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WEAK SOLUTIONS TO KIRCHHOFF TYPE PROBLEMS VIA TOPOLOGICAL DEGREE

Soukaina Yacini, Chakir Allalou*, **Khalid Hilal, Abderrazak Kassidi**

Laboratory LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco

Abstract. In this article, we study the existence of weak solutions to a nonlinear Dirichlet boundary value problem with a Kirchhoff term. The Berkovits topological degree is applied to an abstract Hammerstein equation to investigate the results, in the framework of weighted Sobolev spaces.

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Corresponding author: Chakir Allalou, Laboratory LMACS, Faculty of Sciences and Technologies, Beni-Mellal, Morocco, e-mail: *chakir.allalou@yahoo.fr*
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1 Introduction

In this paper, we are interested in the following Kirchhoff type problem with the Dirichlet boundary value condition

$$\begin{cases} -M\Big(\displaystyle\int_\Omega (A(x,\nabla u) + \frac{1}{q}|u|^q)\,dx\Big)\Big[\operatorname{div} a(x,\nabla u) - |u|^{q-2}u\Big] = \lambda g(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a Lipschitz bounded open domain in $\mathbb{R}^N$, $N \geqslant 1$, and $2 < q < p < \infty$, is a positive real number, λ is a real parameter, $-\operatorname{div} a(x,\nabla u)$ is a Leray-Lions operator, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory's function meets a few conditions that we will review in the section 3. g is defined from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ is a Carathéodory's functions has the growth condition and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Because of the term M, the problem (1) is called a nonlocal issue, implying that the equation in (1) is no longer a pointwise equation. Problem (1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

presented by Kirchhoff (1883). The parameters in (2) have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and P_0 is the initial tension. The Kirchhoff equation (2) is an extension of the classical d'Alembert's wave equation that takes into account the effects of length changes of the string produced by transverse vibrations. This (1) problem models several physical and biological systems where u describes a process which depends on the average of itself, such as the population identity, see Alves et al. (2005). Many authors have investigated Kirchhoff type equations, particularly after Lions work (Lions, 1978), where a functional analysis framework for the problem was proposed; see e.g.(Arosio & Panizzi, 1996; Rasheed et al., 2021; Cavalcanti et al., 2001;

Allalou et al., 2021; Chakib et al., 2016; Chipot & Rodrigues, 1992; Nachaoui et al., 2021b)) for some interesting results and further references. In recent years, various Kirchhoff-type problems have been discussed in many papers see (Temghart et al., 2021; Faraci & Farkas, 2020; Heidarkhani et al., 2017; Fan, 2015). More recently, Cabanillas et al. (2018) have dealt with the $p(x)$-Kirchhoff type equation by topological methods, in the framework of variable exponent Sobolev space.

Inspired by the above work and the results in Cabanillas et al. (2018); Chaharlang & Razani (2021); Abbassi et al. (2020a), we study the existence of a weak solution to the problem (1). The method used to solve (1) is the topological degree, for a class of demicontinuous operators of generalized (S_+) type. The topological degree theory is introduce the first time by Leray & Schauder (1934) in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder (1983) constructed a topological degree for operators of class (S_+) in reflexive Banach spaces. We mention the works Abbassi et al. (2021, 2020b); Cho & Chen (2006); Chaharlang et al. (2020); Chaharlang & Razani (2021); Ellabib et al. (2021); Hayeck et al. (1990); Heidarkhani et al. (2017); Nachaoui et al. (2021a) for more details.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on weighted Sobolev spaces and an outline of Berkovits degree theory. In section 3 we give our basic assumption. In section 4 is devoted we prepare for the proof of the main theorem, we present several similar lemmas. Finally, in the fourth section, we prove the existence of weak solutions of (1).

2 Mathematical preliminaries

2.1 Weighted Lebesgue and Sobolev spaces

In this subsection, we give some definitions and elementary properties for weighted Sobolev spaces with Weight $W^{1,p}(\Omega, w)$.

Let Ω a bounded open set of $\mathbb{R}^N (N \geq 1)$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose for any $0 \leq i \leq N$ in all our considerations that

$$w_i \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega). \tag{3}$$

The weighted Sobolev space $W^{1,p}(\Omega, w)$ is defined as

$$W^{1,p}(\Omega, w) = \Big\{ u \in L^p(\Omega, w_0) \,:\, \partial_i u \in L^p(\Omega, w_i), \quad i = 1, ..., N \Big\}.$$

Note that the derivatives $\partial_i = \dfrac{\partial}{\partial x_i}$ are understood in the sense of distributions. This set of functions forms a Banach space under the standard norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx \Big)^{1/p}. \tag{4}$$

The first condition in (3), we have that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (4). Moreover, the second condition in (3) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0, \ldots, N\}$, p' is the conjugate of p and $p' = \frac{p}{p-1}$. (see Akdim et al. (2001); Akdim & Allalou (2014) for more details).

Now, we shall assume that the expression

$$\|u\| = \Big(\sum_{i=1}^{N}\int_{\Omega}|\partial_i u(x)|^p\, w_i(x)\, dx\Big)^{1/p} \tag{5}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ equivalent to the norm (4).

Note that $(W_0^{1,p}(\Omega, w), \|\cdot\|)$ is a uniformly convex (and thus reflexive) Banach space.

We can find a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{-\frac{p}{(q-1)(q-p)}} \in L^1(\Omega) \tag{6}$$

then the Hardy inequality

$$\Big(\int_{\Omega}|u|^q\sigma\, dx\Big)^{1/q} \le c\Big(\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^p w_i\, dx\Big)^{1/p} \tag{7}$$

holds for any $u \in W_0^{1,p}(\Omega, w)$ with a constant $c > 0$ independent of u, besides the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma) \tag{8}$$

expressed by (8) is compact.

Remark 1. *If we suppose that $w_0(x) \equiv 1$, the integrability condition holds: there exists $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ such that*

$$w_i^{-\nu} \in L^1(\Omega), \text{ for all } i = 1...N \tag{9}$$

and note that the assumption (9) is stronger than (3), then

$$\|u\| = \Big(\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^p\, w_i(x)\, dx\Big)^{1/p} \tag{10}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and equivalent to (4), and also, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega) \tag{11}$$

is compact for all $1 \le q \le p_1^*$ if $p\nu < N(\nu+1)$, and for all $q \ge 1$ if $p\nu \ge N(\nu+1)$, where $p_1 = p\nu/\nu + 1$ and p_1^* is the Sobolev conjugate of p_1 (see Drabek et al. (1996), pp.30-31).

2.2 Classes of mappings and topological degree

Now, we give some results and properties from the Berkovits degree theory, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot,\cdot\rangle$ and given a nonempty subset Ω of X, and $\rightharpoonup$ represents the weak convergence.

Let Y be another real Banach space.

Definition 1. *The operator $F : \Omega \subset X \to Y$ is said to be bounded, if it takes any bounded set into a bounded set.*

Definition 2. *The operator $F : \Omega \subset X \to Y$ is said to be demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.*

Definition 3. *The operator $F : \Omega \subset X \to Y$ is said to be compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 4. *A mapping* $F : \Omega \subset X \to X^*$ *is said to be of type* (S_+), *if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$ *and* $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, *we have* $u_n \to u$.

Definition 5. *The operator* $F : \Omega \subset X \to X^*$ *is said to be quasimonotone, if* $u_n \rightharpoonup u$ *implies* $\limsup\limits_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 6. *Let* $T : \Omega_1 \subset X \to X^*$ *be a bounded operator such that* $\Omega \subset \Omega_1$. *For any operator* $F : \Omega \subset X \to X$, *we say that* F *satisfies condition* $(S_+)_T$, *if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ *and* $\limsup\limits_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, *we have* $u_n \to u$.

Remark 2. *(Zeider, 1990)*

1. *If a mapping is compact in a set, then it is quasi-monotone in that set.*
2. *If the mapping is demi-continuous and of type* (S_+) *in a set, then it is quasimonotone in that set.*

In the following, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \big\{F : \Omega \to X^* \mid\ F \text{ is bounded, demicontinuous and of type}\,(S_+)\big\},$$
$$\mathcal{F}_T(\Omega) := \big\{F : \Omega \to X \mid\ F \text{ is demicontinuous and satifies condition } (S_+)_T\big\}.$$

Lemma 1. *(Berkovits, 2007, Lemmas 2.2 and 2.4) Lets* $T \in \mathcal{F}_1(\overline{G})$ *be continuous and* S: $D_S \subset X^* \to X$ *be demicontinuous such that* $T(\overline{G}) \subset D_s$, *where* G *is a bounded open set in a real reflexive Banach space* X. *Then the following statements are true:*

1. If S *is quasimonotone, then* $I + SoT \in \mathcal{F}_T(\overline{G})$, *where* I *denotes the identity operator.*
2. *If* S *is of type* (S_+), *then* $SoT \in \mathcal{F}_T(\overline{G})$.

Definition 7. *Let* G *is to be a bounded open subset of a real reflexive Banach space* X,$T \in \mathcal{F}_1(\overline{G})$ *be continuous and let* $F, S \in \mathcal{F}_T(\overline{G})$. *We define an affine homotopy* $\Lambda : [0,1] \times \overline{G} \to X$ *by*

$$\Lambda(t,u) := (1-t)Fu + tSu \qquad \textit{for} \qquad (t,u) \in [0,1] \times \overline{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3. *(Berkovits, 2007)* The above affine homotopy satisfies condition $(S_+)_T$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. we give the Berkovits topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$ for more details see Berkovits (2007).

Theorem 1. *Let*

$$M = \big\{(F,G,h) | G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ h \notin F(\partial E)\big\}.$$

There exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ *which satisfies the following properties :*

1. *(* Normalization*) For any* $h \in G$, *we have* $d(I,E,h) = 1$.
2. *(* Additivity*) Let* $F \in \mathcal{F}_T(\overline{G})$. *If* G_1 *and* G_2 *are two disjoint open subsets of* G *such that* $h \notin F(\overline{G}\backslash(G_1 \cup G_2))$ *then we have*
$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h)\ .$$
3. *(* Homotopy invariance*) If* $\Lambda : [0,1] \times \overline{G} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* h*:* $[0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \Lambda(t, \partial G)$ *for all* $t \in [0,1]$, *then the value of* $d(\Lambda(t,\cdot), G, h(t))$ *is constant for all* $t \in [0,1]$.
4. *(* Existence*) if* $d(F,G,h) \neq 0$, *then the equation* $Fu = h$ *has a solution in* G.

3 Essential Assumptions

Throughout the paper, we assume that the following assumption hold true.
$a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory vector-valued function, such that $a(x,\xi) = \nabla_\xi A(x,\xi)$, where $A(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$. Suppose that a and A satisfy the following hypotheses, for a. e. in $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^N$, $(\xi \neq \xi')$:

(A_1) $\quad A(x,0) = 0,$

(A_2) $$\alpha \sum_{i=1}^{N} w_i |\xi_i|^p \leq a(x,\xi)\cdot\xi \leq pA(x,\xi),$$

(A_3) $$|a_i(x,\xi)| \leq \beta\, w_i^{1/p}\Big(k(x) + \sum_{j=1}^{N} w_j^{1/p'}\,|\xi_j|^{p-1}\Big) \qquad \text{for all } i = 1,\cdots,N.$$

(A_4) $$[a(x,\xi) - a(x,\xi')]\cdot(\xi - \xi') > 0,$$

where α, β are some positive constants and $k(x)$ is a positive function in $L^{p'}(\Omega)$(p' is the conjugate exponent of p). Furthermore, the Carathéodory's function g is defined from $\Omega\times\mathbb{R}\times\mathbb{R}^N$ into $\mathbb{R}$ and it is satisfies only the growth condition, for all $t \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

(H_1) $$|g(x,s,\eta)| \leq \varrho\sigma^{1/q}\Big(e(x) + \sigma^{1/q'}|s|^{q/q'} + \sum_{j=1}^{N} w_j^{1/q'}|\eta_j|^{p/q'}\Big),$$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{q'}(\Omega)$.

(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function, for which there exist two positive constant m_0 and m_1 such that $m_0 \leq M(t) \leq m_1$ for all $t \in [0,+\infty[$.

4 Notions of solutions and technical Lemmas

Definition 8. *We say a function $u \in W_0^{1,p}(\Omega,w)$ is to weak solution of problem* (1) *if*

$$M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{q}|u|^q)\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |u|^{q-2}uv\Big] = \lambda\int_\Omega g(x,\ u,\nabla u)v\,dx,$$

for all $v \in W_0^{1,p}(\Omega,w)$.

Lemma 2. *(Akdim et al., 2001) Let $g \in L^r(\Omega,\nu)$ and $g_n \subset L^r(\Omega,\nu)$ be such that $\|g_n\|_{r,\nu} \leq C$, $1 < r < \infty$, If $g_n(x) \to g(x)$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^r(\Omega,\nu)$, where ν is a weight function on Ω.*

Lemma 3. *(Akdim et al., 2001) Assume that (A_2)-(A_4) hold, let $(u_n)_n$ be a sequence in $W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and*

$$\int_\Omega \big[a(x,\nabla u_n) - a(x,\nabla u)\big]\nabla(u_n - u)dx \longrightarrow 0, \tag{12}$$

then $u_n \longrightarrow u$ strongly in $W_0^{1,p}(\Omega,w)$.

Let us consider the following functional

$$\mathcal{E}(u) = \widehat{M}\Big(\int_\Omega \big(A(x,\nabla u) + \frac{1}{q}|u|^q\big)dx\Big), \qquad \text{for all } u \in W_0^{1,p}(\Omega,w)$$

where $\widehat{M}: [0,+\infty[\longrightarrow [0,+\infty[$ be the primitive of the function M, defned by

$$\widehat{M}(t)=\int_0^t M(\xi)d\xi.$$

It is well known that $\mathcal{E}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u\in W_0^{1,p}(\Omega,w)$ is the functional $\mathcal{E}'(u)\in W^{-1,p'}(\Omega,w^*)$ setting by

$$\langle \mathcal{E}'(u)\,,\,v\rangle=\langle Fu\,,\,v\rangle,\qquad \text{for all } u,v\in W_0^{1,p}(\Omega,w),$$

where the operator F acting from $W_0^{1,p}(\Omega,w)$ to its dual $W^{-1,p'}(\Omega,w^*)$ is defined by

$$\langle Fu\,,\,v\rangle=M\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{q}|u|^q)\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v+\int_\Omega |u|^{q-2}uv\Big] \tag{13}$$

for all $u,v\in W_0^{1,p}(\Omega,w)$.

Proposition 1. *Suppose that (M_0),$(A_1)-(A_4)$ hold, then*

(i) F is bounded, strictly monotone, coercive, continuous operator.

(ii) F is of type (S_+).

Proof. $i)$ It is clear that F is continuous, because F is the Fréchet derivative of $\mathcal{E}$.

Now, we prove that the operator F is bounded.

Let $u,v\in W_0^{1,p}(\Omega,w)$, by the Hölder's inequality and (M_0), we obtain

$$\begin{aligned}
|< Fu,v >| &= \Big|M\Big(\int_\Omega \big(A(x,\nabla u)+\frac{1}{q}|u|^q\big)\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v+\int_\Omega |u|^{q-2}uvdx\Big]\Big|\\
&\le m_1\Big(\sum_{i=1}^N\int_\Omega a_i(x,\nabla u)w_i^{-1/p}\,\partial_i v\, w_i^{1/p}dx+\int_\Omega |u|^{q-1}\sigma^{-1/q}|v|\sigma^{1/q}dx\Big)\\
&\le m_1\sum_{i=1}^N\Big(\int_\Omega |a_i(x,\nabla u)\,w_i^{-1/p}|^{p'}dx\Big)^{1/p'}\Big(\int_\Omega |\partial_i v\, w_i^{1/p}\,|^p dx\Big)^{1/p}\\
&+m_1\Big(\int_\Omega ||u|^{q-1}\sigma^{-1/q}|^{q'}dx\Big)^{1/q'}\Big(\int_\Omega ||v|\sigma^{1/q}|^q\Big)^{1/q}dx\\
&\le m_1\Big(\sum_{i=1}^N\int_\Omega \big|a_i(x,\nabla u)\,w_i^{-1/p}\big|^{p'}dx\Big)^{1/p'}\Big(\sum_{i=1}^N\int_\Omega |\partial_i v|^p w_i dx\Big)^{1/p}\\
&+m_1\Big(\int_\Omega |u|^q\sigma^{-q'/q}dx\Big)^{1/q'}\Big(\int_\Omega |v|^q\sigma dx\Big)^{1/q}\\
&\le m_1\Big(\sum_{i=1}^N\int_\Omega \big|a_i(x,\nabla u)\,w_i^{-1/p}\big|^{p'}dx\Big)^{1/p'}\Big(\sum_{i=1}^N\int_\Omega |\partial_i v|^p w_i dx\Big)^{1/p}\\
&+m_1\Big(\big(\int_\Omega |u|^{q\times\frac{p}{q}}dx\big)^{\frac{q}{p}}\big)\big(\int_\Omega |\,\sigma^{-q'/q}\,|^{\frac{p}{p-q}}\,dx\big)^{\frac{p-q}{p}}\Big)^{1/q'}\Big(\int_\Omega |v|^q\sigma dx\Big)^{1/q}\\
&\le m_1\Big(\sum_{i=1}^N\int_\Omega \big|a_i(x,\nabla u)\,w_i^{-1/p}\big|^{p'}dx\Big)^{1/p'}\Big(\sum_{i=1}^N\int_\Omega |\partial_i v|^p w_i dx\Big)^{1/p}\\
&+m_1\Big(\int_\Omega |u|^p dx\Big)^{\frac{q}{pq'}}\Big(\int_\Omega \sigma^{-\frac{p}{(q-1)(p-q)}}\,dx\Big)^{\frac{p-q}{pq'}}\Big(\int_\Omega |v|^q\sigma dx\Big)^{1/q}.
\end{aligned}$$

Thanks to the Hardy inequality (7) and the growth condition (A_2), we can easily show that $\Big(\sum_{i=1}^{N}\int_\Omega \big|a_i(x,\nabla u)\,w_i^{-1/p}\big|^{p'}dx\Big)^{1/p'}$ is bounded for all u in $W_0^{1,p}(\Omega,w)$, and by using the condition (6). We have

$$\begin{aligned}|\langle Fu,v\rangle| &\le C_1\Big(\sum_{i=1}^{N}\int_\Omega|\partial_i v|^p w_i dx\Big)^{1/p}+C_2\|u\|_p^{q/q'}\Big(\sum_{i=1}^{N}\int_\Omega|\partial_i v|^p w_i dx\Big)^{1/p}\\ &\le C_1\|v\|+C_3\|v\|\\ &\le \mathit{const}\,\|v\|\,,\end{aligned}$$

as a result the operator F is bounded.

Next, we prove that F is strictly monotone operator.

For that, we consider the functional $L:\,W_0^{1,p}(\Omega,w)\to\mathbb{R}$ setting by

$$L(u)=\int_\Omega\Big(A(x,\nabla u)+\frac{1}{q}|u|^q\Big)dx\qquad\text{for all}\qquad u\in V,$$

then $L\in C^1(W_0^{1,p}(\Omega,w),\mathbb{R})$ and

$$\langle L'(u),v\rangle=\int_\Omega a(x,\nabla u)\nabla v dx+\int_\Omega |u|^{q-2}uv dx\qquad\text{for all }\ u,\,v\in V.$$

By using (A_4), and taking into the inequality (see Kichenassamy & Veron (1985)), For all $\xi,\eta\in\mathbb{R}^N$,

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\ge\Big(\frac{1}{2}\Big)^p|\xi-\eta|^p,\qquad p\ge 2.\tag{14}$$

We obtain for all $u,\,v\in W_0^{1,p}(\Omega,w)$ with $u\neq v$

$$\langle L'(u)-L'(v),u-v\rangle>0,$$

which implies that L' is strictly monotone. Thus, by Prop. 25.10 in Zeider (1990), L is strictly convex. Furthermore, as M is nondecreasing, then $\widehat{M}$ is convex in $[0,+\infty[$. So, for any $u,v\in X$ with $u\neq v$, and every $s,t\in(0,1)$ with $s+t=1$, we have

$$\widehat{M}(L(su+tv))<\widehat{M}(sL(u)+tL(v))\le s\widehat{M}(L(u))+t\widehat{M}(L(v)).$$

This proves that $\mathcal{E}$ is strictly convex, since $\mathcal{E}'(u)=F(u)$ in $W^{-1,p'}(\Omega,w^*)$ we infer that F is strictly monotone in $W_0^{1,p}(\Omega,w)$. Next, we prove that the operator F is coercive. Let $u\in W_0^{1,p}(\Omega,w)$, according to (A_2) and (M_0), we obtain

$$\begin{aligned}\frac{\langle Fu,u\rangle}{\|u\|}&=\frac{M\Big(\int_\Omega(A(x,\nabla u)+\frac{1}{q}|u|^q)\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla u+\int_\Omega|u|^q dx\Big]}{\|u\|}\\ &\ge\alpha\,m_0\,\frac{\sum_{i=1}^{N}\int_\Omega|\partial_i u|^p\,w_i dx}{\|u\|}+m_0\frac{\int_\Omega|u|^q dx}{\|u\|}\\ &\ge\alpha\,m_0\|u\|^{p-1},\end{aligned}$$

which means that

$$\frac{\langle Fu,u\rangle}{\|u\|}\to\infty\quad\text{as}\quad\|u\|\to\infty,$$

therefore F is coercive.

$ii)-$ We verify that the operator F is of type (S_+). Let $(u_n)_n$ be a sequence in $W_0^{1,p}(\Omega,w)$ such that

$$\begin{cases} u_n \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega,w) \\ \limsup\limits_{n\to\infty}\langle Au_n, u_n-u\rangle \leq 0. \end{cases} \tag{15}$$

We will show that $u_n \to u$ in $W_0^{1,p}(\Omega,w)$.

On the one hand, in fact $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,w)$, so $(u_n)_n$ is a bounded sequence in $W_0^{1,p}(\Omega,w)$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,w)$, under the strict monotonicity of F we get

$$\limsup_{n\to\infty}\langle Fu_n - Fu,\ u_n-u\rangle = \lim_{n\to\infty}\langle Fu_n - Fu,\ u_n-u\rangle = 0. \tag{16}$$

Then

$$\lim_{n\to\infty}\langle Fu_n,\ u_n-u\rangle = 0,$$

which means

$$\lim_{n\to\infty} M\Big(\int_\Omega (A(x,\nabla u_n)+\frac{1}{q}|u_n|^q)\,dx\Big)\Big[\int_\Omega a(x,\nabla u_n)\nabla(u_n-u)+\int_\Omega |u_n|^{q-2}u_n(u_n-u)dx\Big] = 0. \tag{17}$$

On the other hand, by (A_1) we have for any $x\in\Omega$ and $\xi\in\mathbb{R}^n$

$$A(x,\xi) = \int_0^1 \frac{d}{ds}A(x,s\xi)ds = \int_0^1 a(x,s\xi)\xi ds.$$

By combining (A_3), Fubini's theorem and Young's inequality we have

$$\begin{aligned}
\int_\Omega A(x,\nabla u_n)dx &= \int_\Omega\int_0^1 a(x,s\nabla u_n)\nabla u_n ds\,dx \\
&= \int_0^1\int_\Omega a(x,s\nabla u_n)\nabla u_n dx\,ds \\
&= \int_0^1 \Big[\sum_{i=1}^N \int_\Omega a_i(x,s\nabla u_n)w_i^{-1/p}\partial_i u_n w_i^{1/p}dx\Big]ds \\
&\leq \int_0^1 \Big[C_{p'}\sum_{i=1}^N\int_\Omega \big|a_i(x,s\nabla u_n)\big|^{p'}w_i^{-p'/p}dx + C_p\sum_{i=1}^N\int_\Omega|\partial_i u_n|^p w_i dx\Big]ds \\
&\leq C_1 + C'\int_0^1\sum_{i=1}^N\int_\Omega |s\,\partial_i u_n|^p w_i\,dx\,ds + C_p\|u_n\|^p \\
&\leq C_1 + C_2\sum_{i=1}^N\int_\Omega|\partial_i u_n|^p w_i\,dx + C_p\|u_n\|^p \\
&\leq C\Big(\|u_n\|^p+1\Big).
\end{aligned}$$

and $\frac{1}{q}\int_\Omega \big|u_n\big|^q dx$ is bounded.

Then, we infer that $\Big(\int_\Omega \big(A(x,\nabla u_n)+\frac{1}{q}|u_n|^q\big)dx\Big)_{n\geq1}$ is bounded.

As M is continuous, up to a subsequence there is $t_0\geq 0$ such that

$$M\Big(\int_\Omega (A(x,\nabla u_n)+\frac{1}{q}\mid u_n\mid^q)dx\Big) \longrightarrow M(t_0)\geq m_0 \qquad \text{as} \qquad n\to\infty. \tag{18}$$

From (17) and (18), we get

$$\Big[\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\nabla(u_n-u)dx+\int_\Omega |u_n|^{q-2}u_n(u_n-u)dx\Big]=0.$$

Using the compact embedding $W_0^{1,p}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, we have $\lim_{n\to\infty}\int_\Omega |u_n|^{q-2}u_n(u_n-u)dx=0$ Then

$$\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\nabla(u_n-u)dx=0$$

In light of Lemma 3, we obtain

$$u_n\longrightarrow u \qquad \text{strongly in } \ W_0^{1,p}(\Omega,w),$$

which implies that F is of type (S_+).

□

Lemma 4. *If the condition (H_1) hold, then the operator $S: W_0^{1,p}(\Omega,w)\to W^{-1,p'}(\Omega,w^*)$ defined by*

$$\langle S\,u,\ v\rangle=-\lambda\int_\Omega g(x,u,\nabla u)\,v\,dx, \qquad \textit{for all } u,\ v\in W_0^{1,p}(\Omega,w)$$

is compact.

Proof. Let us consider the operator $\phi:\ W_0^{1,p}(\Omega,w)\to L^{q'}(\Omega,\sigma^*)$ as follows

$$\psi u(x):=-\lambda g(x,u,\nabla u) \quad \text{for} \quad u\in W_0^{1,p}(\Omega,w)) \quad \text{and} \quad x\in\Omega$$

is bounded and continuous. For this, let $u\in W_0^{1,p}(\Omega,w)$, using the growth condition (H_1) we have

$$\begin{aligned}
\|\psi\,u\|_{q',\sigma^*}^{q'} &\le \int_\Omega |\lambda g(x,u,\nabla u)|^{q'}\sigma^* dx\\
&\le (\varrho\lambda)^{q'}\int_\Omega \sigma^{q'/q}\big(e(x)+|u|^q\sigma+\sum_{i=1}^N w_i|\partial_i u|^p\big)\sigma^{1-q'}dx\\
&\le (\varrho\lambda)^{q'}\int_\Omega e(x)^{q'}dx+(\varrho\lambda)^{q'}\Big(\int_\Omega |u|^q\sigma dx+\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i\,dx\Big)\\
&\le (\varrho\lambda)^{q'}\int_\Omega e(x)^{q'}dx+C\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i\,dx+(\varrho\lambda)^{q'}\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i\,dx\\
&\le (\varrho\lambda)^{q'}\|e\|_{q'}^{q'}+C\|u\|^p+(\varrho\lambda)^{q'}\|u\|^p\\
&\le C_T(\|u\|^p+1),
\end{aligned} \tag{19}$$

where $C_T=\max\big((\varrho\lambda)^{q'}\|e\|_{q'}^{q'},(\varrho\lambda)^{q'}+C\big)$.

Therefore ψ is bounded on $W_0^{1,p}(\Omega,w)$.

It remains to prove that ψ is continuous, let $u_n\to u$ in $W_0^{1,p}(\Omega,w)$, then $u_n\to u$ in $L^p(\Omega,w_0)$ and $\nabla u_n\to\nabla u$ in $\prod_{i=1}^N L^p(\Omega,w_i)$. Hence, there exist a subsequence still denoted by (u_n) and measurable functions θ in $L^p(\Omega,w_0)$ and ς in $\prod_{i=1}^N L^p(\Omega,w_i)$ such that

$$u_n(x)\to u(x) \quad \text{and} \quad \nabla u_n(x)\to\nabla u(x),$$

$$|u_n(x)| \leq \theta(x) \quad \text{and} \quad |\nabla u_n(x)| \leq |\varsigma(x)|$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since g satisfies the Carathéodory condition, we obtain

$$g(x, u_n(x), \nabla u_n(x)) \to g(x, u(x), \nabla u(x)) \quad \text{a.e. } x \in \Omega. \tag{20}$$

By (H_1) we get that

$$|g(x, u_n(x), \nabla u_n(x))| \leq \varrho\sigma^{1/q}(e(x) + \sigma^{1/q'}|\theta(x)|^{q/q'} + \sum_{i=1}^{N} w_i^{1/q'}|\varsigma(x)|^{p/q'})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

As we have

$$\varrho\sigma^{1/q}(e(x) + \sigma^{1/q'}|\theta(x)|^{q/q'} + \sum_{i=1}^{N} w_i^{1/q'}|\varsigma(x)|^{p/q'}) \in L^{q'}(\Omega, \sigma^*),$$

then, by using (20), we obtain

$$\int_\Omega |g(x, u_k(x), \nabla u_k(x)) - g(x, u(x), \nabla u(x))|^{p'}\sigma^* dx \longrightarrow 0.$$

Applying the dominated convergence Theorem, we get

$$\psi u_k \to \psi u \quad \text{in} \quad L^{q'}(\Omega, \sigma^*).$$

Then, the entire sequence $\psi u_n)$ converges to ψu in $L^{q'}(\Omega, \sigma^*)$ and then ψ is continuous.
Since the embedding $I : W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma)$ is compact, in light of (Rudin, 1991, Theorem 4.19) the adjoint operator $I^* : L^{q'}(\Omega, \sigma^*) \hookrightarrow W^{-1,p'}(\Omega, w^*)$ is also compact.
It follows that $S = I^* o\psi$ is compact. □

5 Main results

Theorem 2. *Suppose that the hypotheses* $(A_1) - (A_4)$*,* (H_1) *and* (M_0) *hold. Then, problem* (1) *has a weak solution* u *in* $W_0^{1,p}(\Omega, w)$*.*

Proof. Let $u \in W_0^{1,p}(\Omega, w)$ be a weak solutions of the problem (1) if and only if

$$Fu = -Su, \tag{21}$$

where F, S be two operators as defined in (13) and Lemma 4 respectively.

On the one hand, from Proposition 1 the operator F given in (13) is strictly monotone, bounded, continuous, coercive and satisfies condition (S_+). Then, by using the Minty-Browder Theorem (see Zeider (1990), Theorem 26 A), the inverse operator $T := F^{-1} : W^{-1,p'}(\Omega, w^*) \to W_0^{1,p}(\Omega, w)$ exists and is bounded. Moreover, it is continuous and satisfies condition (S_+).
On the other hand, notice by Lemma 4 the operator S is bounded, quasimonotone and continuous.

Hence, equation (21) is equivalent to the abstract Hammerstein equation

$$u = Tv \quad \text{and} \quad v + SoTv = 0. \tag{22}$$

To solve the equations (22), we will employ the Berkovits topological degree seen in section above. For this, let us consider the set

$$B := \{ v \in W^{-1,p'}(\Omega, w^*) \mid \quad v + tSoTv = 0 \quad \text{for some} \quad t \in [0,1]\}.$$

We first proove that that the set B is bounded in $W^{-1,p'}(\Omega, w^*)$.

Let $v \in B$ and take $u := Tv$. According to (A_2), (H_1) and by the Hardy inequality and the Young's inequality, we obtain

$$
\begin{aligned}
m_0\|Tv\|^p = m_0 \sum_{i=1}^{N} \int_\Omega |\nabla u|^p \, w_i \, dx &\le \frac{1}{\alpha} M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{q}|u|^q) \, dx \Big) \Big[\int_\Omega a(x,\nabla u)\nabla u + \int_\Omega |u|^q \Big] \\
&= \frac{1}{\alpha}\langle Fu, u\rangle = \frac{1}{\alpha}\langle v, Tv\rangle \\
&\le \ \frac{t}{\alpha}|\langle S\circ Tv, Tv\rangle| \le \frac{t}{\alpha}\int_\Omega |\lambda g(x,u,\nabla u)| \, u \, dx \\
&\le \frac{|\lambda| t}{\alpha}\int_\Omega \varrho \sigma^{1/q}(e(x) + \sigma^{1/q'}|u|^{q/q'} + \sum_{i=1}^{N} w_i^{1/q'}|\partial_i u|^{p/q'}) u dx \\
&\le C_{q'}\int_\Omega \Big| e(x) + |u|^{\frac{q}{q'}}\sigma^{\frac{1}{q'}} + \sum_{i=1}^{N} |\partial_i u|^{\frac{p}{q'}} w_i^{\frac{1}{q'}} \Big|^{q'} dx + C_q \int_\Omega \Big| u\sigma^{\frac{1}{q}}\Big|^q dx \\
&\le C_1 \int_\Omega |e(x)|^{q'} + |u|^q\sigma + \sum_{i=1}^{N} |\partial_i u|^p w_i dx + C_q \int_\Omega |u|^q \sigma dx \\
&\le C_1||e||_{q'}^{q'} + C_1\int_\Omega |u|^q\sigma dx + C_1\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx + C_q\int_\Omega |u|^q\sigma dx \\
&\le C_1||e||_{q'}^{q'} + C_2\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx\Big)^{q/p} + C_1\sum_{i=1}^{N}\int_\Omega |\nabla u|^p w_i dx + C_3\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx\Big)^{q/p} \\
&\le C_0 + C_2\|u\|^q + C_1\|u\|^p + C_3\|u\|^q \\
&\le C_0 + Cst\big(\|Tv\|^q + \|Tv\|^p\big) \\
&\le C_m\big(\|Tv\|^q + \|Tv\|^p + 1\big).
\end{aligned}
$$

This implies that $\{Tv \setminus v \in B\}$ is bounded.
Since the operator S is bounded and from (22), it follows that the set B is bounded in $W^{-1,p'}(\Omega, w^*)$. Then, there exists a positive constant R such that

$$\|v\|_{W^{-1,p'}(\Omega,w^*)} < R \quad \text{for all} \quad v \in B.$$

Thus

$$v + tSoTv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

By Lemma 1, we get

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = FoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Consider an affine homotopy Λ from $[0,1]\times\overline{B_R(0)}$ into $W^{-1,p'}(\Omega, w^*)$ by

$$\Lambda(t,v) := v + tSoTv \quad \text{for} \quad (t,v) \in [0,1]\times\overline{B_R(0)}.$$

Using the homotopy invariance and normalization property of the degree d stated in Theorem 1, we have

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

consequently, we can find a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

it follows that $u = Tv$ is a weak solution of (1). This ends the proof. $\square$

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