathbb{C}, \ \nu > -1. \tag{5} $$

El-Shahed & Salem (2015) introduced a generalized Wright function as

$$ W^{\gamma,\delta}_{\nu,\mu}(r) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \ \Gamma(\nu n + \mu)} \frac{r^n}{n!}, \tag{6} $$

where $\nu, \ \delta, \mu, \gamma \in \mathbb{C}$; $\nu > -1$, $\delta > 0$ with $r \in \mathbb{C}$ and $|r| < 1$ with $\nu = -1$, $(\gamma)_n$ is the Pochhammer symbol (Rainville, 1960) and $\Gamma(.)$ is the gamma function (Rainville, 1960).

They also introduced the two auxiliary functions of generalized Wright type as

$$ M^{\gamma,\delta}_{\nu}(r) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(1 - \nu(n+1)) \ (\delta)_n} \frac{(-1)^n \ r^n}{n!}, \tag{7} $$

$$ F^{\gamma,\delta}_{\nu}(r) = \sum_{n=1}^{\infty} \frac{(\gamma)_n}{(\delta)_n \ \Gamma(-\nu n)} \frac{(-1)^n \ r^n}{n!}. \tag{8} $$

Lemma 1. *In the following, we mention some relation of the generalized Wright function $W^{\gamma,\delta}_{\nu,\mu}$ with other notable special functions as follows:*

1. *By replacing ν with $-\nu$, r with $-r$, setting $\mu = 1 - \nu$ in (6), we write in view of (7)*

$$ W^{\gamma,\delta}_{-\nu,1-\nu}(-r) = M^{\gamma,\delta}_{\nu}(r). \tag{9} $$

2. By replacing ν with $-\nu$, r with $-r$, setting $\mu = 0$ in (6)*, we write in view of* (8)

$$W^{\gamma,\delta}_{-\nu,0}(-r) = F^{\gamma,\delta}_{\nu}(r). \tag{10}$$

3. By replacing r with $-r$ in (6) *and using* (3)*, we write*

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\, W^{\gamma,\delta}_{\nu,\mu}(-r) = H^{1\ 1}_{1\ 3}\left[r \middle| \begin{matrix} (1-\gamma,1) \\ \\ (0,1),\ (1-\mu,\nu),\ (1-\delta,1) \end{matrix} \right]. \tag{11}$$

4. By replacing r with $-r$, $\nu = 1$ in (6)*, we relate the generalized Wright function with Meijer-G function (Andrews, 1985) as:*

$$W^{\gamma,\delta}_{\nu,\mu}(-r)\, \frac{\Gamma(\gamma)}{\Gamma(\delta)} = G^{1\ 1}_{1\ 3}\left[r \middle| \begin{matrix} 1-\gamma \\ \\ 0, 1-\mu, 1-\delta \end{matrix} \right]. \tag{12}$$

5. By replacing $\nu = 0$ and $\gamma = 1$ in (6)*, we can write*

$$\frac{\Gamma(\mu)}{\Gamma(\delta)}\, W^{1,\delta}_{0,\mu}(r) = E_{1,\delta}(r), \tag{13}$$

where $E_{1,\delta}(r)$ is the classical Mittag-Leffler functions (Wiman, 1905; Mittag-Leffler, 1903).

Lemma 2. *We require for the current investigation, the following result stated by MacRobert (1961) and Edward (1992):*

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\left[at + b(1-t)\right]^{-\zeta-\eta} dt = \frac{1}{a^{\zeta} b^{\eta}}\, \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)}, \tag{14}$$

provided $\mathbb{R}(\eta) > 0$, $\mathbb{R}(\zeta) > 0$, the constants a, b and the expression $at + b(1-t)$, where $0 \le t \le 1$ are non-zero, and

$$\int_0^1 \int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\, ds\, dt = \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)}\,. \tag{15}$$

Several authors including (Choi & Agarwal, 2013) and (Choi et. al., 2014) have been researching integral formulae involving special functions (such as Bessel and generalized Bessel functions) over the past few decades. (El-Shahed & Salem, 2015) obtained an extension of Wright function and found its essential properties such integral transforms, integral representation and its representations in terms of other special functions. In the sequel, (Khan et. al, 2015, 2016a, 2019, 2020a,b,c, 2016b; Kamarujjama & Khan, 2019),(Khan et. al, 2017), (Belafhal et.al, 2020) and (Vinod & Kanak, 2017) developed specific integral formulas that included generalized Bessel-Maitland function, Struve function, Mittag-Leffler function and Whittaker function. Besides, in terms of Meijer G and Fox- H functions, the Wright function can express by considering different values of parameters, so in section 3, we obtained some integral formulas from our key results as special cases.

The paper is organized as follows. In section 1, along with the introduction, we give two lemmas defining the integral formulas and the relationships of the generalized Wright function with other special functions. In section 2, we derive some integral formulas involving generalized Wright function, whose solutions are expressed in terms of generalized Wright hypergeometric function and generalized hypergeometric function. In section 3, we derive several new and interesting integral formulas involving different special functions which occur as special cases of generalized Wright function. In the conclusion section, we remark the flexibility of the generalized Wright function which shows the possibility of obtaining several other novel results involving some well-known special functions.

2 Main Results

Theorem 1. *Let $\gamma, \delta, \zeta, \eta, \nu, \mu \in \mathbb{C}$, then the undermentioned integral formula holds true:*

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\left[at+b(1-t)\right]^{-\zeta-\eta}\ W_{\nu,\mu}^{\gamma,\delta}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m\right]dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{1}{a^\zeta b^\eta}\ {}_3\Psi_3\left[\begin{array}{ccc|} (\gamma,1), & (\zeta,m), & (\eta,m); \\ & & x \\ (\delta,1), & (\mu,\nu), & (\zeta+\eta,2m); \end{array}\right]. \tag{16}$$

where $\mathbb{R}(\delta) > 0,\ \mathbb{R}(\mu) > 0,\ \mathbb{R}(\nu) > 0,\ \mathbb{R}(\gamma) > 0,\ \mathbb{R}(\zeta) > 0,\ \mathbb{R}(\eta) > 0,\ m \in \mathbb{N}$ and the particular expression $at + b(1-t),\ 0 \le t \le 1$ is non zero.

Proof. Denoting left hand side of (16) by I and using (6), we get

$$I = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\left[at+b(1-t)\right]^{-\zeta-\eta}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ (ab)^{mn}\ (1-t)^{mn}(t)^{mn}}{(\delta)_n\ \Gamma(\nu n+\mu)\ [at+b(1-t)]^{2mn}}\frac{x^n}{n!}dt \tag{17}$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_n\ (ab)^{mn}}{(\delta)_n\ \Gamma(\nu n+\mu)}\frac{x^n}{n!}\int_0^1 t^{\zeta+\ mn-1}(1-t)^{\eta+\ mn-1}\left[at+b(1-t)\right]^{-\zeta-\eta-\ 2mn}dt. \tag{18}$$

Using (14) to simplify the above integral, we have

$$I = \frac{1}{a^\zeta b^\eta}\ \frac{\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)\ \Gamma(\zeta+mn)\ \Gamma(\eta+mn)}{\Gamma(\delta+n)\ \Gamma(\mu+\nu n)\ \Gamma(\zeta+\eta+2mn)}\frac{x^n}{n!}, \tag{19}$$

which is on using (2), gives (16). □

Theorem 2. *Let $\nu, \mu, \zeta, \eta, \gamma, \delta \in \mathbb{C}$, then the undermentioned integral formula holds true:*

$$\int_0^1\int_0^1 s^\zeta(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\ W_{\nu,\mu}^{\gamma,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ {}_3\Psi_3\left[\begin{array}{ccc|} (\gamma,1), & (\zeta,m), & (\eta,m); \\ & & x \\ (\delta,m), & (\mu,\nu), & (\zeta+\eta,2m); \end{array}\right]. \tag{20}$$

where $\mathbb{R}(\delta) > 0, \mathbb{R}(\mu) > 0, \mathbb{R}(\nu) > 0, \mathbb{R}(\gamma) > 0, \mathbb{R}(\zeta) > 0, \mathbb{R}(\eta) > 0,\ m \in \mathbb{N}$ and $0 \le t \le 1$.

Proof. Denoting the left side of (20) by I and then using (6), we get

$$I = \int_0^1\int_0^1 s^\zeta(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\ W_{\nu,\mu}^{\gamma,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt \tag{21}$$

$$= \int_0^1\int_0^1 s^\zeta(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ s^{mn}(1-t)^{mn}\ (1-s)^{mn}}{(\delta)_n\ \Gamma(\nu n+\mu)\ (1-st)^{2mn}}\frac{x^n}{n!}\,ds\,dt, \tag{22}$$

which on rearranging, we write

$$I = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\nu n+\mu)\ (\delta)_n}\frac{x^n}{n!}\int_0^1\int_0^1 s^{\zeta+mn}(1-t)^{\zeta+mn-1}(1-s)^{\eta+mn-1}(1-st)^{1-\zeta-\eta+mn}ds\,dt. \tag{23}$$

Using (15) to simplify the above integral, we have

$$I = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\;\Gamma(\zeta+mn)\;\Gamma(\mu+mn)}{\Gamma(\delta+n)\;\Gamma(\mu+\nu n)\;\Gamma(\nu+\mu+2mn)} \frac{x^n}{n!}, \tag{24}$$

which on using (2), yields (20). □

Theorem 3. *Let $\zeta, \eta, \gamma, \delta, \nu, \mu \in \mathbb{C}$ with $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\delta) > 0$, $\mathbb{R}(\mu) > 0$, $\mathbb{R}(\nu) > 0$, $\mathbb{R}(\zeta) > 0$, $\mathbb{R}(\eta) > 0$, $m \in \mathbb{N}$ and the expression $at + b(1-t)$, $0 \le t \le 1$ is non zero, then the undermentioned integral formula holds true:*

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\, W_{\nu,\mu}^{\gamma,\delta}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m\right]dt$$

$$= \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)\;\Gamma(\mu)}\;\frac{1}{a^{\zeta}b^{\eta}}\;\; {}_{2m+1}F_{2m+\nu+1}\left[\begin{matrix} \Delta(m;\zeta), & \Delta(m;\eta), & \gamma; \\ \Delta(\nu;\mu), & \Delta(2m;\zeta+\eta), & \delta; \end{matrix}\;\frac{x}{4^m\nu^\nu}\right], \tag{25}$$

$$\textit{where,} \qquad \Delta(k;\lambda) = \frac{\lambda}{k}.\frac{\lambda+1}{k}\ldots\frac{\lambda+k-1}{k} \qquad (k \ge 1).$$

Proof. In order to prove the required result (25), using the following formulas:

$$\Gamma(\nu+n) = \Gamma(\nu)(\nu)_n \tag{26}$$

and

$$(\lambda)_{kn} = k^{kn}\left(\frac{\lambda}{k}\right)_n\left(\frac{\lambda+1}{k}\right)_n \cdots \left(\frac{\lambda+k-1}{k}\right)_n, \tag{27}$$

and after a little simplification, we can obtain the required result.

□

Theorem 4. *Let $\zeta, \eta, \gamma, \delta, \nu, \mu \in \mathbb{C}$ with $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\delta) > 0$, $\mathbb{R}(\mu) > 0$, $\mathbb{R}(\nu) > 0$, $\mathbb{R}(\zeta) > 0$, $\mathbb{R}(\eta) > 0$, $m \in \mathbb{N}$. Then the undermentioned integral formulas holds true:*

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\; W_{\nu,\mu}^{\gamma,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt$$

$$= \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\mu)\;\Gamma(\zeta+\eta)}\;\; {}_{2m+1}F_{2m+\nu+1}\left[\begin{matrix} \Delta(m;\zeta), & \Delta(m;\eta), & \gamma; \\ \Delta(\nu;\mu), & \Delta(2m;\zeta+\eta), & \delta; \end{matrix}\;\frac{x}{4^m\nu^\nu}\right]. \tag{28}$$

$$\textit{where,} \qquad \Delta(k;\lambda) = \frac{\lambda}{k}.\frac{\lambda+1}{k}\ldots\frac{\lambda+k-1}{k} \qquad (k \ge 1).$$

Proof. This can be illustrated by the same method as in the proof of Theorem 3 . So we're omitting every detail.

3 Special Cases

In this section, we consider new integral formulas that occur as special cases of our main results. With all the conditions stated with Theorem 1, Theorem 2, Theorem 3 and Theorem 4 are satisfied, then the following integral formulas holds true:

(i) On replacing ν with $-\nu$, x with $-x$, setting $\mu = 1-\nu$ in (16) or with the help of (9), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\,M_{\nu}^{\gamma,\delta}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m\right]dt$$

$$= \frac{1}{a^{\zeta}b^{\eta}}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,1), & (\zeta,m), & (\eta,m); \\ (\delta,1), & (1-\nu,-\nu), & (\zeta+\eta,2m);\end{matrix}\;-x\right]. \tag{29}$$

(ii) On replacing ν with $-\nu$, x with $-x$, setting $\mu = 0$ in (16) or with the help of (10), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\,F_{\nu}^{\gamma,\delta}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m\right]dt$$

$$= \frac{1}{a^{\zeta}b^{\eta}}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,1), & (\zeta,m), & (\eta,m); \\ (\delta,1), & (0,-\nu), & (\zeta+\eta,2m);\end{matrix}\;-x\right]. \tag{30}$$

(iii) On replacing x with $-x$ in (16) or with the help of (11), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}$$

$$\times H_{1\,3}^{1\,1}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m \middle|\; \begin{matrix}(1-\gamma,1) \\ (0,1), \; (1-\mu,\nu), \; (1-\delta,1)\end{matrix}\right]dt$$

$$= \frac{1}{a^{\zeta}b^{\eta}}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,1), & (\zeta,m), & (\eta,m); \\ (\delta,1), & (\mu,\nu), & (\zeta+\eta,2m);\end{matrix}\;-x\right]. \tag{31}$$

(iv) On replacing x with $-x$ and $\nu = 1$ in (16) or with the help of (12), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\,G_{1\,3}^{1\,1}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m \middle|\; \begin{matrix}1-\gamma \\ 0, 1-\mu, 1-\delta\end{matrix}\right]dt$$

$$= \frac{1}{a^{\zeta}b^{\eta}}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,1), & (\zeta,m), & (\eta,m); \\ (\delta,1), & (\mu,1), & (\zeta+\eta,2m);\end{matrix}\;-x\right]. \tag{32}$$

(v) On replacing $\nu = 0$ and $\gamma = 1$ in (16) or with the help of (13), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\,E_{1,\delta}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m\right]dt$$

$$= \frac{1}{a^{\zeta}b^{\eta}}\ \Gamma(\mu)\ {}_3\Psi_3\left[\begin{array}{ccc}(1,1), & (\zeta,m), & (\eta,m); \\ & & \\ (\delta,1), & (\mu,0), & (\zeta+\eta,2m);\end{array}\ x\right]. \tag{33}$$

(vi) On replacing ν by $-\nu$, x by $-x$ and $\mu = 1-\nu$ in (20) and using (9), we get

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}M_{\nu}^{\gamma,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ {}_3\Psi_3\left[\begin{array}{ccc}(\gamma,1), & (\zeta,m), & (\eta,m); \\ & & \\ (\delta,m), & (1-\nu,-\nu), & (\zeta+\eta,2m);\end{array}\ -x\right]. \tag{34}$$

(vii) On replacing ν by $-\nu$, x by $-x$, and $\mu = 0$ in (20) and using (10), we get

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}\ F_{\nu}^{\gamma,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ {}_3\Psi_3\left[\begin{array}{ccc}(\gamma,1), & (\zeta,m), & (\eta,m); \\ & & \\ (\delta,m), & (0,-\nu), & (\zeta+\eta,2m);\end{array}\ -x\right]. \tag{35}$$

(viii) On replacing x with $-x$ in (20) and using (11), we get

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}$$

$$\times H_{1\,3}^{1\,1}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\ \middle|\ \begin{array}{c}(1-\gamma,1) \\ \\ (0,1),\ (1-\mu,\nu),\ (1-\delta,1)\end{array}\right]ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ {}_3\Psi_3\left[\begin{array}{ccc}(\gamma,1), & (\zeta,m), & (\eta,m); \\ & & \\ (\delta,m), & (\mu,\nu), & (\zeta+\eta,2m);\end{array}\ -x\right]. \tag{36}$$

(ix) On replacing x by $-x$ and $\nu = 1$ in (20) and in view of (12), we get

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}$$

$$\times G_{1\,3}^{1\,1}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\ \middle|\ \begin{array}{c}1-\gamma \\ \\ 0,\ 1-\mu,\ 1-\delta\end{array}\right]ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ \frac{\Gamma(\delta)}{\Gamma(\gamma)}\ {}_3\Psi_3\left[\begin{array}{ccc}(\gamma,1), & (\zeta,m), & (\eta,m); \\ & & \\ (\delta,m), & (\mu,1), & (\zeta+\eta,2m);\end{array}\ -x\right]. \tag{37}$$

(x) On replacing $\nu = 0$ and $\gamma = 1$ in (20) and using (13), we get

$$\int_0^1\int_0^1 s^{\zeta}(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}E_{1,\delta}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m\right]ds\,dt$$

$$= \Gamma(\mu)\, {}_3\Psi_3 \left[\begin{array}{c} (1,1),\, (\zeta,m),\, (\eta,m); \\ \\ (\delta,m),\, (\mu,0),\, (\zeta+\eta,2m); \end{array} x \right]. \tag{38}$$

(xi) On replacing x by $-x$ in (25) or with the help of (11), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}$$
$$\times H^{1\,1}_{1\,3}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m \middle| \begin{array}{c} (1-\gamma,1) \\ \\ (0,1),\ \ (1-\mu,\nu),\ \ (1-\delta,1) \end{array} \right] dt \tag{39}$$

$$= \frac{1}{a^\zeta b^\eta}\, \frac{\Gamma(\delta)\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\gamma)\Gamma(\mu)\Gamma(\zeta+\eta)}\ {}_{2m+1}F_{2m+\nu+1}\left[\begin{array}{c} \Delta(m;\zeta),\, \Delta(m;\eta),\, \gamma; \\ \\ \Delta(\nu;\mu),\, \Delta(2m;\zeta+\eta),\, \delta; \end{array} \frac{-x}{4^m\nu^\nu} \right]. \tag{40}$$

(xii) On replacing x by $-x$ and $\nu = 1$ in (25) and using (12), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\,[at+b(1-t)]^{-\zeta-\eta}\, G^{1\,1}_{1\,3}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}^m \middle| \begin{array}{c} 1-\gamma \\ \\ 0,\ \ 1-\mu,\ \ 1-\delta \end{array} \right] dt$$

$$= \frac{1}{a^\zeta b^\eta}\, \frac{\Gamma(\delta)}{\Gamma(\gamma)}\, \frac{\Gamma(\zeta)\ \Gamma(\eta)}{\Gamma(\zeta+\eta)\ \Gamma(\mu)}\ {}_{2m+1}F_{2m+2}\left[\begin{array}{c} \Delta(m;\zeta),\ \ \Delta(m;\eta),\ \ \ \gamma; \\ \\ \Delta(1;\mu),\ \ \Delta(2m;\zeta+\eta),\ \ \ \delta; \end{array} \frac{-x}{4^m} \right]. \tag{41}$$

(xiii) On replacing x by $-x$ in (28) and using (11), we get

$$\int_0^1\int_0^1 s^\zeta(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta}$$
$$\times H^{1\,1}_{1\,3}\left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m \middle| \begin{array}{c} (1-\gamma,1) \\ \\ (0,1),\, (1-\mu,\nu),\, (1-\delta,1) \end{array} \right] ds\,dt \tag{42}$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\, \frac{\Gamma(\zeta)\ \Gamma(\eta)}{\Gamma(\zeta+\eta)\ \Gamma(\mu)}\ {}_{2m+1}F_{2m+\nu+1}\left[\begin{array}{c} \Delta(m;\zeta),\ \ \Delta(m;\eta),\ \ \ \gamma; \\ \\ \Delta(\nu;\mu),\ \ \Delta(2m;\zeta+\eta),\ \ \ \delta; \end{array} \frac{-x}{4^m\nu^\nu} \right]. \tag{43}$$

(xiv) On replacing x by $-x$ and $\nu = 1$ in (28) and using (12), we get

$$\int_0^1\int_0^1 s^\zeta(1-t)^{\zeta-1}(1-s)^{\eta-1}(1-st)^{1-\zeta-\eta} G^{1\,1}_{1\,3}\times$$
$$\times \left[x\left\{\frac{s(1-t)(1-s)}{(1-st)^2}\right\}^m \middle| \begin{array}{c} 1-\gamma \\ \\ 0,\ \ 1-\mu,\ \ 1-\delta \end{array} \right] ds\,dt$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\, \frac{\Gamma(\zeta)\ \Gamma(\eta)}{\Gamma(\zeta+\eta)\ \Gamma(\mu)}\ {}_{2m+1}F_{2m+2}\left[\begin{array}{c} \Delta(m;\zeta),\ \ \Delta(m;\eta),\ \ \ \gamma; \\ \\ \Delta(1;\mu),\ \ \Delta(2m;\zeta+\eta),\ \ \ \delta; \end{array} \frac{-x}{4^m} \right]. \tag{44}$$

4 Conclusion

We have investigated a new generalization of Wright function and found their connections with other functions important in the literature of special functions. e.g. Meijer G-function, Fox-H function and Mittag-Leffler function etc. Some known and new interesting results are obtained by specializing the suitable values the parameters involved.
e.g. for $m = 1$ and x replaced by $-x$ in (16), we get

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\left[at+b(1-t)\right]^{-\zeta-\eta}$$
$$\times H^{1\,1}_{1\,3}\left[x\left\{\frac{abt(1-t)}{[at+b(1-t)]^2}\right\}\middle|\begin{array}{l}(1-\gamma,1)\\ \\(0,1),\ (1-\mu,\nu),\,(1-\delta,1)\end{array}\right]dt$$
$$= \frac{1}{a^{\zeta}b^{\eta}}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\,\frac{\Gamma(\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{array}{lll}(\gamma,1), & (\zeta,1), & (\eta,1);\\ & & \\ (\delta,1), & (\mu,\nu), & (\zeta+\eta,2);\end{array}\; -x\right]. \tag{45}$$

and their hypergeometric representation is as follows:

$$= \frac{\Gamma(\delta)\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\gamma)\Gamma(\mu)\Gamma(\zeta+\eta)}\;\frac{1}{a^{\zeta}b^{\eta}}\;\;{}_3F_{3+\nu}\left[\begin{array}{cccc} & \zeta, & \eta, & \gamma;\\ \Delta(\nu;\mu), & \frac{(\zeta+\eta)}{2} & \frac{(\zeta+\eta+1)}{2}, & \delta;\end{array}\;\frac{-x}{4\nu^{\nu}}\right]. \tag{46}$$

Therefore the similar type of new interesting results are obtained from the main results with different arguments. We also conclude from the present analysis that the results obtained in this paper are general and can be specialized in providing additional exciting and potentially useful formulas that include integral transform and special functions.

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