Advanced Mathematical Models & Applications
Vol.6, No.3, 2021, pp.252-265

OPTIMAL CONTROL PROBLEM FOR THE STATIONARY QUASI-OPTICS EQUATION WITH A SPECIAL GRADIENT TERM

G. Yakub[1], **N.S. Ibrahimov**[2*], **M. Zengin**[1]

[1]Kafkas University, Kars, Turkey
[2]Lankaran State University, Lankaran, Azerbaijan

Abstract. In this paper, we consider the problem of optimal control with a final and integral quality criteria on the boundary of the domain for the multidimensional linear stationary quasi-optics equation with a special gradient term. As a control we consider the refraction and absorption coefficients of the medium of propagation of light beams. The theorems of the existence and uniqueness of the solution for the considered problem are proved.

Keywords: quasi-optics equation, optimal control problem, refraction and absorption coefficients, boundary functional.

AMS Subject Classification: 35D, 35M, 35Q.

Corresponding author: Natig Ibrahimov, Lankaran State University, 50 Hazi Aslanov St., Lankaran, AZ 4200, Azerbaijan, e-mail: *natiq_ibrahimov@mail.ru*

Received: 2 July 2021; Revised: 19 October 2021; Accepted: 22 November 2021;
Published: 27 December 2021.

1 Introduction

Optimal control problems for the linear and nonlinear stationary quasi-optics equations or linear and nonlinear nonstationary Schrödinger equations often arise in quantum mechanics, nuclear physics, nonlinear optics, and in other areas of modern physics and technology, and the study of these problems is of both theoretical and practical interest (Küçük et al., 2019; Lions & Magenes, 1972; Tikhonov & Arsenin, 1979). One of these actual problems is the problem of the motion of charged particles the potential of which is unknown and must be determined. It is known that if a charged particle in the constant uniform magnetic field moves and the direction of the magnetic field is chosen along the axis z, then the motion of the particle occurs in the plane $(x, y) \in E_2$ and this motion is usually described by the two-dimensional linear Schrödinger equation with a special gradient term (see Küçük et al. (2019), p. 82). Similar optimal control problems for the linear nonstationary Schrödinger equation with a special gradient term were previously studied in papers Goebel (1979); Yagub et al. (2017). Note that optimal control problems for the linear and nonlinear nonstationary Schrödinger equations without a special gradient term were previously studied in detail in, for example, papers Butkovsky & Samoilenko (1984); Vorontsov & Shmalgauzen (1985); Zhuravlev (2001); Yagubov et al. (2012); Iskenderov & Yagubov (1988); Zhang (2018); Pashaev et al. (2020) Iskenderov et al. (2017, 2016); Ibragimov (2012); Vasiliev (1981); Yosida (1967). Optimal control problems for the nonlinear nonstationary Schrödinger equation with a special gradient term and with a real-valued potential, when the potential plays the role of control and is sought in the class of measurable bounded functions and the coefficient in the nonlinear part of the equation is a purely imaginary number, investigated in papers (De la Vega & Rial, 2018; Aronna et al., 2019). At the same time it should be noted that the optimal control problem for the three-dimensional nonlinear non-stationary Schrödinger equation with a special gradient term and with a real-valued potential, when the potential depends on both spatial and time variables plays the role of control and is sought in the class of measurable bounded functions and the coefficient in the nonlinear part of the equation is a complex number, was first investigated in Ibragimov (2010c). However, the problems of the optimal control problem for the stationary quasi-optics equation or the nonstationary Schrödinger equation with a special gradient term, when the quality criteria is an integral over the domain boundary, have been relatively less studied. A similar problem of optimal control only for the one-dimensional nonlinear Schrödinger equation with a special gradient term and with a complex potential, when the quality criteria is an integral over the boundary of the domain, was first studied in Ibragimov (2010b).

The optimal control problem for the multidimensional linear Schrödinger equation with a special gradient term and with a complex potential depending only on spatial variables, when the quality criteria is an integral over the boundary of the domain and the controls are real and imaginary parts of the complex potential and are selected from the class of measurable bounded functions depending on spatial variables was investigated in Yagub et al. (2015). The present work is devoted to the study of the optimal control problem for a multidimensional linear stationary quasi-optics equation with a special gradient term, when the quality criteria is final and integral over the boundary of the domain. As the control are considered the refraction and absorption coefficients of the medium that are taken from the class of the quadratic-summable functions having quadratic-summable derivatives depending on the distance variable z. It should be noted that identification problems for linear and nonlinear stationary equations of quasi-optics without a special gradient term were previously studied in detail in Yagubov & Musaeva (1997); Baudouin et al. (2005); Aksoy et al. (2017).

2 Problem statement

Let D be a bounded convex domain from the n-dimensional Euclidean space R^n, with smooth enough boundary Γ, $x = (x_1, x_2, ..., x_n)$ is an arbitrary point of the domain D, $L > 0$ is a given number, $0 \le z \le L$, $\Omega_z = D \times (0, z), \Omega = \Omega_L, S = \Gamma \times (0, L)$ is a lateral surface of Ω; $C^k([0, L], B)$ is a Banach space of the k- times continuously differentiable on the interval $[0, L]$ functions with values from the Banach space B; $L_p(D)$ is a Lebesque space of the functions, summable over the module with order $p \ge 1$; $L_2(0, L; B)$ is a Banach space of the functions defined and quadratic summable over the order on the interval $[0, L]$ with values from the Banach space B; $L_\infty(0, L; B)$ is a Banach space of the functions measurable and bounded on $(0, L)$ with values from the Banach space B; the Sobolev spaces $W_p^k(D), W_p^{k,m}(\Omega), p \ge 1,\ k \ge 0,\ m \ge 0$ are defined as, for example in Yagub & Boztepe (2018); Yagub et al. (2019); Ibragimov (2010a).

Consider the problem of minimizing the functional

$$J_\alpha(v) = \beta \|\psi - y\|^2_{L_2(S)} + \beta_0 \|\psi(., L) - y_0\|^2_{L_2(D)} + \alpha \|v - \omega\|^2_H \quad (1)$$

on the set

$$V = \left\{ v = v(z) = (v_0(z), v_1(z)) : \ v_m \in W_2^1(0, L), \|v_m\|_{W_2^1(0,L)} \le b_m, m = 0, 1 \right\}$$

under the conditions

$$i\frac{\partial \psi}{\partial z} + a_0 \Delta \psi + i a_1(x) \nabla \psi - a(x)\psi + v_0(z)\psi + i v_1(z)\psi = f(x, z), \ \ (x, z) \in \Omega, \quad (2)$$

$$\psi(x, 0) = \varphi(x), \ x \in D, \left.\frac{\partial \psi}{\partial \nu}\right|_S = 0, \quad (3)$$

where $i = \sqrt{-1}$; L $> 0,\ b_m > 0,\ m = 0, 1, a_0 > 0,\ \alpha \geq 0, \beta \geq 0, \beta_0 \geq 0$ are the given numbers such that $\beta + \beta_0 \neq 0$; $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator; $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right)$ is the nubla operator; ν is the outward normal to the boundary Γ; $a(x)$ is a bounded measurable function satisfying the condition

$$0 < \mu_0 \leq a(x) \leq \mu_1, \ \overset{o}{\forall} x \in D, \ \ \mu_0, \mu_1 = const > 0, \tag{4}$$

$a_1(x) = (a_{11}(x), a_{12}(x), ..., a_{1n}(x))$ is a given vector-function the components of which satisfy the conditions

$$|a_{1j}(x)| \leq \mu_2, \ \left| \frac{\partial a_{1j}(x)}{\partial x_k} \right| \leq \mu_3, \ j, k = \overline{1, n}, \overset{0}{\forall} x \in D$$

$$a_1(x)|_\Gamma = 0, \quad \mu_2, \mu_3 = const > 0; \tag{5}$$

$\varphi(x),\ f(x, z), y(\xi, z), y_0(x)$ are complex-values functions satisfying

$$\varphi \in W_2^2(D), \ \frac{\partial \varphi}{\partial \nu}\bigg|_\Gamma = 0 \ , f \in W_2^{0,1}(\Omega) \ ; \tag{6}$$

$$y \in L_2(S), y_0 \in L_2(D) \ ; \tag{7}$$

$\omega \in H$ is a given element, where $H \equiv W_2^1(0, L) \times W_2^1(0, L)$; the symbol $\overset{0}{\forall}$ means "for almost all".

The problem of determining the function $\psi = \psi(x, z) \equiv \psi(x, z; v)$ from conditions (2), (3) for each $v \in V$ is the second initial-boundary value problem for a multidimensional linear stationary quasi-optics equation with a special gradient term.

Definition 1. *For each $v \in V$ as the solution of the second initial-boundary value problem* (2), (3) *we mean the function $\psi = \psi(x, z) \equiv \psi(x, z; v)$ from the space $B_1 \equiv C^0([0, L], W_2^2(D)) \bigcap \bigcap C^1([0, L], L_2(D))$that satisfies equation* (2) *for almost all $x \in D$and anyz $\in [0, L]$, and the initial and boundary conditions* (3) *for almost all $x \in D$ and for almost all $(x, z) \in S$, correspondingly.*

Initial-boundary value problems for linear and nonlinear non-stationary Schrö- dinger equations with a special gradient term were previously studied in papers Iskenderov & Yagubov (2007); Iskenderov et al. (2012); Barbu et al. (2018); Yagub et al. (2017); Ibragimov (2010c); Yagub et al. (2016); Yagubov et al. (2017); Iskenderov et al. (2018). Using the methodology of those papers, the following statement was proved:

Theorem 1. *Let the functionsa (x), $a_1(x)$, $\varphi(x)$, $f(x, z)$ satisfy conditions* (4)-(6). *Then initial boundary value problem* (2), (3) *for each $v \in V$has a unique solution from the space B_1 and this solution satisfies the estimate:*

$$\|\psi(\cdot, z)\|^2_{W_2^2(D)} + \left\| \frac{\partial \psi(\cdot, z)}{\partial z} \right\|^2_{L_2(D)} \leq c_0 \left(\|\varphi\|^2_{W_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)} \right), \forall z \in [0, L], \tag{8}$$

where $A_0 > 0$ is a constant not depending on z .

It follows from this theorem and from the embedding of the space B_1 into the spaces $L_2(S), L_2(D)$ that functional (1) makes sense in the considered class of solutions B_1.

3 Existence and uniqueness of a solution to the optimal control problem

In this section, we study the existence and uniqueness of the solution to the optimal control problem (1) - (3). Therefore, we first establish a result on the existence of a unique solution to the problem. For this purpose, we give a well-known theorem on the existence and uniqueness of a solution to nonconvex optimization problem.

Theorem 2. *(Goebel, 1979). Let $\tilde{X}$ be a uniformly convex space, U is a closed bounded set from $\tilde{X}$, the functional $I(v)$ be lower semicontinuous and lower bounded on U, $\alpha > 0$, $\beta \geq 1$be a given number. There exists dense subset G of the space $\tilde{X}$ such that for any $\omega \in G$ the functional*

$$J_\alpha(v) = I(v) + \alpha \|v - \omega\|_{\tilde{X}}^\beta$$

reaches its lowest value at U. If $\beta > 1$ then the lowest value of the functional $J_\alpha(v)$ on $U$is reached on the unique element.

Using this theorem, we prove the following statement:

Theorem 3. *Let the functions $a(x)$, $a_1(x)$, $\varphi(x)$, $f(x,z)$, $y(\xi,z)$, $y_0(x)$ satisfy conditions* (4)-(7). *Let, in addition $\omega \in H$. Then there exists a dense subset G of the space H such that, for any $\omega \in G$ at $\alpha > 0$ optimal control problem* (1)-(3) *has a unique solution.*

Proof. First, we prove the continuity of the functional $J_0(v)$ on the set V.

$$J_0(v) = \beta \|\psi - y\|_{L_2(S)}^2 + \beta_0 \|\psi(.,L) - y_0\|_{L_2(D)}^2. \tag{9}$$

Let the increment $\delta v \in H \equiv W_2^1(0,L) \times W_2^1(0,L)$ of any control $v \in V$ be such that $v + \delta v \in V$ and $\delta\psi = \delta\psi(x,z) \equiv \psi(x,z;v+\delta v) - \psi(x,z;v)$, where $\psi(x,z;v)$ is a solution to initial boundary value problem (2), (3) at $v \in V$. From conditions (2), (3) follows that the function $\delta\psi = \delta\psi(x,z)$ is a solution to the following initial-boundary value problem

$$i\frac{\partial \delta\psi}{\partial z} + a_0\Delta\delta\psi + ia_1(x)\nabla\delta\psi - a(x)\delta\psi + (v_0(z) + \delta v_0(z))\,\delta\psi + i\,(v_1(z) + \delta v_1(z))\,\delta\psi =$$

$$= -\delta v_0(z)\,\psi(x,z) - i\delta v_1(z)\,\psi(x,z)\,,\ (x,z) \in \Omega, \tag{10}$$

$$\delta\psi(x,0) = 0, x \in D,\ \left.\frac{\partial \delta\psi}{\partial \nu}\right|_S = 0, \tag{11}$$

where $\psi_\delta = \psi_\delta(x,z) \equiv \psi(x,z;v+\delta v)$ is a solution to initial-boundary value problem (2), (3) at$v + \delta v \in V$, $\delta v \in B$.

Let us establish an estimate for the solution of the initial-boundary value problem (10), (11). For this purpose, we multiply both sides of equation (10) by the function $\delta\bar{\psi}(x,z)$ and integrate the obtained equality over the domain Ω_z. Then using the formula for integration by parts and the boundary condition from (11), we have

$$\int_{\Omega_z}\left(i\frac{\partial\delta\psi}{\partial z}\delta\bar{\psi} - a_0|\nabla\delta\psi|^2 + ia_1(x)\nabla\delta\psi\delta\bar{\psi} -\right.$$

$$\left.-a(x)|\delta\psi|^2 + (v_0(\tau) + \delta v_0(\tau))|\delta\psi|^2\right)dxd\tau +$$

$$+i\int_{\Omega_z}(v_1(\tau) + \delta v_1(\tau))|\delta\psi|^2 dxd\tau =$$

$$= -\int_{\Omega_z} \delta v_0(\tau)\,\psi\delta\bar{\psi}dxd\tau - i\int_{\Omega_z} \delta v_1(\tau)\,\psi\delta\bar{\psi}dxd\tau, \forall z \in [0, L]\,.$$

Subtracting its complex conjugation from this equality and applying the Cauchy-Bunyakovsky inequality, using the initial and boundary conditions from (11), as well as the condition on the function $a_1(x)$ we obtain the following inequality

$$\|\delta\psi(.,z)\|^2_{L_2(D)} \le (n\mu_2 + 2)\int_0^z \|\delta\psi(.,\tau)\|^2_{L_2(D)}\,d\tau +$$

$$+\int_{\Omega_z} |\delta v_0(\tau)|^2\,|\psi(x,\tau)|^2\,dxd\tau + \int_{\Omega_z} |\delta v_1(\tau)|^2\,|\psi(x,\tau)|^2\,dxd\tau, \forall z \in [0, L]\,.$$

Hence, by virtue of estimate (8) and applying Gronwall's lemma (see Ladyzhenskaya et al. (1967), pp. 30-31), we obtain the estimate

$$\|\delta\psi(.,z)\|^2_{L_2(D)} \le c_1\left(\|\delta v_0\|^2_{L_2(0,L)} + \|\delta v_1\|^2_{L_2(0,L)}\right), \forall z \in [0, L]\,, \tag{12}$$

where $c_1 > 0$ is a constant not depending on δv.

Now we establish an estimate for the first partial derivatives with respect to the variables $x_j, j = \overline{1,n}$ of the solution to initial-boundary value problem (10), (11). For this purpose, we multiply both sides of (10) by the function $L\delta\bar{\psi}(x,z)$and integrate the resulting equality over the domain Ω_z. Then we have

$$\int_{\Omega_z}\left(i\frac{\partial\delta\psi}{\partial z}L\delta\bar{\psi} - a_0\,|L\delta\psi|^2 + ia_1(x)\,\nabla\delta\psi L\delta\bar{\psi} + (v_0(\tau) + \delta v_0(\tau))\,\delta\psi L\delta\bar{\psi}\right)dxd\tau +$$

$$+i\int_{\Omega_z}(v_1(\tau) + \delta v_1(\tau))\,\delta\psi L\delta\bar{\psi}dxd\tau =$$

$$= -\int_{\Omega_z} \delta v_0(\tau)\,\psi L\delta\bar{\psi}dxd\tau - i\int_{\Omega_z} \delta v_1(\tau)\,\psi L\delta\bar{\psi}dxd\tau, \forall z \in [0, L]\,,$$

where $L\delta\bar{\psi}(x,z) = -a_0\Delta\psi(x,z) + a(x)\,\delta\bar{\psi}(x,z)$.

Using the formula for integration by parts and the boundary condition (11), this equality can be written in the form:

$$\int_{\Omega_z}\left(ia_0\frac{\partial\nabla\delta\psi}{\partial z}\nabla\delta\bar{\psi} + ia(x)\,\frac{\partial\delta\psi}{\partial z}\delta\bar{\psi} - a_0\,|L\delta\psi|^2\right)dxd\tau +$$

$$+\int_{\Omega_z}\left(-ia_0a_1(x)\,\nabla\delta\psi\Delta\delta\bar{\psi} + ia_1(x)\,\nabla\delta\psi a(x)\,\delta\bar{\psi}\right)dxd\tau +$$

$$+a_0\int_{\Omega_z}((v_0(\tau) + \delta v_0(\tau))\,\nabla\delta\psi)\,\nabla\delta\bar{\psi}dxd\tau +$$

$$+ia_0\int_{\Omega_z}((v_1(\tau) + \delta v_1(\tau))\,\nabla\delta\psi)\,\nabla\delta\bar{\psi}dxd\tau +$$

$$+\int_{\Omega_z}((v_0(\tau) + \delta v_0(\tau))\,\delta\psi)\,a(x)\,\delta\bar{\psi}dxd\tau +$$

$$+i\int_{\Omega_z}((v_1(\tau) + \delta v_1(\tau))\,\delta\psi)\,a(x)\,\delta\bar{\psi}dxd\tau =$$

$$= -a_0\int_{\Omega_z}(\delta v_0(\tau)\,\nabla\psi)\,\nabla\delta\bar{\psi}dxd\tau - ia_0\int_{\Omega_z}(\delta v_1(\tau)\,\nabla\psi)\,\nabla\delta\bar{\psi}dxd\tau -$$

$$-\int_{\Omega_z}(\delta v_0(\tau)\,\psi)\,a(x)\,\delta\bar{\psi}dxd\tau - i\int_{\Omega_z}(\delta v_1(\tau)\,\psi)\,a(x)\,\delta\bar{\psi}dxd\tau, \forall z \in [0, L]\,.$$

Subtracting its complex conjugation from this equality, it is easy to obtain the validity of the equality

$$\begin{aligned}
&\int_{\Omega_z}\left(a_0\frac{\partial}{\partial z}\left\|\nabla\delta\psi\right\|^2_{R^n}-a_0\left(a_1(x)\nabla\delta\psi\Delta\delta\bar{\psi}+a_1(x)\nabla\delta\bar{\psi}\Delta\delta\psi\right)+\right.\\
&\left.+a(x)\frac{\partial}{\partial z}\left|\delta\psi\right|^2\right)dxd\tau+\\
&+2a_0\int_{\Omega_z}\left(v_1(\tau)+\delta v_1(\tau)\right)\left\|\nabla\delta\psi\right\|^2_{R^n}dxd\tau+\\
&+2\int_{\Omega_z}\left(a(x)\left(v_1(\tau)+\delta v_1(\tau)\right)\left|\delta\psi\right|^2\right)dxd\tau+\\
&2\textstyle\int_{\Omega_z}Re\left(a_1(x)\nabla\delta\psi a(x)\delta\bar{\psi}\right)dxd\tau=\\
&=-2a_0\int_{\Omega_z}Im\left(\nabla\left(\delta v_0(\tau)\psi\right)\nabla\delta\bar{\psi}\right)dxd\tau-\\
&2a_0\textstyle\int_{\Omega_z}Re\left(\nabla\left(\delta v_1(\tau)\psi\right)\nabla\delta\bar{\psi}\right)dxd\tau-\\
&-2\int_{\Omega_z}a(x)\delta v_0(\tau)Im\left(\psi\delta\bar{\psi}\right)dxd\tau-\\
&2\textstyle\int_{\Omega_z}a(x)\delta v_1(\tau)Re\left(\psi\delta\bar{\psi}\right)dxd\tau,\forall z\in[0,L].
\end{aligned}\tag{13}$$

Now we transform the second term on the left-hand side of this equality as follows:

$$\begin{aligned}
&-a_0\int_{\Omega_z}\left(\left(a_1(x)\nabla\delta\psi\Delta\delta\bar{\psi}+a_1(x)\nabla\delta\bar{\psi}\Delta\delta\psi\right)\right)dxd\tau=\\
&=a_0\int_{\Omega_z}\sum_{j=1}^n\sum_{k=1}^n 2\frac{\partial a_{1k}(x)}{\partial x_j}Re\left(\frac{\partial\delta\psi}{\partial x_k}\frac{\partial\delta\bar{\psi}}{\partial x_j}\right)dxd\tau+\\
&+a_0\int_{\Omega_z}\sum_{j=1}^n\sum_{k=1}^n\frac{\partial}{\partial x_k}\left(a_{1k}(x)\left|\frac{\partial\delta\psi}{\partial x_j}\right|^2\right)dxd\tau-\\
&-a_0\int_{\Omega_z}\sum_{j=1}^n\sum_{k=1}^n\frac{\partial a_{1k}(x)}{\partial x_k}\left|\frac{\partial\delta\psi}{\partial x_j}\right|^2dxd\tau,\forall z\in[0,L].
\end{aligned}$$

Taking into account the boundary conditions for the vector function $a_1(x)$ and $\delta\psi(x,z)$, it is easy to see that the second term on the right-hand side is equal to zero. Therefore, the last equality may be written as follows

$$\begin{aligned}
&-a_0\int_{\Omega_z}\left(\left(a_1(x)\nabla\delta\psi\Delta\delta\bar{\psi}+a_1(x)\nabla\delta\bar{\psi}\Delta\delta\psi\right)\right)dxd\tau=\\
&=a_0\int_{\Omega_z}\sum_{j=1}^n\sum_{k=1}^n 2\frac{\partial a_{1k}(x)}{\partial x_j}Re\left(\frac{\partial\delta\psi}{\partial x_k}\frac{\partial\delta\bar{\psi}}{\partial x_j}\right)dxd\tau-\\
&-a_0\int_{\Omega_z}\sum_{j=1}^n\sum_{k=1}^n\frac{\partial a_{1k}(x)}{\partial x_k}\left|\frac{\partial\delta\psi}{\partial x_j}\right|^2dxd\tau,\forall z\in[0,L].
\end{aligned}$$

Taking into account this equality on the left-hand side of equality (13) we obtain:

$$a_0 \int_{\Omega_z} \frac{\partial}{\partial z} \left\| \nabla \delta\psi\left(.,\tau\right)\right\|_{R^n}^2 d\tau + \int_{\Omega_z} a\left(x\right) \frac{\partial}{\partial z} \left\| \delta\psi\left(.,\tau\right)\right\|_{R^n}^2 d\tau =$$

$$= -a_0 \int_{\Omega_z} \sum_{j=1}^{n} \sum_{k=1}^{n} 2 \frac{\partial a_{1k}\left(x\right)}{\partial x_j} Re\left(\frac{\partial \delta\psi}{\partial x_k} \frac{\partial \delta\bar{\psi}}{\partial x_j}\right) dx d\tau +$$

$$+ a_0 \int_{\Omega_z} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial a_{1k}\left(x\right)}{\partial x_k} \left|\frac{\partial \delta\psi}{\partial x_j}\right|^2 dx d\tau -$$

$$-2a_0 \int_{\Omega_z} \left(v_1\left(\tau\right) + \delta v_1\left(\tau\right)\right) \left\|\nabla \delta\psi\right\|_{R^n}^2 dx d\tau -$$

$$-2 \int_{\Omega_z} \left(a\left(x\right)\left(v_1\left(\tau\right) + \delta v_1\left(\tau\right)\right) \left|\delta\psi\right|^2\right) dx d\tau -$$

$$-2 \int_{\Omega_z} Re\left(a_1\left(x\right) \nabla \delta\psi a\left(x\right) \delta\bar{\psi}\right) dx d\tau -$$

$$-2a_0 \int_{\Omega_z} Im\left(\left(\delta v_0\left(\tau\right) \nabla \psi\right) \nabla \delta\bar{\psi}\right) dx d\tau -$$

$$-2a_0 \int_{\Omega_z} Re\left(\left(\delta v_1\left(\tau\right) \nabla \psi\right) \nabla \delta\bar{\psi}\right) dx d\tau -$$

$$-2 \int_{\Omega_z} a\left(x\right) \delta v_0\left(\tau\right) Im\left(\psi \delta\bar{\psi}\right) dx d\tau -$$

$$-2 \int_{\Omega_z} a\left(x\right) \delta v_1\left(\tau\right) Re\left(\psi \delta\bar{\psi}\right) dx d\tau, \forall z \in \left[0, L\right].$$

From this equality, under the accepted assumptions on the coefficients of the equation, using the initial conditions $\delta\psi\left(x,0\right) = 0, \nabla \delta\psi\left(x,0\right) = 0, \overset{0}{\forall} x \in D$, as well as the Cauchy-Bunyakovsky inequality, it is easy to establish the validity of the following inequality

$$a_0 \left\|\nabla \delta\psi\left(.,z\right)\right\|_{L_2(D)}^2 + \mu_0 \left\|\delta\psi\left(.,z\right)\right\|_{L_2(D)}^2 \leq$$

$$\leq 3a_0\mu_3 n \int_0^z \left\|\nabla \delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau +$$

$$+2a_0 \left\|v_1 + \delta v_1\right\|_{L_\infty(0,L)} \int_0^z \left\|\nabla \delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau +$$

$$+2\mu_1 \left\|v_1 + \delta v_1\right\|_{L_\infty(0,L)} \int_0^z \left\|\delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau +$$

$$+\mu_1\mu_2\sqrt{n} \int_0^z \left\|\nabla \delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau + \mu_1\mu_2\sqrt{n} \int_0^z \left\|\delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau +$$

$$+a_0 \left(\left\|\delta v_0\right\|_{L_\infty(0,L)}^2 + \left\|\delta v_1\right\|_{L_\infty(0,L)}^2\right) \int_{\Omega_z} \left\|\nabla \psi\left(x,\tau\right)\right\|_{R^n}^2 dx d\tau +$$

$$+2a_0 \int_0^z \left\|\nabla \delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau +$$

$$+\mu_1 \left(\left\|\delta v_0\right\|_{L_\infty(0,L)}^2 + \left\|\delta v_1\right\|_{L_\infty(0,L)}^2\right) \int_{\Omega_z} \left|\psi\left(x,\tau\right)\right|^2 dx d\tau +$$

$$+2\mu_1 \int_0^z \left\|\delta\psi\left(.,\tau\right)\right\|_{L_2(D)}^2 d\tau, \forall z \in \left[0, L\right]. \tag{14}$$

Due to the fact that $\delta v \in H$ and $v+\delta v \in V$ we can establish the validity of the inequalities

$$\|\delta v_m\|_{L_\infty(0,L)} \leq c_2 \|\delta v_m\|_{W_2^1(0,l)} \leq c_3, m=0,1, \tag{15}$$

$$\|v_m+\delta v_m\|_{L_\infty(0,L)} \leq c_4 \|v_m+\delta v_m\|_{W_2^1(0,l)} \leq c_2 b_m, m=0,1. \tag{16}$$

By virtue of these inequalities and estimates (8), (12), from inequality (14) we can obtain the following inequality

$$a_0 \|\nabla\delta\psi(.,z)\|^2_{L_2(D)} \leq A_5 \left(\|\delta v_0\|^2_{W_2^1(0,L)} + \|\delta v_1\|^2_{W_2^1(0,L)}\right) +$$
$$+c_6 \int_0^z \|\nabla\delta\psi(.,\tau)\|^2_{L_2(D)} d\tau, \forall z \in [0,L].$$

If we divide both sides of this inequality by a_0 then apply Gronwall's lemma, then we obtain the validity of the estimate

$$\|\nabla\delta\psi(.,z)\|^2_{L_2(D)} \leq c_7 \left(\|\delta v_0\|^2_{W_2^1(0,L)} + \|\delta v_1\|^2_{W_2^1(0,L)}\right), \forall z \in [0,L], \tag{17}$$

where $A_7>0$ is a constant not depending on δv. Summing this with estimate (12) and taking into account that $\delta v \in H$ we obtain

$$\|\delta\psi(.,z)\|^2_{W_2^1(D)} \leq c_8 \|\delta v\|^2_H, \forall z \in [0,L], \tag{18}$$

from which follows the estimate

$$\|\delta\psi\|^2_{W_2^{1,0}(\Omega)} \leq c_9 \|\delta v\|^2_H, \tag{19}$$

where $A_9>0$ is a constant not depending on δv. From this estimate, by virtue of the trace theorem (see Ibragimov (2011), p.170), we can establish

$$\|\delta\psi\|^2_{L_2(S)} \leq c_{10} \|\delta v\|^2_H, \tag{20}$$

where $A_{10}>0$ is a constant not depending on δv.

Now let's consider the increment of the functional $J_0(v)$on any element $v \in V$. By formula (9) we have

$$\delta J_0(v) = J_0(v+\delta v) - J_0(v) =$$
$$= 2\beta \int_S Re\left[(\psi(\xi,z) - y(\xi,z))\,\delta\bar{\psi}(\xi,z)\right] d\xi dz + \beta\|\delta\psi\|^2_{L_2(S)} + \tag{21}$$
$$+2\beta_0 \int_D Re\left[(\psi(x,L) - y_0(x))\,\delta\bar{\psi}(x,L)\right] dx + \beta_0 \|\delta\psi(.,L)\|^2_{L_2(D)}.$$

From this formula, applying the Cauchy-Bunyakovsky inequality and the trace theorem, using estimates (8), (19), (20) and conditions (7), we obtain the inequality

$$|\delta J_0(v)| \leq c_{11}\left(\|\delta v\|_H + \|\delta v\|^2_H\right), \forall v \in V,$$

where $A_{11}>0$ is a constant not depending on δv.

This inequality implies the continuity of the functional $J_0(v)$ on the set V. The set V is a closed, bounded and convex set in the uniform convex space H (Mikhailov, 1983). Then, by virtue of Theorem 2, there exists a dense subset G of the space H such that for any $\omega \in G$ and $\alpha>0$ optimal control problem (1)-(3) has a unique solution. Theorem 3 is proved.

Now let us show that for $\alpha \geq 0$ and for any $\omega \in H$ optimal control problem (1) - (3) has at least one solution.

Theorem 4. *Let the conditions of Theorem 1 be satisfied. Then, for* $\alpha \geq 0$ *and for any* $\omega \in H$ *optimal control problem* (1) - (3) *has at least one solution.*

Proof. Take any minimizing sequence $\left\{v^k\right\} \subset V$:

$$\lim_{k\to\infty} J_\alpha\left(v^k\right) = J_{\alpha*} = \inf_{v\in V} J_\alpha(v).$$

Let $\psi_k = \psi_k(x,z) \equiv \psi\left(x,z;v^k\right)$, $k = 1,2,....$ By virtue of Theorem 1, for each $v^k \in V$ initial-boundary value problem (2), (3) has a unique solution $\psi_k(x,z)$ from the space B_1, and this solution satisfies the estimate

$$\|\psi_k(\cdot,z)\|^2_{W_2^2(D)} + \left\|\frac{\partial\psi_k(\cdot,z)}{\partial t}\right\|^2_{L_2(D)} \leq c_0\left(\|\varphi\|^2_{W_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)}\right),$$
$$\forall z \in [0,L], k = 1,2,..., \tag{22}$$

where the right-hand side of the estimate does not depend on k.

Since V is a bounded set of the Hilbert space $H \equiv W_2^1(0,L) \times W_2^1(0,L)$, from the sequence $\left\{v^k\right\} \subset V$ one can choose such a subsequence $\left\{v^{k_p}\right\}$, (which for the sake of simplicity we again denote by $\left\{v^k\right\}$), that satisfies

$$v_m^k \to v_m,\ m = 0,1 \text{ weekly in } L_2(0,L), \tag{23}$$

$$\frac{dv_m^k}{dz} \to \frac{dv_m}{dz},\ m = 0,1 \text{ weekly in } L_2(0,L) \tag{24}$$

at $k \to \infty$. Moreover, V is a closed convex set from H. Therefore, V is a weakly closed set i.e. $v \in V$. In addition, due to the compact embedding of space $H \equiv W_2^1(0,L) \times W_2^1(0,L)$ into space $C[0,L] \times C[0,L]$ we can write the following relation:

$$\lim_{k\to\infty} v_m^k(z) = v_m(z), m = 0,1 \tag{25}$$

uniformly relatively to $z \in [0,L]$.

It follows from estimate (22) that the sequence $\{\psi_k(x,z)\}$ is uniformly bounded in the norm of the space B_1. Then from this sequence one can choose a subsequence $\left\{\psi_{k_p}(x,z)\right\}$ (which for the sake of simplicity we again denote by $\{\psi_k(x,z)\}$), such that

$$\psi_k(\cdot,z) \to \psi(\cdot,z) \text{ weekly in } W_2^2(D); \tag{26}$$

$$\frac{\partial\psi_k(\cdot,z)}{\partial z} \to \frac{\partial\psi(\cdot,z)}{\partial z} \text{ weekly in } L_2(D), \tag{27}$$

for each $z \in [0,L]$ at $k \to \infty$.

It is clear that each element $\{\psi_k(x,z)\}$ from B_1 satisfies the identity

$$\int_D \left(i\frac{\partial\psi_k(x,z)}{\partial z} + a_0\Delta\psi_k(x,z) + ia_1(x)\nabla\psi_k(x,z) - a(x)\psi_k(x,z) + \right.$$
$$\left. + v_0^k(z)\psi_k(x,z) + iv_1^k(z)\psi_k(x,z) - f(x,z)\right)\bar{\eta}(x)\,dx = 0, \forall z \in [0,L], k = 1,2,... \tag{28}$$

for any function $\eta = \eta(x)$ from $L_2(D)$, initial condition

$$\psi_k(x,0) = \varphi(x), \overset{0}{\forall} x \in D, k = 1,2,... \tag{29}$$

and boundary condition

$$\left.\frac{\partial \psi_k}{\partial \nu}\right|_S = 0, k = 1, 2, ... \tag{30}$$

Due to the compactness of the embedding of the space B_1 into $C^0\left([0, L], L_2(D)\right)$ we have

$$\left\|\psi_k(\cdot, z) - \psi(\cdot, z)\right\|_{L_2(D)} \to 0, \tag{31}$$

uniformly relatively to $z \in [0, L]$ at $k \to \infty$. Using this and limit relations (25), we can establish the validity of the relations

$$\int_D v_m^k(z)\, \psi_k(x, z)\, \bar{\eta}(x)\, dx = \int_D v_m(z)\, \psi(x, z)\, \bar{\eta}(x)\, dx, m = 0, 1, \tag{32}$$

for each $z \in [0, L]$ and for any $\eta \in L_2(D)$ at $k \to \infty$. Using limit relations (26), (27), and (32), passing to the limit in the integral identity (25), we obtain the identity:

$$\int_D \left(i\frac{\partial \psi(x, z)}{\partial z} + a_0 \Delta\psi(x, z) + ia_1(x)\nabla\psi(x, z) - a(x)\psi(x, z) + \right.$$

$$\left. + v_0(z)\psi(x, z) + iv_1(z)\psi(x, z) - f(x, z)\right)\bar{\eta}(x)\, dx = 0 \tag{33}$$

for each $z \in [0, L]$ and for any function $\eta = \eta(x)$ from $L_2(D)$. Hence it follows that the limit function $\psi(x, z)$ for each $z \in [0, L]$ and for almost all $x \in D$ satisfies (2). Satisfaction of the initial condition follows from the limit relation (31) at$z = 0$ the initial condition (29) and from the inequality:

$$\|\psi(\cdot, 0) - \varphi\|_{L_2(D)} \leq \|\psi(\cdot, 0) - \psi_k(\cdot, 0)\|_{L_2(D)} + \|\psi_k(\cdot, 0) - \varphi\|_{L_2(D)}.$$

Finally, let us prove that the limit function $\psi(x, z)$ satisfies the second boundary condition from (3). Indeed, by virtue of the theorem on traces (see Yagub et al. (2019), p.98; Ibragimov (2010a)) for $\{\psi_k(x, z)\}$ from the space B_1 the following relation is valid

$$\left.\frac{\partial \psi_k}{\partial \nu}\right|_S \in L_2(S), k = 1, 2, ... \tag{34}$$

Therefore, we can state that the following limit relation is valid

$$\left.\frac{\partial \psi_k}{\partial \nu}\right|_S \to \left.\frac{\partial \psi}{\partial \nu}\right|_S \; weekly\ in\ L_2(S) \tag{35}$$

at $k \to \infty$. Then, using this and boundary condition (30), from the equality

$$\int_S \frac{\partial \psi(\xi, z)}{\partial \nu} g(\xi, z)\, d\xi dz = \int_S \left(\frac{\partial \psi(\xi, z)}{\partial \nu} - \frac{\partial \psi_k(\xi, z)}{\partial v}\right) g(\xi, z)\, d\xi dz +$$

$$+ \int_S \frac{\partial \psi_k(\xi, z)}{\partial \nu} g(\xi, z)\, d\xi dz, \forall g \in L_2(S)$$

with passing to limit, we obtain the validity of the boundary condition

$$\frac{\partial \psi(\xi, z)}{\partial \nu} = 0, \overset{0}{\forall}(\xi, z) \in S.$$

Thus, we have proved that the limit function $\psi(x, z)$ is a solution to the initial-boundary value problem (2), (3) corresponding to the limit function $v \in V$ i.e. $\psi = \psi(x, z) \equiv \psi(x, z; v)$. In addition, this function satisfies estimate (8), which immediately follows from estimate (22) with passing to limit along weakly converging subsequences $\{\psi_k(x, z)\}$. By virtue of Theorem 1, such a solution uniquely belongs to the space B_1. Due to the compactness of the embedding of the space B_1 into the space $L_2(S)$, we obtain the validity of the limit relation

$$\psi_k \to \psi \text{ strongly in } L_2(S) \text{ at } k \to \infty. \quad (36)$$

Using this relation and (31), as well as the weak lower semicontinuity of the norms of the spaces$L_2(S)$,$L_2(D)$, H for $\forall\alpha \geq 0$ and $\forall\omega \in H$ we obtain

$$J_{\alpha*} \leq J_\alpha(v) \leq \underline{\lim}_{k\to\infty} J_\alpha(v_k) = \inf_{v\in V} J_\alpha(v) = J_{\alpha*}.$$

Hence it follows that $v \in V$ is the solution to optimal control problem (1)-(3) for $\forall\alpha \geq 0$ and $\forall\omega \in H$. Theorem 4 is proved.

4 Conclusion

Proved in this paper solvability theorems form the theoretical basis for the numerical solution of this optimal control problem for the Schrödinger equation. Along with these results, it is also possible to apply variational methods to solve inverse problems of determining the refractive index and absorption in the stationary uranium of quasi-optics, which describes the motion of charged particles or light beams in an inhomogeneous medium (Küçük et al., 2019; Lions & Magenes, 1972). It is known that the above considered optimal control problem for $\alpha = 0$ is from the class of ill-posed problems, in other words, this problem is unstable (see the example in Iskenderov et al. (2016)). Along with these, it should be noted that when $\alpha = 0$ i.e. when the influence of the term of the functional with a coefficient α is canceled, the solution of the problem under consideration is not only non-unique and, as shown above, unstable. Therefore, the results obtained above regarding the solvability of the considered optimal control problem make it possible to develop a stable algorithm for solving this problem (Ladyzhenskaya, 1973).

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