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EXISTENCE OF A SOLUTION TO THE NONLINEAR BRIDGE PROBLEM WITH A TIME-VARYING DELAY*

Akbar B. Aliev[1,2], **Yeter M. Farhadova**[1†]

[1]Institute of Mathematics and Mechanics, Baku, Azerbaijan
[2]Azerbaijan State Oil and Industry University, Baku, Azerbaijan

Abstract. In this paper we consider the mathematical model of the bridge problem with time-varying delay. The existence and uniqueness of the solution is investigated by modelling this problem as the Cauchy problem for an operator coefficient equation in a certain space.

Keywords: bridge problem, aerodynamic resistance force, time-varying delay, mixed problem.

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[†]**Corresponding author:** Yeter M. Farhadova, Institute of Mathematics and Mechanics, Baku, Azerbaijan, e-mail: *ferhadova.yeter@gmail.com*

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1 Introduction

For many years, suspension bridges have an exclusive place among other structural systems due to their outstanding architectural appearance. Due to the dominating tension stresses, suspension bridges assure covering the longest spans in the world. In suspension bridges, large main cables (normally two) hang between the towers and are anchored at each end to the ground. The main cables, which are free to move on bearings in the towers, bear the load of the bridge deck. Before the deck is installed, the cables are under tension from their own weight. Along the main cables smaller cables or rods connect to the bridge deck, which is lifted in sections. As this is done, the tension in the cables increases, as it does with the live load of traffic crossing the bridge. The tension on the main cables is transferred to the ground at the anchorages and by downwards compression on the towers.

There has always been great interest in modelling suspension bridges because of their efficiency and remarkable architectural appearance (Lazer & McKenna, 1987; McKenna & Walter, 1990; Ahmed & Harbi,1998; Aliev & Farhadova, 2021; 2022). The systems described by PDE's with time delays have been an active field of research over the last years (Dafermos, 1970a; 1970b; Xu et al., 2006; Nicaise et al., 2009; Nicaise & Pignotti, 2006; 2011; Rahmoune, 2021; Ferreira et al., 2022; Chellaouna & Boukhatem, 2020) and references therein. Since time delay may destroy stability (Xu et al., 2006; Nicaise et al., 2009; Nicaise & Pignotti, 2006; 2011; Rahmoune, 2021) even if it is very small, the stabilization problem of systems with time delays has been a popular topic in the mathematical control theory and engineering.

In this paper, we consider the mixed problem with a time-varying delay in linear aerodynamic resistance force in the bridge problem and prove the theorem on the existence and uniqueness of the solution of considered problem.

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2 Statement of the problem. Existence and uniqueness of the solution

We consider the following mathematical model for the oscillations of the bridge with a time-varying delay

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + [u-v]_+ + \lambda_1 u_t(x,t) + \\ +\lambda_2 u_t(x,t-\tau_1(t)) = h_1(t,x), \\ v_{tt}(x,t) - v_{xx}(x,t) - [u-v]_+ + \mu_1 v_t(x,t) + \\ +\mu_2 v_t(x,t-\tau_2(t)) = h_2(t,x), \end{cases} \tag{1}$$

where $0 \leq x \leq l$, $t > 0$, $u(x,t)$ is state function of the road bed and $v(x,t)$ is that of the main cable; $\tau_1(t)$, $\tau_2(t)$ are time-varying delays, λ_1, λ_2, μ_1, μ_2 are real numbers, $[a]_+ = \max\{a, 0\}$.

Let's define the following initial and boundary conditions for the system (1).

$$\begin{cases} u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = v(0,t) = v(l,t) = 0, \ t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,l), \\ u_t(x,t-\tau_1(t)) = f_{01}(x,t-\tau_1(t)), \quad x \in (0,l), \ t \in (0,\tau_1(0)), \\ v(x,0) = v_0(x), v'(x,0) = v_1(x), \ x \in (0,l), \\ v_t(x,t-\tau_2(t)) = f_{02}(x,t-\tau_2(t)), \quad x \in (0,l), \ t \in (0,\tau_2(0)). \end{cases} \tag{2}$$

Problem (1) - (2) will be investigated under the following conditions:

$$\left.\begin{array}{c} \tau_i(\cdot) \in W_2^2(0,T), \quad \tau_i'(t) \leq d_i < 1 \\ 0 < \tau_{i0} \leq \tau_i(t) \leq \tau_{i1}, 0 \leq t \leq T, \quad i = 1,2 \end{array}\right\} \tag{3}$$

$$h_i(\cdot) \in W_2^1([0,+\infty); L_2(0,l)), \ i = 1,2. \tag{4}$$

For investigating the problem (1)- (2), we introduce the following notations:

$$H^k(a,b) = \left\{ y : \ y, y', ..., y^{(k)} \in L_2(a,b) \right\},$$

$$\widehat{H}^k(a,b) = \left\{ y : \quad y \in H^k(a,b), \quad y^{(2s)}(a) = y^{(2s)}(b) = 0, \ s = 0,1,..., \left[\frac{k}{2}\right] \right\},$$

where $[r]$ is the integer part of the number r . We will denote the space $\widehat{H}^k(0,l)$ as $\widehat{H}^k$.

The following theorem on the existence and uniqueness of the problem (1), (2) is true.

Theorem 1. *Assume that the conditions (3)- (4) are satisfied. Then for any* $u_0 \in \widehat{H}^2$, $u_1 \in L_2(0,1)$, $v_0 \in \widehat{H}^1$, $v_1 \in L_2(0,1)$, $f_{0i}(\cdot, -\cdot\tau_{i0}), f_{0i\rho}(\cdot, -\cdot\tau_{i0}) \in L_2((0,l) \times (0,1))$, $i = 1,2$, *the problem (1)-(2) has a unique solution* $(u(x,t), \ v(x,t))$, *where*

$$u(\cdot) \in C\left([0,+\infty), \widehat{H}^2\right) \cap C^1\left([0,+\infty), L_2(0,1)\right),$$

$$v(\cdot) \in C\left([0,+\infty), \widehat{H}^1\right) \cap C^1\left([0,+\infty), L_2(0,1)\right).$$

Moreover, if $u_0 \in \widehat{H}^4$, $u_1 \in \widehat{H}^2$, $v_0 \in \widehat{H}^2$, $v_1 \in \widehat{H}^1$, $f_{0i}(\cdot, -\cdot\tau_1) \in L_2((0,l) \times (0,1))$, $f_{0i\rho}(\cdot, -\cdot\tau_1) \in L_2((0,l) \times (0,1))$, *then the solution of (1) satisfies*

$$u(\cdot) \in C([0,+\infty), \widehat{H}^4) \cap C^1([0,+\infty), \widehat{H}^2) \cap C^2([0,+\infty), L_2(0,1)),$$

$$v(\cdot) \in C([0,+\infty), \widehat{H}^2) \cap C^1([0,+\infty), L_2(0,1)) \cap C^2([0,+\infty), L_2(0,1)).$$

3 Proof of Theorem 1

In order to establish the existence of a unique solution to (1)-(2), we introduce the new variables (Showalter, 1997; Barbu, 1976):

$$\left.\begin{array}{l} z_1\left(x,\rho,t\right)=u_t(x,t-\tau_1(t)\rho), \rho\in(0,1), x\in(0,l), t>0, \\ z_2\left(x,\rho,t\right)=v_t(x,t-\tau_2(t)\rho), \rho\in(0,1), x\in(0,l), t>0. \end{array}\right\} \tag{5}$$

Obviously, z_1 and z_2 are solutions to the following problems:

$$\left\{\begin{array}{l} \tau_i(t)z_{it}\left(x,\rho,t\right)+(1-\rho\tau_i'(t))z_{i\rho}\left(x,\rho,t\right)=0, \quad \rho\in(0,1),\ x\in(0,l),\ t>0 \\ z_i\left(x,\rho,0\right)=f_{0i}(x,-\rho\tau_i(0)),\ x\in(0,l), \rho\in(0,1),\ i=1,2. \end{array}\right. \tag{6}$$

So, problem (1)-(2) takes the system equation

$$\left\{\begin{array}{l} u_{tt}\left(x,t\right)+u_{xxxx}\left(x,t\right)+\left[u-v\right]_+ +\lambda_1 u_t\left(x,t\right)+\lambda_2 z_1\left(x,1,t\right)=0, \text{ in } \left(0,l\right)\times\left(0,\infty\right), \\ v_{tt}(x,t)-v_{xx}(x,t)-\left[u-v\right]_+ +\mu_1 v_t(x,t)+\mu_2 z_2(x,1,t)=0, \text{ in } (0,l)\times(0,\infty), \\ \tau_i(t)z_{it}\left(x,\rho,t\right)+(1-\ \rho\tau_i'(t))\ z_{i\rho}\left(x,\rho,t\right)=0, \text{ in } \left(0,l\right)\times\left(0,1\right)\times\left(0,\infty\right), i=1,2 \end{array}\right. \tag{7}$$

with boundary conditions

$$\left\{\begin{array}{l} u\left(0,t\right)=u_{xx}\left(0,t\right)=u\left(l,t\right)=u_{xx}\left(l,t\right)=0, \\ v\left(0,t\right)=v\left(l,t\right)=0, \\ z_i\left(0,\rho,t\right)=z_i\left(l,\rho,t\right)=0, \quad i=1,2 \end{array}\right. \tag{8}$$

and initial conditions

$$\left\{\begin{array}{l} u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in(0,l), \\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \quad x\in(0,l), \\ z_i\left(x,\rho,0\right)=f_{0i}(x,-\rho\tau_i(0)), \quad x\in(0,l), \rho\in(0,1),\ i=1,2. \end{array}\right. \tag{9}$$

We introduce the following space:

$$\mathcal{H}=\widehat{H}^2\times L_2\left(0,l\right)\times\widehat{H}^1\times L_2\left(0,l\right)\times L_2((0,1)\times(0,l))\times L_2((0,1)\times(0,l)),$$

equipped with the scalar product

$$\langle\omega,\widetilde{\omega}\rangle=\int_0^l u_{1xx}\tilde{u}_{1xx}dx+\int_0^l u_2\tilde{u}_2dx+\int_0^l u_3\tilde{u}_3dx+\int_0^l u_4\tilde{u}_4dx+$$

$$+\eta_1\int_0^l\int_0^1 z_1\tilde{z}_1d\rho dx+\eta_2\int_0^l\int_0^1 z_2\tilde{z}_2d\rho dx,$$

for all $\omega=(u_1,\ u_2,u_3,u_4,\ z_1,z_2)^T$, $\widetilde{\omega}=(\ \tilde{u}_1,\ \tilde{u}_2,\ \ \tilde{u}_3,\ \ \tilde{u}_4,\ \tilde{z}_1,\widetilde{z_2})^T\in\mathcal{H}$, where

$$\eta_i>\frac{\tau_{i1}\left|\mu_i\right|}{1-d_i},\ i=1,2. \tag{10}$$

Let's define the following operators A_0, $A_1(\cdot)$ and $G(\cdot)$ in the space $\mathcal{H}$. The linear operator A_0 is defined by

$$A_0(t)\omega=\left(\begin{array}{c} -u_2 \\ u_{1xxxx}+\lambda_1 u_2+\lambda_2 z_1\left(\cdot,1\right) \\ -u_4 \\ -u_{3xx}+\mu_1 u_4+\mu_2 z_2\left(\cdot,1\right) \\ \frac{1-\rho\tau'_1(t)}{\tau_1(t)}z_{1\rho} \\ \frac{1-\rho\tau'_2(t)}{\tau_2(t)}z_{2\rho} \end{array}\right)$$

with domain

$$D\left(A_0(t)\right)=\left\{\omega:\omega=(u_1,u_2,u_3,u_4,z_1,z_2)^T\in\mathcal{H},\ u_1\in\widehat{H}^4,\quad u_2\in\widehat{H}^2,\quad u_3\in\widehat{H}^2,\right.$$

$$\left. u_4\in\widehat{H}^1,\quad z_i,z_{i\rho}\in L_2((0,1)\times(0,l)),\ z_1(x,0)=u_2(x),z_2(x,0)=u_4(x),\ 0<x<l,\ i=1,2\right\}.$$

It is obvious that $D\left(A_0(t)\right)$ does not depend on t and $D\left(A_0(t)\right)=D\left(A_0(0)\right)$.

The nonlinear operators $A_1(\cdot)$ and $G(\cdot)$, acting from $\mathcal{H}$ into the space $\mathcal{H}$ are respectively defined as

$$A_1(\omega)=\begin{pmatrix}0\\ [u_1-u_3]_+\\ 0\\ -\ [u_1-u_3]_+\\ 0\\ 0\end{pmatrix},\ G(t)=\begin{pmatrix}0\\ h_1(t,x)\\ 0\\ h_2(t,x)\\ 0\\ 0\end{pmatrix}.$$

Let $u_1\left(t\right)=u\left(t\right)$, $u_2\left(t\right)=u_t\left(t\right)$, $u_3\left(t\right)=v\left(t\right)$, $u_4\left(t\right)=v_t\left(t\right)$, $z_1(t)=z_1(\cdot,t)$, $z_2(t)=z_2(\cdot,t)$ and denote by

$$\omega=\omega(t)=\left(u_1(t),\ u_2(t),u_3(t),u_4(t),\ z_1(t),z_2(t)\right)^T,$$

$$\omega(0)=\omega_0=\left(u_{10},\ u_{20},\ u_{30},u_{40},\ z_1(\cdot,-\rho\tau_1(0)),z_2(\cdot,-\rho\tau_1(0))\right).$$

Then problem (7)-(9) can be rewritten as an initial- value problem

$$\begin{cases}\omega'+A_0(t)\omega+A_1(\omega)=G(t),\\ \omega(0)=\omega_0.\end{cases}\tag{11}$$

We have the following result on the existence and uniqueness of solutions to the problem (11).

Theorem 2. *Assume that the conditions* (3)-(4) *are satisfied. Then for any $\omega_0\in\mathcal{H}$, the problem* (11) *has a unique solution*

$$\omega(\cdot)\in C([0,+\infty)\ ,\ \mathcal{H}\ ).$$

Moreover, if $\omega_0\in D(A_0(0))$, then the solution of (11) satisfies

$$\omega(\cdot)\in C^1([0,+\infty)\ ,\mathcal{H})\cap C([0,+\infty)\ ,D\left(A_0(0)\right).$$

To prove the Theorem 2, we should prove the following Lemmas using known results for operator equations in the monographs (Kato, 1985; Showalter, 1997).

Lemma 1. *For every $t\in\lceil 0,\infty)$, $A_{0\gamma},(t)$ is the maximal dissipative operator, where $A_{0\gamma}(t)=A_0(t)+\gamma I$, I is the identity operator,*

$$\gamma=\min\left\{\lambda_1-\frac{|\lambda_2|}{2}-\eta_1\frac{1}{2\tau_{10}}-\frac{\max_{t>0}|\tau_1'(t)|}{2\tau_{10}},\ \mu_1-\frac{|\mu_2|}{2}-\eta_2\frac{1}{2\tau_{20}}-\frac{\max_{t>0}|\tau_2'(t)|}{2\tau_{20}}\right\}.$$

Proof. We start by showing that $-A_0(t)$ is dissipative. So, for $\omega=(u_1,u_2,u_3,u_4,z_1,z_2)^T\in D\left(A_0\right)$, we have

$$\langle A_0(t)\omega,\omega\rangle=-\int_0^l u_{1xx}(x)u_{2xx}(x)dx+$$

$$+\int_0^l u_2(x)\left(u_{1xxxx}(x)+\ \lambda_1u_2(x)+\lambda_2z_1\left(x,1\right)\right)dx+$$

$$+\int_0^l u_{3x}(x)u_{4x}(x)dx+\int_0^l u_4(x)\left(-u_{3xx}(x)+\ \mu_1u_4(x)+\ \mu_2z_2(x,1)\right)dx+$$

$$+\eta_1\int_0^l\int_0^1\frac{1-\rho{\tau'}_1(t)}{\tau_1(t)}\;z_1(x,\rho)z_{1\rho}(x,\rho)d\rho dx+$$

$$+\eta_2\int_0^l\int_0^1\frac{1-\rho{\tau'}_2(t)}{\tau_2(t)}\;z_2(x,\rho)z_{2\rho}(x,\rho)d\rho dx=$$

$$=\lambda_1\int_0^l|u_2(x)|^2dx+\lambda_2\int_0^l u_2(x)z_1(x,1)dx+$$

$$+\mu_1\int_0^l|u_4(x)|^2dx+\mu_2\int_0^l u_4(x)z_2\,(x,1)\,dx+$$

$$+\eta_1\int_0^l\int_0^1\frac{1-\rho{\tau'}_1(t)}{\tau_1(t)}\;z_1\,(x)\,z_{1\rho}\,(x,\rho)\,d\rho dx+$$

$$+\eta_2\int_0^l\int_0^1\frac{1-\rho{\tau'}_2(t)}{\tau_2(t)}\;z_2(x)z_{2\rho}(x,\rho)d\rho dx. \tag{12}$$

Using the Gronwall lemma, we obtain the following inequalities

$$\left|\lambda_2\int_0^l u_2(x)z_1(x,1)dx\right|\le\frac{|\lambda_2|}{2}\int_0^l|u_2(x)|^2dx+\frac{|\lambda_2|}{2}\int_0^l|z_1\,(x,1)|^2dx, \tag{13}$$

$$\left|\mu_2\int_0^l u_4(x)z_2(x,1)dx\right|\le\frac{|\mu_2|}{2}\int_0^l|u_4(x)|^2dx+\frac{|\mu_2|}{2}\int_0^l|z_2\,(x,1)|^2dx. \tag{14}$$

Considering (4) and (6), we get the following equality

$$\int_0^l\int_0^1\frac{1-\rho\tau_i'(t)}{\tau_i(t)}z_i(x,\rho)z_{i\rho}(x,\rho)d\rho dx=\frac{1-\tau_i'(t)}{2\tau_i(t)}\int_0^l|z_i(x,1)|^2dx-$$

$$-\frac{1}{2\tau_i(t)}\int_0^l|z_i(x,0)|^2dx+\frac{\tau_i'(t)}{2\tau_i(t)}\int_0^l\int_0^1|z_i(x,\rho)|^2d\rho dx.$$

It follows from (4),(5) and (9) that

$$\langle A_0(t)\omega,\omega\rangle\ge\left[\lambda_1-\frac{|\lambda_2|}{2}-\eta_1\frac{1}{2\tau_1\,(t)}\;\right]\int_0^l|u_2(x)|^2dx+$$

$$+\left[\mu_1-\frac{|\mu_2|}{2}-\eta_2\frac{1}{2\tau_2\,(t)}\;\right]\int_0^l|u_4(x)|^2dx+$$

$$+\left[\eta_1\frac{1-{\tau'}_1\,(t)}{2\tau_1\,(t)}\;-\frac{|\lambda_2|}{2}\right]\int_0^l|z_1(x,1)|^2dx+$$

$$+\left[\eta_2\frac{1-{\tau'}_2\,(t)}{2\tau_2\,(t)}\;-\frac{|\mu_2|}{2}\right]\int_0^l|z_2(x,1)|^2dx\ge$$

$$\left[\lambda_1-\frac{|\lambda_2|}{2}-\eta_1\frac{1}{2\tau_{10}}\;\right]\int_0^l|u_2(x)|^2dx+$$

$$+\left[\mu_1-\frac{|\mu_2|}{2}-\eta_2\frac{1}{2\tau_{20}}\;\right]\int_0^l|u_4(x)|^2dx+$$

$$+\left[\eta_1\frac{1-d_1}{2\tau_{11}}\;-\frac{|\lambda_2|}{2}\right]\int_0^l|z_1(x,1)|^2dx+\left[\eta_2\frac{1-d_2}{2\tau_{21}}\;-\frac{|\mu_2|}{2}\right]\int_0^l|z_2(x,1)|^2dx-$$

$$-\sum_{i=1}^{2}\frac{\max_{t>0}|\tau_i'(t)|}{2\tau_{i0}}\int_0^l\int_0^1|z_i(x,\rho)|^2d\rho dx. \quad (15)$$

From (10)-(15) we obtain that

$$\langle A_0(t)\omega,\omega\rangle \geq \gamma\|\omega\|_{\mathcal{H}}^2. \quad (16)$$

It follows from (16) that

$$\langle A_{0\gamma}(t)\omega,\omega\rangle \geq 0.$$

Next, we show that for every $t\in[0,\infty)$, $A_{0\gamma}(t)$ is a maximal dissipative operator. To do this, we will prove that for any $t\in[0,T]$ and $G=(g_1,g_2,g_3,g_4,g_5,g_6)^T\in\mathcal{H}$, the problem

$$\omega + A_0(t)\omega = G \quad (17)$$

has a solution $\omega=(u_1,u_2.u_3,u_4,z_1,z_2)\in D(A_0(t))=D(A_0(0))$.

Writing equation (16) in coordinates, we obtain the following initial-boundary value problem for systems of equations

$$\left.\begin{array}{r} u_1(x)-u_2(x)=g_1(x)\\ u_2(x)+u_{1xxxx}(x)+\lambda_1u_2(x)+\lambda_2z_1(x,1)=g_2(x)\\ u_3(x)-u_4(x)=g_3(x)\\ u_4(x)-u_{3xx}(x)+\mu_1u_4(x)+\mu_2z_2(x,1)=g_4(x)\\ z_1(x,\rho)+\frac{1-\rho{\tau'}_1(t)}{\tau_1(t)}z_{1\rho}(x,\rho)=g_5(x,\rho)\\ z_2(x,\rho)+\frac{1-\rho{\tau'}_2(t)}{\tau_2(t)}z_{2\rho}(x,\rho)=\ g_6(x,\rho) \end{array}\right\},\ 0\leq x\leq l,\ 0\leq\rho\leq1, \quad (18)$$

with boundary conditions

$$u_i(0)=u_i(l)=0, i=1,...,4 \quad (19)$$

$$u_{1xx}(0)=u_{1xx}(l)=0, \quad (20)$$

$$z_i(0,\rho)=z_i(l,\rho),\quad \rho\in(0,1),\quad i=1,2, \quad (21)$$

and initial conditions

$$z_1(x,0)=u_2(x),\ 0\leq x\leq l \quad (22)$$

$$z_2(x,0)=u_4(x),\ 0\leq x\leq l. \quad (23)$$

By virtue of (22), from the fifth equation of system (18), we get that

$$z_1(x,\rho)+\frac{1-\rho{\tau'}_1(t)}{\tau_1(t)}z_{1\rho}(x,\rho)=g_5(x,\rho),$$

$$z_1(x,0)=u_1(x)-g_1(x).$$

From this follows that

$$z_{1\rho}+\frac{\tau_1(t)}{1-\rho{\tau'}_1(t)}z_1=\tilde{g}_5, \quad (24)$$

$$z_1(x,0)=u_1(x)-g_1(x), \quad (25)$$

where $\tilde{g}_5=\frac{\tau_1(t)}{1-\rho{\tau'}_1(t)}\ g_5$.

Solving (24), (25) we get

$$z_1 = z_1(x,\rho) = \left(1-\rho{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}\left[u_1(x) - g_1(x) + \tau_1(t)\int_0^\rho \left(1-s{\tau'}_1(t)\right)^{-\frac{\tau_1(t)}{{\tau'}_1(t)}-1} g_5(x,s)ds\right].$$

Hence, it is clear that

$$z_1(x,1) = \left(1-{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}\{u_1(x) - g_1(x)\}+$$

$$+\left(1-{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}\tau_1(t)\int_0^1 \left(1-s{\tau'}_1(t)\right)^{-\frac{\tau_1(t)}{{\tau'}_1(t)}-1} g_5(x,s)ds. \tag{26}$$

Similarly, we have

$$z_2(x,1) = \left(1-{\tau'}_2(t)\right)^{\frac{\tau_2(t)}{{\tau'}_2(t)}}\{u_3(x) - g_3(x)\}+$$

$$+\left(1-{\tau'}_2(t)\right)^{\frac{\tau_2(t)}{{\tau'}_2(t)}}\tau_2(t)\int_0^1 \left(1-s{\tau'}_2(t)\right)^{-\frac{\tau_2(t)}{{\tau'}_2(t)}-1} g_6(x,s)ds. \tag{27}$$

From (18),(26) and (27) it follows that $u(x)$ is a solution to the following boundary value problem

$$u_{1xxxx}(x) + k_1(t)u_1(x) = \psi_1, \tag{28}$$

$$u_1(0) = u_1(l) = u_{1xx}(0) = u_{1xx}(l) = 0, \tag{29}$$

where

$$k_1(t) = 1 + \lambda_1 + \lambda_2\left(1-{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}$$

$$\psi_1 = [1+\lambda_1+\lambda_2\left(1-{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}]g_1 + g_2+$$

$$+\lambda_2\tau_1(t)\left(1-{\tau'}_1(t)\right)^{\frac{\tau_1(t)}{{\tau'}_1(t)}}\int_0^1 \left(1-s{\tau'}_1(t)\right)^{-\frac{\tau_1(t)}{{\tau'}_1(t)}-1} g_5(x,s)ds.$$

Similarly, from (19),(20) (26) and (27), we get

$$-u_{1xx}(x) + k_2(t)u_1(x) + (1+\mu_1+\mu_2)u_1(x) = \psi_3, \tag{30}$$

$$u_1(0) = u_1(l) = 0, \tag{31}$$

where

$$\psi_3 = [1+\mu_1+\mu_2\left(1-{\tau'}_2(t)\right)^{\frac{\tau_2(t)}{{\tau'}_2(t)}}]g_3 + g_4+$$

$$+\mu_2\tau_2(t)\left(1-{\tau'}_2(t)\right)^{\frac{\tau_2(t)}{{\tau'}_2(t)}}\int_0^1 \left(1-s{\tau'}_1(t)\right)^{-\frac{\tau_2(t)}{{\tau'}_2(t)}-1} g_6(x,s)ds.$$

We define the following bilinear form in the space $V = \widehat{H}^2 \times \widehat{H}^1$:

$$B(U,\ W) = \int_0^l u_{1xx}w_{1xx}dx + \int_0^l u_{3x}\tilde{w}_{1x}dx + \kappa_1(t)\int_0^l u_1w_1dx + \kappa_2(t)\int_0^l u_3\tilde{w}_1dx,$$

and the linear form

$$L(\tilde{w}) = \left\langle \tilde{f}, \tilde{u}\right\rangle = \int_0^l \psi_1 w_1dx + \int_0^l \psi_3 w_3dx$$

where $\tilde{u}=(u_1,u_3)$, $\tilde{w}=(w_1,\ w_3)$ and $\tilde{g}=(\psi_1,\psi_3\ )$. Let's show that $B(\tilde{u},\ \tilde{w})$ and $L(W)$ satisfy the conditions of Lax-Milgram theorem. By using Hölder's inequality, we obtain

$$B\left(\tilde{u},\tilde{w}\right)\le\left(\int_0^l\left|u_{1xx}\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_{1xx}\right|^2dx\right)^{\frac{1}{2}}+$$

$$+\left(\int_0^l\left|u_{3x}\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_{3x}\right|^2dx\right)^{\frac{1}{2}}+$$

$$+\max_{0\le t\le T}k_1\left(t\right)\left(\int_0^l\left|u_1\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_1\right|^2dx\right)^{\frac{1}{2}}+$$

$$+\max_{0\le t\le T}k_2\left(t\right)\left(\int_0^l\left|u_3\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_3\right|^2dx\right)^{\frac{1}{2}}.$$

Thus $B(\tilde{u},\ \tilde{w})$ is continuous, acting from $V\times V$ to R.

Estimating from below, we obtain

$$B\left(\tilde{u},\tilde{u}\right)=\int_0^l\left|u_{1xx}\right|^2dx+\int_0^l\left|u_{3x}\right|^2dx+\kappa_1(t)\int_0^l\left|u_1\right|^2dx+$$

$$+\kappa_2(t)\int_0^l\left|u_3\right|^2dx\ge C_0\ \left\|\tilde{u}\right\|_V^2\ ,$$

where $C_0=\min\{1,\ \min_{0\le t\le T}\kappa_1(t)\ ,\min_{0\le t\le T}\kappa_2(t)\}$.

On the other hand,

$$\left|L(\tilde{w})\right|\le\left|\int_0^l\psi_1w_1dx\right|+\left|\int_0^l\psi_3w_3dx\right|\le\frac{1}{2}$$

$$\left(\int_0^l\left|\psi_1\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_1\right|^2dx\right)^{\frac{1}{2}}+\left(\int_0^l\left|\psi_3\right|^2dx\right)^{\frac{1}{2}}\left(\int_0^l\left|w_3\right|^2dx\right)^{\frac{1}{2}}.$$

Thus, the bilinear form B and the linear functional L satisfy the conditions of the Lax-Milgram theorem (Evans, 2010). So, there exists a unique $\tilde{u}=(u_1,u_3)\in\widehat{H}^2\times\widehat{H}^1$ satisfying

$$B(\tilde{u},\ \tilde{w})=L(\tilde{w}),\quad\forall\ \tilde{w}\in\widehat{H}^2\times\widehat{H}^1.\tag{32}$$

Consequently, $u_2=u_1-f_1\in\widehat{H}^2$, $u_4=u_3-f_3\in\widehat{H}^1$ and

$$z_1\left(\cdot,\rho\right),z_2\left(\cdot,\rho\right),\ z_{1\rho}\left(\cdot,\rho\right),\ z_{2\rho}\left(\cdot,\rho\right)\in L_2(0,l).$$

Using (27) and (28), we get $z_1\left(\cdot,\rho\right)$, $z_2\left(\cdot,\rho\right)\in L_2((0,l)\times(0,l))$. Thus, (17) has a unique solution $\omega=(u_1,\ u_2,u_3,u_4,\ z_1,z_2)^T\in\mathcal{H}$. □

Lemma 2. *The linear operator $A_0(t)$ is strongly continuously differentiable.*

Proof. Let $\omega=(u_1,\ u_2,u_3,u_4,\ z_1,z_2)^T\in D(A_0(0))$. Then from the definition of $A_0(t)$ we get

$$A'_0(t)\omega=\begin{pmatrix}0\\0\\0\\0\\\frac{-\rho\tau''_1(t)\tau_1(t)-[1-\rho\tau'_1(t)]}{\tau_1^2(t)}z_{1\rho}\\\frac{-\rho\tau''_2(t)\tau_2(t)-[1-\rho\tau'_2(t)]}{\tau_2^2(t)}z_{2\rho}\end{pmatrix}.$$

From this we have

$$\|A'_0(t)\omega\|_{\mathcal{H}}^2 = \int_0^1 \int_0^l \left| \frac{-\rho\tau''_1(t)\,\tau_1(t) - [1-\rho\tau'_1(t)]}{\tau_1^2(t)} z_{1\rho}(x,\rho) \right|^2 dxd\rho +$$

$$+ \int_0^1 \int_0^l \left| \frac{-\rho\tau''_2(t)\,\tau_1(t) - [1-\rho\tau'_2(t)]}{\tau_2^2(t)} z_{2\rho}(x,\rho) \right|^2 dxd\rho \le$$

$$\le \max_{0\le t\le T} \left| \frac{-\rho\tau''_1(t)\,\tau_1(t) - [1-\rho\tau'_1(t)]}{\tau_1^2(t)} \right|^2 \cdot \int_0^1 \int_0^l |z_{1\rho}(x,\rho)|^2 dxd\rho \le$$

$$\le \max_{0\le t\le T} \left| \frac{-\rho\tau''_2(t)\,\tau_1(t) - \left[1-\rho\tau'_2(t)\right]}{\tau_2^2(t)} \right|^2 \cdot \int_0^1 \int_0^l |z_{2\rho}(x,\rho)|^2 dxd\rho \le$$

$$\le M\{\int_0^1 \int_0^l |z_{1\rho}(x,\rho)|^2 dxd\rho + \int_0^1 \int_0^l |z_{2\rho}(x,\rho)|^2 dxd\rho\},$$

where

$$M = \max_{0\le t\le T} \left\{ \left| \frac{-\rho\tau''_1(t)\,\tau_1(t) - [1-\rho\tau'_1(t)]}{\tau_1^2(t)} \right|^2, \left| \frac{-\rho\tau''_2(t)\,\tau_1(t) - \left[1-\rho\tau'_2(t)\right]}{\tau_2^2(t)} \right|^2 \right\}.$$

□

Using the definition of the norm in the space $\mathcal{H}$ and Lemma 2, we obtain that

$$\left\| A'_0(t)\omega \right\|_{\mathcal{H}} \le c \, \|A_0(0)\omega\|_{\mathcal{H}}.$$

Lemma 3. *The nonlinear operator $A_1(\cdot)$ satisfy Lipschitz condition.*

Proof. Since $\left|[\alpha]_+ - [\beta]_+\right| \le |\alpha - \beta|$, for any α, β, we get

$$\|A_1(\omega_2) - A_1(\omega_1)\|_{\mathcal{H}} \le \|\omega_2 - \omega_1\|_{\mathcal{H}}.$$

The definition of $G(\cdot)$ also implies the following assertion. □

Lemma 4. $G(\cdot) \in W_2^1(0,T;\ \mathcal{H})$

According to Lemma 1-4, the operator $A_0(t) + A_1(\cdot)$ and the function $G(t)$ satisfy all conditions of the theorems on the existence and uniqueness of solutions to the Cauchy problem for operator differential equations (Kato, 1985; Showalter, 1997; Barbu, 1976).

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