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SOME PROPERTIES OF MOBIUS FUNCTION GRAPH $\mathcal{M}^{(1)}$

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Abstract. Throughout this work, a new graph is called the Mobius Function Graph $\mathcal{M}^{(1)}$ is introduced. Three ways of determining the prime-counting function by using this graph are presented. Also, some properties of this graph are proved. Moreover, the domination number, chromatic number, independence number, and clique number of this graph are determined. Finally, a comparison was made between the two numbers the domination and independence.

Keywords: Mobius Function Graph $\mathcal{M}^{(1)}$, domination number, independence number, and chromatic number.
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1 Introduction

In recent years, the science of graph theory has begun to take a great place because it has become a new language spoken by all sciences, as its use has become common in all sciences such as medicine, engineering, physics, chemistry, and others. The most important applications of this type of graph is to find new relationships between numbers that serve the science of number theory, such as finding the exact number of prime numbers that are less than a certain number. Graph $G(V, E)$ depends on two sets, the first is called the vertex set (V) and the other is called edge set (E). In mathematics, graph theory deals with various fields as topological graphs (Al'Dzhabri et al., 2021; Jabor & Omran, 2020) fuzzy graphs (Omran & Ibrahim, 2021; Yousif & Omran, 2021), general graphs (Abdlhusein & Al-Harere, 2021; Ibrahim & Omran, 2021a,b; Omran et al., 2022; Omran & Shalaan, 2021), topological indices (Alsinai et al., 2021a,b,c,d,e; Afzal et al., 2021), and others. The most important concepts in the graph theory is: dominating set, independent set, and coloring problem of vertex set. These concepts resulted in important numbers that graph theory deals with, which are: domination number, independence number, and chromatic number. A dominating set is a subset of the vertex set such that each vertex out of this subset is adjacent to at least one vertex of this subset. A domination number is the minimum cardinality of dominating sets and denoted by $\gamma(G)$ (Kahat et al., 2021). The subset of the vertex set is called Independent if every two vertices in it are non-adjacent. The maximum cardinality of all independent sets is called the independence number of the graph G and denoted by $\beta\ (G)$ (Haynes et al., 1998). A vertex-coloring of G is an assignment of colors to all its vertices such that all pairs of adjacent vertices are assigned different colors. The chromatic number, denoted by $\chi\ (G)$ is the smallest number of colors necessary for coloring G (Brooks, 1941). The order of largest complete (each vertex in it is adjacent to all other vertices in it) subgraph of a graph G is called the clique number. In this work, we will focus our attention on the relationship of graph theory with number theory (Al-Maamori & Hilberdink, 2015; Al-Maamori, 2017; AL-Ameedee et al., 2020) by defining a new graph whose vertices depend on the natural numbers and its edges on a known function, which is a Mobius function. Through the results we obtained are calculating the prime-counting function (which is counting the number of prime numbers less than or equal to the order of a graph and denoted by $\pi(n)$) for three different ways. Moreover, some properties of this graph are discussed. Also, the independence number, chromatic number, and domination number are determined. Finally, the relationship between the independence number with the domination number is obtained. The reader can find all concepts not included here in Brooks (1941); Lesniak & Chartrand (2005); Harary (1969).

2 Main results

Definition 1. *The Mobuis function defined by $\mathcal{M}(1) = 1$, for $n > 1$, then write*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = \Pi_{i=1}^{k} p_i^{\alpha_i},$$

so

$$\mathcal{M}(n) = \begin{cases} (-1)^k, & if\ a_1 = a_2 = \dots = a^k = 1; \\ 0, & otherwise. \end{cases}$$

Theorem 1. *If G is a Mobius function graph $\mathcal{M}^{(1)}$, then*

$$\pi(n) = deg(v_2) + 1 - \left\{ u : u = \Pi_{i=1}^{k} P_i,\ k\ is\ odd;\ P_i \neq 2 \forall i \right\}$$

where P_i are distinct primes numbers and labeled of $f(v_i) = i$.

Proof. If the vertices of a graph take labeled $f(v_i) = i$, then the vertex labeled 2 is adjacent to all vertices that have labeled primes number, since $\mathcal{M}(2P_i) = 1$, $\forall i$. Moreover, $\mathcal{M}\left(2\Pi_{i=1}^{k} P_i\right) = 1$, where k *is odd and* $P_i \neq 2 \forall i$, so the vertex labeled 2 is adjacent to all vertices that are labeled with the form

$$\Pi_{i=1}^{k} P_i;\ k\ is\ odd\ and\ P_i \neq 2 \forall i.$$

Thus,

$$\pi(n) = deg(v_2) + 1 - \left\{ u : u = \Pi_{i=1}^{k} P_i,\ i\ is\ odd;\ P_i \neq 2 \forall i \right\}$$

(for example, see Figure 1.). □

Theorem 2. *The clique number to Mobius function graph $\mathcal{M}^{(1)}$is $\pi(n)$.*

Proof. Let v_i and v_j be any two vertices such that i and j are prime numbers, then $\mathcal{M}(ij) = 1$ (By definition 1 and ij means i product j), so the vertex v_i is adjacent to the vertex v_j. Thus, the vertices which have prime numbers constitute an induced subgraph isomorphism to a complete. Now, suppose that there is a vertex say v_r that is adjacent to all vertices mentioned above, then the label of this vertex must be of the form $\Pi_{i=1}^{k \leq n} P_i$ and k is odd greater than one. Thus, the minimum value of k is three, so take $f(v_r) = P_1 P_2 P_3$, where $P_i, = 1, 2, 3$ are prme numbers. One can be concluded that this vertex is not adjacent to the vertices have labeled P_1, P_2, and P_3 and this is a contradiction. Therefore, the clique number to Mobius function graph $\mathcal{M}^{(1)}$ is $\pi(n)$. □

Proposition 1. *If G is a Mobius function graph $\mathcal{M}^{(1)}$ is non trivial, then $\chi(G) = \pi(n)$.*

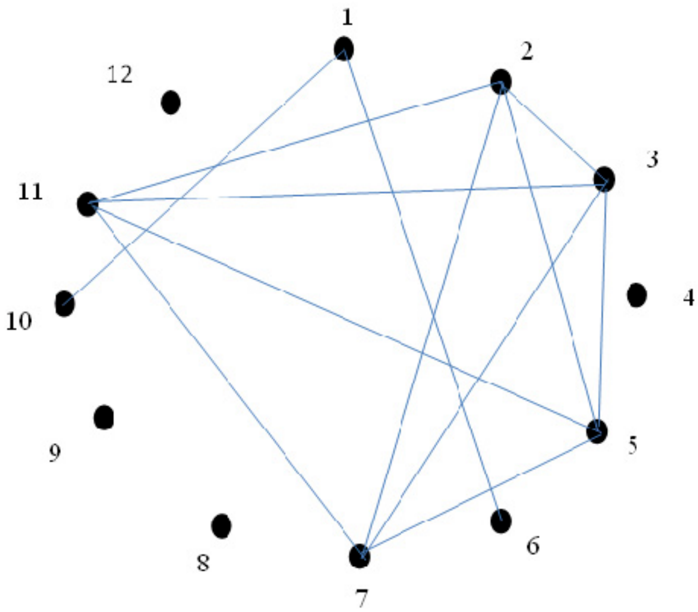

Figure 1: The Mobius function graph $\mathcal{M}^{(1)}$ of order 12

Proof. According to Theorem 2, the clique number is equal to $\pi(n)$, so $\pi(n)$ color is needed to color the largest subgraph isomorphic to complete graph of order $\pi(n)$. The other vertices can be colored by the same color which is used previously. Thus, $\chi(G) = \pi(n)$. □

Proposition 2. *Let G be a Mobius function graph $\mathcal{M}^{(1)}$, then*

1. $N(v_1) = \left\{v_i;\ f(v_i) = \Pi_{i=1}^{k} P_i;\ k\, is\, even\right\}$
2. *All vertices have labeled not free square prime are isolated vertices.*
3. *The vertex v is pendant if it is adjacent to the vertex v_1 and there is a common prime factor with labeled of other vertices that adjacent to the vertex v_1*
4. *If G be a Mobius function graph $\mathcal{M}^{(1)}$ of order $n > 1$, then graph G is disconnected.*

Proof. 1. The labeled of the vertex v_1 is one, so all vertices in the set

$$\left\{v_i;\ f(v_i) = \Pi_{i=1}^{k} P_i; k\ is\ even\right\}$$

are adjacent to the vertex v_1, since

$$\mathcal{M}\left(1 * \Pi_{i=1}^{k} P_i\right) = \mathcal{M}\left(\Pi_{i=1}^{k} P_i\right) = 1,$$

where k is even. Thus, the required statement is obtained.

2. One can be concluded that each vertex that is labeled has a square prime is not adjacent to all other vertices by definition of Mobius function. Thus, the result is getting.

3. Let the vertex v_r be adjacent to the vertex v_1, then this vertex has labeled $\Pi_{i=1}^{k} P_i$;$k\ is\ even$. If there is a vertex that adjacent to the vertex v_r says v_s, so the label of this vertex is $\Pi_{i=1}^{j} P_i; j\ is$ even. If there is a common prime factor, then the vertex v_s is adjacent to the vertex v_1 otherwise the two vertices are not adjacent. Therefore, the vertex is a pendant vertex if has no common factor prime with all vertices have a characteristic similar to that of the vertex v_s. Thus, the result is obtained.

4. Three cases are discussed below:

1) If $n = 2$, then the graph $\mathcal{M}^{(1)}$ is isomorphic to the null graph of order 2, then the graph $\mathcal{M}^{(1)}$ is disconnected.

2) If $n = 3$, then $\mathcal{M}^{(1)} \equiv K_2 \cup K_1$, so the result is obtained.

3) If $n \geq 4$, then the vertex v_4 is the isolated vertex since it is not a free square $\square$

Proposition 3. *Let G be a Mobius function graph $\mathcal{M}^{(1)}$ of order n. Then*

$$\beta(G) = \begin{cases} 1, & \text{if } n = 1 \ ; \\ 2, & \text{if } \ n = 2, 3 \ ; \\ 3, & \text{if } n = 4, 5, 6 \ ; \\ |N_e^n| + |S - N_e^n|, & \text{if } n \geq 7, \end{cases}$$

where S is the set of the isolated vertices in G,and N_e^n is the set of even numbers less than or equal n

Proof. Two cases appeared as follows.
Case 1. One can conclude the result when $n = 1, 2, \ldots, 6$.
Case 2. Let v_r is an isolated vertex, then this vertex belongs to all independent set (I) of G, thus $S \subseteq I$ so, $\beta(G) \geq |S|$. Now, if a vertex v_r is not an isolated vertex, so this vertex is free square prime. Let H be the set of vertices that labeled even number, let u and v be any two vertices of the set H, so $f(uv) = 2^2 P_z P_w$. Therefore the vertex u is not adjacent to the vertex v, so the set H is independent. Note that $S \cap H \neq \emptyset$, then $\beta(G) \geq |H| + |S - H|$. Again, let u_1 be a vertex not belonging in the sets S and H, so the labeled of this vertex is $P_j \Pi_{i=1}^{r} P_i, \ P_j \neq 2$. It is obvious that there is at least one $P_i, \ i = 1, \ldots, r$ such that divides one vertex of the set H, since $2 < P_j$. Therefore, $\beta(G) = |H| - |S - H|$. $\square$

Theorem 3. *Let G be a Mobius function graph $\mathcal{M}^{(1)}$ of order n, then*

$$\gamma\left(\mathcal{M}^{(1)}\right) = \begin{cases} 1, & \text{if } n = 1 \ ; \\ 2, & \text{if } \ n = 2, 3 \ ; \\ |S| + 2, & \text{otherwise.} \end{cases}$$

Proof. Depending on the number of vertices four cases are discussed as the following.
Case 1. If $n = 1$, then it is obvious that $\gamma\left(\mathcal{M}^{(1)}\right) = 1$.
Case 2. If $n = 2$, then the Mobius function graph is isomorphic to null graph of order 2, then $\gamma\left(\mathcal{M}^{(1)}\right) = 2$.
Case 3. If $n = 3$, then Mobius function graph is isomorphic to graph $(K_2 \cup K_1)$ and $\gamma\left(\mathcal{M}^{(1)}\right) = 2$.
Case 4. If $n \geq 4$, then let S be the set of all isolated vertices, then the set S contains in each dominating set. So, the remained vertices have non square prime factor and these vertices have two kinds as the following: Subcases 1. The vertices of the form $\Pi_{i=1}^{r} P_i$ where r is even, all these vertices are adjacent to the vertex v_1 according to proposition 2.3(1). Subcases 2. The vertices of the form $\Pi_{i=1}^{s} P_i$ where s is odd, all these vertices are adjacent to the vertex v_2 according to Theorem 1. From all cases above, the result is obtained (as an example, see Figure 2).

$\square$

Proposition 4. *Let $\mathcal{M}^{(1)}$ and $u, v \in \mathcal{M}^{(1)}$, $g.c.d\,(u, v) = 1$. Then $\mathcal{M}(u.v) = \mathcal{M}(u)\,\mathcal{M}(v)$.*

Proof. Let G be a Mobius function graph $\mathcal{M}^{(1)}$ and $,v \in \mathcal{M}^{(1)}$, $g.c.d\,(u, v) = 1$. Then four cases are appeared as the following.
Case 1. If $\mathcal{M}(u) = \mathcal{M}(v) = 1$, then $u = \Pi_{i=1}^{s} P_i andv = \Pi_{j=1}^{r} P_i$, where r and s are even number, so $u.v = \Pi_{i=1}^{s} P_i \Pi_{j=1}^{r} P_i = \Pi_{k=1}^{s+r} P_i$ and it is obvious that $s+r$ is even, since $g.c.d\,(u, v) = 1$. Thus, $\mathcal{M}(u.v) = 1 = \mathcal{M}(u)\,\mathcal{M}(v)$.

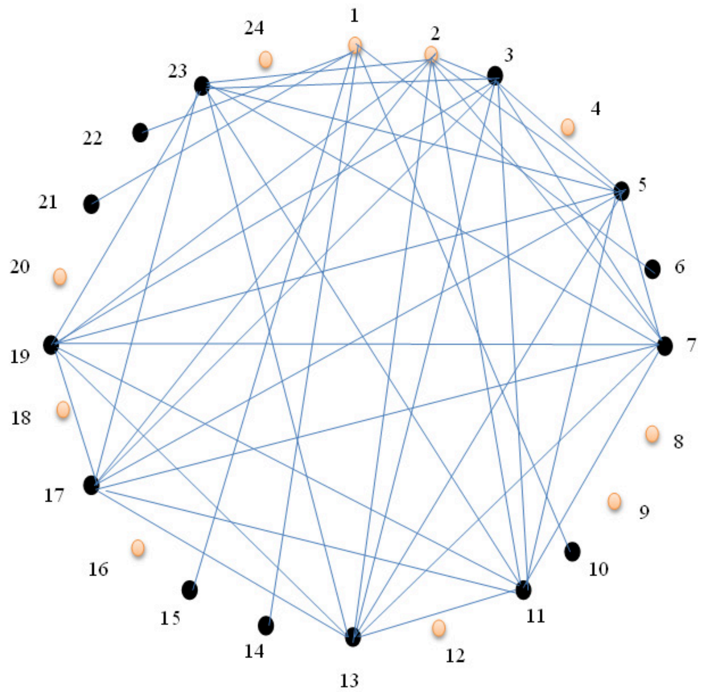

Figure 2: The domination number of $\mathcal{M}^{(1)}$ of order n

Case 2. If $\mathcal{M}(u) = 1$ and $\mathcal{M}(v) = -1$, then $u = \Pi_{i=1}^{s} P_i; s \text{ is even}$ and $v = \Pi_{j=1}^{r} P_i; r \text{ is odd}$, then $u.v = \Pi_{i=1}^{s} P_i \Pi_{j=1}^{r} P_i = \Pi_{k=1}^{s+r} P_i$ and it is obvious that $s + r$ is odd, since $g.c.d(u, v) = 1$. Thus, $\mathcal{M}(u.v) = -1 = \mathcal{M}(u).\mathcal{M}(v)$.
Case 3. If $\mathcal{M}(u) = -1$ and $\mathcal{M}(v) = -1$, then $u = \Pi_{i=1}^{s} P_i; s \text{ is odd}$ and $v = \Pi_{j=1}^{r} P_i; r \text{ is odd}$, then $u.v = \Pi_{i=1}^{s} P_i \Pi_{j=1}^{r} P_i = \Pi_{k=1}^{s+r} P_i$ and it is obvious that $s + r$ is even, since $g.c.d(u, v) = 1$. Thus, $\mathcal{M}(u.v) = 1 = \mathcal{M}(u).\mathcal{M}(v)$.
Case 4. If there is factor prime $P_i^{\alpha i}$; $\alpha i > 1$ is divided the labeled of at least one vertex from two vertices u or v say u, then $\mathcal{M}(u) = 0$. Now, the factor prime $P_i^{\alpha i}$ is divided the labeled of the product of two vertices u and v. Thus, $\mathcal{M}(u.v) = 0 = \mathcal{M}(u).\mathcal{M}(v)$.
From all cases above, the result is done

□

Theorem 4. *Let G be a non-trivial Mobius function graph $\mathcal{M}^{(1)}$. Then*

1. $\beta(G) + \gamma(G) \geq n$, $\forall n$ *except* $n = 7$.

2. $\gamma(G) \leq \beta(G)$.

Proof. (i) The result is obvious if $n = 1, 2, 3$ by proposition 5, and Theorem 6, so there are cases as follows.

1. If $n = 4$, then the Mobius graph is isomorphic to graph $N_2 \cup K_2$, so this graph contains two isolated vertices which have labeled 1 and 4 and two adjacent vertices labeled 2 and 3. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 > n = 4$.

2. If $n = 5$, then the Mobius graph is isomorphic to graph $N_2 \cup K_3$, so this graph contains two isolated vertices which have labeled 1 and 4 and three vertices of the graph K_3 labeled $2, 3$, and 5. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 > n = 5$.

3. If $n = 6$, then the Mobius graph is isomorphic to graph $N_1 \cup K_2 \cup K_3$, so this graph contains one isolated vertex which have labeled 4 and three vertices of the graph K_3 that

labeled $2, 3$ and 5 and two vertices of the graph K_2 that labeled $1, 6$. Thus, $\beta(G)+\gamma(G) = 3 + 3 = 6 = n$.

4. If $n = 7$, then the Mobius graph is isomorphic to graph $N_1 \cup K_2 \cup K_4$, so this graph contains one isolated vertex which have labeled 4 and four vertices of the graph K_4 that labeled $2, 3, 5$ and 7 and two vertices of the graph K_2 that labeled $1, 6$. Thus, $\beta(G) + \gamma(G) = 3 + 3 = 6 < n$.

5. If $n \geq 7$, then $\beta(G) + \gamma(G) = |S| + |N_e^n| + |N_e^n - S| + |S| + 2$
$= |S| + |N_e^n| + |N_e^n| - |S| + |S| + 2$
$= 2|N_e^n| + |S| + 2 = 2\left\lfloor \frac{n}{2} \right\rfloor + |S| + 2 \geq n$.

(ii) One can be concluded easily that the inequality is correct when $n = 1, 2, \ldots, 6$. So, $\forall n \geq 7$, $\gamma(G) = |S| + 2 < |S| + |N_e^n| + |S - N_e^n| = \beta(G)$. Thus, the required statement is obtained.

□

3 Conclusion

In the paper three new ways to determine the prime-counting function by using a new graph is called the Mobius Function Graph $\mathcal{M}^{(1)}$ are given. Also, some properties of this graph are proved. In addition, the domination, independence, and clique numbers are calculated, a comparison between the two numbers the domination and independence is made.

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