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OPTIMIZATION OF DODGSON'S CONDENSATION METHOD FOR RECTANGULAR DETERMINANT CALCULATIONS*

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Abstract. In this paper, we present an optimization of the rectangular determinant calculation using the generalized Dodgson's method. Based on a generalization of Dodgson's method for reducing the order of the determinant to the square determinant of the second order, nine different cases were found to find the pivot block. In this paper, we provide that the pivot block can be built from the elements of matrix A by removing any two rows and any two columns. Since, regarding the execution time the advantage of this modification comes into place in cases where there are several zero elements in the rectangular determinant, hence by removing any two rows and any two columns with the highest number of non-zero elements we can create a pivot block with the highest number of zero elements, which is calculated faster. For this purpose, we have also created an algorithm for finding the two rows and two columns with the greatest number of non-zero elements which are excluded from pivot block. The suggested approach is evaluated algorithmically on a PC and results are compared to existing algorithms, we found that this approach is executed faster than the existing algorithms. Another advantage is in cases where the pivot block equals to zero, in these cases we can create another pivot block, since division with zero is not allowed.

Keywords: Rectangular determinants, pivotal condensation, algorithm optimization, execution time.
AMS Subject Classification: 08A70, 15A15, 11C20, 65F40, 65Y04, 68Q87, 68W40, 97N80.
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1 Definition of determinant of rectangular matrices

Let A be rectangular matrix of order $m \times n$:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

The determinant of matrix is defined as follows, by Stojakovic (1952):

$$det_n A = \sum_{(j)}^{\binom{r}{n}} \left\{ \sum_{(l)}^{\binom{s}{n}} \left[\sum_{(\sigma)}^{n!} (-1)^{J(\sigma)} \cdot a_{\rho_1 \sigma_1} \cdots a_{\rho_n \sigma_n} \right]_{(i)} \right\}_{(j)}. \tag{1}$$

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Cullis (1913) gave the definition of rectangular determinants, which later was improved by Radic (1966), as following:

$$det A = \sum_{j_1<j_2<\cdots<j_m} (-1)^{r|s} \cdot \begin{vmatrix} a_{1j_1} & \cdots & a_{1j_m} \\ \vdots & \ddots & \vdots \\ a_{mj_1} & \cdots & a_{mj_m} \end{vmatrix}. \tag{2}$$

Stanimirovic et al. (1997), provided another definition of rectangular determinants, as following:

$$det_{(\varepsilon,p)} A = \sum_{\alpha_1<\cdots<\alpha_p \beta_1<\cdots<\beta_p} \epsilon^{(\alpha_1+\cdots+\alpha_p)+(\beta_1+\cdots+\beta_p)} A \begin{pmatrix} \alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_p \end{pmatrix}. \tag{3}$$

Bayat (2020), generalized the rectangular determinant definition as following: Determinant of $A \in \mathbb{C}^{m\times n}$ is a function $det_{(\overrightarrow{\varepsilon},p)} : \mathbb{C}^{m\times n} \to \mathbb{C}$ defined by:

$$\underset{(\overrightarrow{\varepsilon},p)}{det}(A) = \begin{cases} \sum_{\substack{I\in Q_{p,m} \\ J\in Q_{p,n}}} \overrightarrow{\varepsilon} det(A[I,J]), & \text{if } 1 \le p \le min\{m,n\} \\ 1, & \text{if } p=0 \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where scalars $\overrightarrow{\varepsilon}_{I,J}$ are components of vector $\overrightarrow{\varepsilon} \in \mathbb{C}^k$ for $k = \binom{m}{p}\binom{n}{p}$.

Depending on the value of $\overrightarrow{\varepsilon}_{I,J}$ there are different above-mentioned definitions. The Stojakovic definition is for $\overrightarrow{\varepsilon}_{I,J} = 1$ and $I \in Q_{p,m}$ and $J \in Q_{p,n}$. For $n = m = p$ and $\overrightarrow{\varepsilon} = 1$, there is definition of square determinant. The Radic definition is if $\varepsilon = -1$, and Stanimirovic and Stankovic definition is in case of $\overrightarrow{\varepsilon}_{I,J} = \varepsilon^{\sum_{l=1}^{p}(i_l+j_l)}$ for $I \in Q_{p,m}$ and $J \in Q_{p,n}$.

2 Dodgson's generalization formula

Amiri et al. (2010) generalized the Dodgson algorithm for non-square matrices as follows

$$det\left(A_{\substack{1\le i\le m \\ 1\le j\le n}}\right) \cdot det\left(A_{\substack{i\ne m-1,m \\ j\ne n-1,n}}\right) = det\left(A_{\substack{i\ne m \\ j\ne n}}\right) \cdot det\left(A_{\substack{i\ne m-1 \\ j\ne n-1}}\right) -$$

$$det\left(A_{\substack{i\ne m \\ j\ne n-1}}\right) \cdot det\left(A_{\substack{i\ne m-1 \\ j\ne n}}\right) + det\left(A_{i\ne m-1,m}\right) \cdot det\left(A_{j\ne n-1,n}\right). \tag{5}$$

Later Bayat (2020), gave another case of generalization of Dodgson's formula for use in rectangular determinant calculation, while pivot block is considered the inner determinant of matrix A, which is presented in Theorem 1.

Theorem 1. *Bayat (2020) (Generalized Dodgson's formula) Let A be $m \times n$ a rectangular matrix. Then for $p = min(m,n) \ge 2$, we have*

$$det\left(A_{\substack{1\le i\le m \\ 1\le j\le n}}\right) \cdot det\left(A_{\substack{1<i<m \\ 1<j<n}}\right) = \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\le i<m \\ 1\le j<n}}\right) \cdot \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i\le m \\ 1<j\le n}}\right) -$$

$$\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\le i<m \\ 1<j\le n}}\right) \cdot \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i\le m \\ 1\le j<n}}\right) + \underset{(\varepsilon,p)}{det}\left(A_{\substack{1\le i\le m \\ 1<j<n}}\right) \cdot \underset{(\varepsilon,p-2)}{det}\left(A_{\substack{1<i<m \\ 1\le j\le n}}\right). \tag{6}$$

Proof. See Theorem 5.1 in Bayat (2020). □

The following we have developed computer algorithm (*det_Dodgson*) for Theorem 1.

Since this method is applied for $m \geq 3$, and $m \leq n-2$, m-number of rows, n-number of columns of the matrix. The following is presented on the pseudocode of Theorem 1.

P 1: Algorithm (*det_Dodgson*) for generalized Dodgson method to calculate rectangular determinants

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Step 1: Checking for conditions:
 if $m < 3$ or $m = n-1$
  Calculate rectangular determinant with known methods, like Laplace, Radic, Chios-like, etc.
 else if $m = n$
  Calculate square determinant with known methods.
 else
Step 2: Calculate submatrices:
 Calculate submatrices presented on Theorem 1, calling *det_Dodgson* algorithm until the conditions on step 1 are met, as
  $d1 = det\_Dodgson(A(1:m-1, 1:n-1));$
  $d2 = det\_Dodgson(A(1:m-1, 2:n));$
  $d3 = det\_Dodgson(A(2:m, 1:n-1));$
  $d4 = det\_Dodgson(A(2:m, 2:n));$
  $d5 = det\_Dodgson(A(2:m-1, 1:n));$
  $d6 = det\_Dodgson(A(1:m, 2:n-1));$
  $d7 = det\_Dodgson(A(2:m-1, 2:n-1));$
Step 3: After calculating submatrices, calculate the result of the determinant as
 $d = (d1 * d4 - d2 * d3 + d5 * d6)/d7.$

Recently, in 2022 we have found 9 different cases of Dodgson's generalization formula for rectangular determinant calculation, which is provided in Theorem 2.

Theorem 2. *Salihu et al. (2022) The pivot block* $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i<m\\1<j<n}}\right)$ *of Bayat's formula can be any block of order* $(m-2)\times(n-2)$ *from the given determinant, and the following cases are:*

Case 1: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\1\leq j\leq n-2}}\right)$;

Case 2: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\2\leq j\leq n-1}}\right)$;

Case 3: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\3\leq j\leq n}}\right)$;

Case 4: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\1\leq j\leq n-2}}\right)$;

Case 5: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\2\leq j\leq n-1}}\right)$;

Case 6: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\3\leq j\leq n}}\right)$;

Case 7: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\1\leq j\leq n-2}}\right)$;

Case 8: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\2\leq j\leq n-1}}\right)$;

Case 9: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\3\leq j\leq n}}\right)$.

Proof. See theorem 3 in Salihu et al. (2022). □

The pseudocode of each case from theorem 2 is like pseudocode presented in P 1, and changes in steps 2 for each case. For example, the pseudocode for Case 1 is changed as following:

P 2: Modified algorithm (*det_Dodgson*) based on theorem 2 (as example is considered Case 1)

Step 1: Checking for conditions:
  if $m < 3$ or $m = n - 1$
    Calculate rectangular determinant with known methods, like Laplace, Radic, Chios-like, etc.
  else if $m = n$
    Calculate square determinant with known methods.
  else
Step 2: Calculate submatrices:
  Calculate submatrices presented on Theorem 1, calling *det_Dodgson* algorithm until the conditions on step 1 are met, as following:
    $d1 = det\_Dodgson(A(1 : m - 1, 1 : n - 1));$
    $d2 = det\_Dodgson(A(1 : m - 1, [1 : n - 2 \quad n]));$
    $d3 = det\_Dodgson(A([1 : m - 2 \quad m], 1 : n - 1));$
    $d4 = det\_Dodgson(A([1 : m - 2 \quad m], [1 : n - 2 \quad n]));$
    $d5 = det\_Dodgson(A(1 : m - 2, 1 : n));$
    $d6 = det\_Dodgson(A(1 : m, 1 : n - 2));$
    $d7 = det\_Dodgson(A(1 : m - 2, 1 : n - 2));$
Step 3: After calculating submatrices, calculate the result of the determinant as following:
  $d = (d1 * d4 - d2 * d3 + d5 * d6)/d7.$

The pseudocode presented in P 2 represents Case 1 of theorem 2. However, the same algorithm can be used for each case of theorem 2, with changes in step 2 while selecting pivot block and reflecting that pivot block in each submatrix.

The above-mentioned theorem and pseudocode, has its advantage in cases of matrices with several zero elements. Where we have developed the algorithm that finds pivot block with highest number of zero elements. Which is presented in pseudocode P 3 Salihu et al. (2022).

P 3: Find the block of order $(m - 2) \times (n - 2)$ with highest number of zero elements

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Step 1: Insert the rectangular determinant A
Step 2: Calculate number of nonzero elements for each row/column
  Initialize R for rows and C for columns
  Create loop for i from 1 to m
    Create loop for j from 1 to n
      if $A(i, j) \neq 0$
        $R(i) = R(i) + 1;$
        $C(i) = C(i) + 1;$
      end
    end
  end
Step 3: Find the best case with the highest number of zero elements
  Initialize first case: $k = 1$
  if $C(2) + C(n - 1) < C(1) + C(n)$
    $k = 2;$

else if $C(1) + C(2) < C(n-1) + C(n)$
$k = 3$;
end
if $R(2) + R(m-1) < R(1) + R(m)$
$k = k + 3$;
else if $R(1) + R(2) > R(m-1) + R(m)$
$k = k + 6$;
end
Step 4: Return best case

Furthermore, we have improved the generalization of Dodgson's formula, where we generalized that pivot block is formed with elimination two rows and two columns. The results are presented in next section, more specifically in Theorem 3 and pseudocode P 4 and P 5.

However, the presented formulas on theorem 1, theorem 2, and theorem 3 does not hold for cases of order $(n-1) \times n$, for these cases Rezaifar's method holds Rezaifar et al. (2007), which is defined for square determinants, with little modification on order of determinants Salihu et al. (2022).

3 Main Results

Theorem 3. *Suppose that A is rectangular matrix of order $m \times n, m > 2$ and $m < n-1$, its determinant can be calculated using formula below:*

$$\underset{(\varepsilon,p)}{det}\left(A_{\substack{1\le i\le m\\1\le j\le n}}\right)\cdot\underset{(\varepsilon,p-2)}{det}\left(A_{\substack{i\neq k,l\\j\neq r,s}}\right)=\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq l\\j\neq s}}\right)\cdot\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq k\\j\neq r}}\right)-$$

$$\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq l\\j\neq r}}\right)\cdot\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq k\\j\neq s}}\right)+\underset{(\varepsilon,p)}{det}\left(A_{\substack{1\le i\le m\\j\neq r,s}}\right)\cdot\underset{(\varepsilon,p-2)}{det}\left(A_{\substack{i\neq k,l\\1\le j\le n}}\right) \quad (7)$$

Where k, l are any two rows of the matrix A and $k \neq l$. While r, s are any two columns of the matrix A and $r \neq s$.

For $m \leq 2$ and $m \geq n-1$, the theorem 3 does not hold.

Proof. The proof can be easily seen based on determinant properties. □

P 4: Modified algorithm (*det_Dodgson*) based on Theorem 3

Step 1: Checking for conditions:
if $m < 3$ or $m = n - 1$
Calculate rectangular determinant with known methods, like Laplace, Radic, Chios-like, etc.
else if $m = n$
Calculate square determinant with known methods.
else
Step 2: Calculate submatrices:
Calculate submatrices presented on Theorem 1, calling *det_Dodgson* algorithm until the conditions on step 1 are met, as following:
$d1 = det\_Dodgson(A([1:i-1 \quad i+1:m],[1:k-1 \quad k+1:n]))$;
$d2 = det\_Dodgson(A([1:i-1 \quad i+1:m],[1:l-1 \quad l+1:n]))$;

$d3 = det\_Dodgson(A([1:j-1 \quad j+1:m],[1:k-1 \quad k+1:n]));$
$d4 = det\_Dodgson(A([1:j-1 \quad j+1:m],[1:l-1 \quad l+1:n]));$
$d5 = det\_Dodgson(A([1:i-1 \quad i+1:j-1 \quad j+1:m],[1:n]));$
$d6 = det\_Dodgson(A([1:m],[1:k-1 \quad k+1:l-1 \quad l+1:n]));$
$d7 = det\_Dodgson(A([1:i-1 \quad i+1:j-1 \quad j+1:m],[1:k-1 \quad k+1:l-1 \quad l+1:n]));$

Step 3: After calculating submatrices, calculate the result of the determinant as
$d = (d1 * d4 - d2 * d3 + d5 * d6)/d7.$

Algorithms in P 1, P 2, and P 4 have the same time complexity, due to the calculation of the same number of submatrices of the same order. New method considers matrices with several zero elements. Since pivot block has the most zero elements, the two rows/columns with the most non-zero elements are not included. The pivot block with the most zero-elements is faster for calculation. We created a method to figure out the two rows/columns having the most non-zero elements in the pivot block.

The pseudocode below finds two rows/columns with the most non-zero elements. Which are considered while constructing the pivot block.

P 5: Find the block of order $(m-2) \times (n-2)$ with highest number of zero elements.

Step 1: Calculate number of nonzero elements for each row/column
  Initialize R for rows and C for columns
  Create loop for i from 1 to m
    Create loop for j from 1 to n
      if $A(i,j) \neq 0$
        $R(i) = R(i) + 1;$
        $C(i) = C(i) + 1;$
      end
    end
  end
Step 2: Find two rows with highest non-zero elements
  Initialize first two rows: $a = 1; b = 2$
  Create loop for k from 3 to m
    if $R(a) < R(k)$
      $a = k;$
    end
    if $R(b) < R(k)$ && $k \sim= a$
      $b = k;$
    end
  end
  Switch if $a > b$
    $temp = a;$
    $a = b;$
    $b = temp;$
Step 3: Find two columns with highest non-zero elements
  Initialize first two rows: $c = 1; d = 2$
  Create loop for l from 3 to n
    if $C(c) < C(l)$
      $c = l;$
    end
    if $C(d) < R(l)$ && $l \sim= c$

$d = l;$
end
end
Switch if $c > d$
$temp = c;$
$c = d;$
$d = temp;$
Step 4: Return index of two rows/columns with highest non-zero elements.

3.1 Computer execution time simulations

For the rectangular determinant computation, MATLAB on a Lenovo e15-gen 1 PC was used. With an Intel Core i7-1051U 1.8Ghz, 16 GB DDR4 of RAM.

In this simulation we evaluated two comparisons, first we compared algorithm P 1 based on theorem 1 with the newly presented algorithm in P 4, which is based on theorem 3. The second comparison was performed between algorithm P 2, which is based on Theorem 2 with the algorithm P 4, which is based on theorem 3. For the first comparison is used algorithm P 5 to identify two rows/columns with highest non-zero elements in order to eliminate from pivot block, which are used in algorithm P 4, while for the second comparison also is used the algorithm P 3 for finding block with the highest number of zero elements for use in algorithm P 2, and algorithm P5 to identify two rows/columns with highest non-zero elements in order to eliminate from pivot block.

The determinants are generated with integer numbers from 0 to 99999. We took care to generate randomly rectangular determinant of order $m \times n$, with an average of 39% of zero elements.

Table 1: First comparison of execution time, between algorithms P 1 and P 4

Order	Theorem 1	Theorem 3			1-2	1-4	(1/2-1)%	(1/4-1)%
	P 1	P 4	P 5	2 + 3				
	1	2	3	4				
5×10	0.01865	0.01745	0.00023	0.01768	0.00120	0.00097	6.88%	5.50%
5×15	0.12981	0.12239	0.00017	0.12256	0.00742	0.00725	6.06%	5.91%
5×20	0.60377	0.57086	0.00016	0.57101	0.03292	0.03276	5.77%	5.74%
6×10	0.06028	0.05669	0.00016	0.05685	0.00359	0.00343	6.34%	6.03%
6×15	0.68182	0.65026	0.00013	0.65038	0.03156	0.03144	4.85%	4.83%
6×20	4.66428	4.45273	0.00012	4.45285	0.21155	0.21143	4.75%	4.75%
7×10	0.09169	0.08203	0.00083	0.08286	0.00966	0.00884	11.78%	10.66%
7×15	3.14885	3.00258	0.00070	3.00329	0.14626	0.14556	4.87%	4.85%
7×20	27.09440	23.63900	0.00087	23.63987	3.45540	3.45453	14.62%	14.61%
8×10	0.29525	0.27573	0.00070	0.27643	0.01952	0.01882	7.08%	6.81%
8×15	10.93367	10.33271	0.00011	10.33283	0.60095	0.60084	5.82%	5.81%
9×15	53.10393	46.57677	0.00004	46.57681	6.52716	6.52712	14.01%	14.01%
10×15	113.62292	101.44142	0.00410	101.44552	12.18150	12.17740	12.01%	12.00%

For this simulation in both comparisons, we have tested for order from 5×7 to 5×20, 6×8 to 6×20, 7×9 to 7×20, 8×10 to 8×16, 9×11 to 9×15, and 10×12 to 10×15, some of the results for the first comparison are presented on table 1, while figure 1 presents result graphically. Some of the results of the second comparison are presented on table 2, while figure 2 presents all results graphically.

As it can be seen from Table 1 and Figure 1, in all tested cases the newly presented theorem/algorithm is executed faster than the algorithm based on theorem 1, with an average of 9.13% improvement compared to the theorem 1. If we consider also the time used for generating pivot block with highest number of zero elements, then this improvement is decreased to 8.86%.

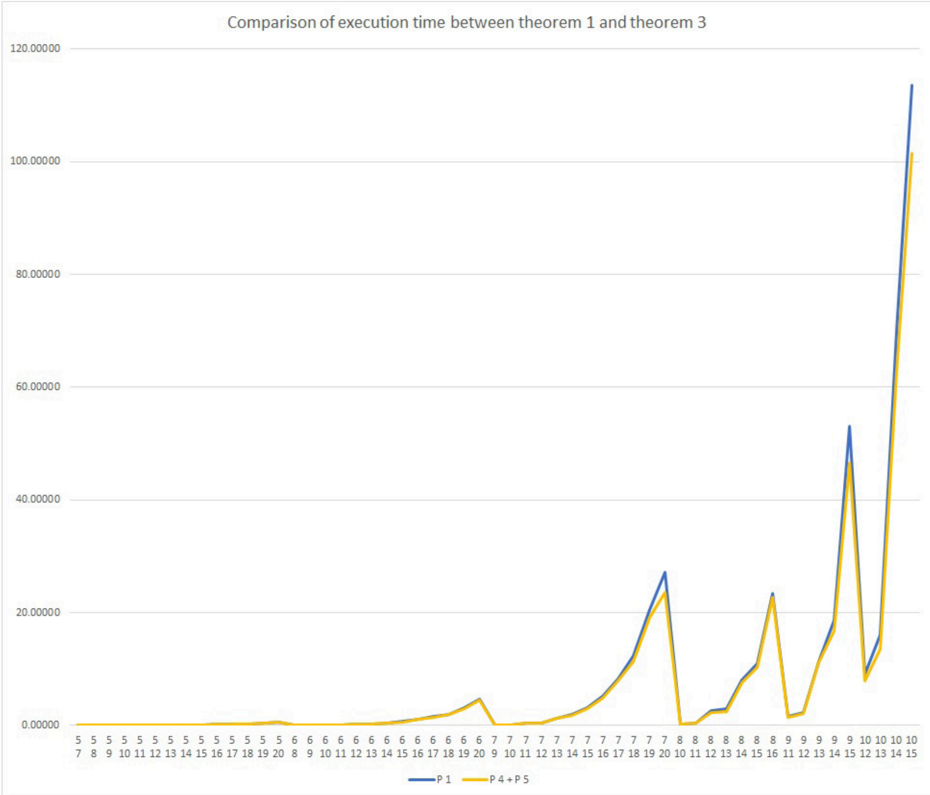

Figure 1: Comparison of execution time of determinant calculation between Theorem 1 and Theorem 3

Table 2: Second comparison of execution time, between algorithms P 2 and P 4

	Theorem 2			Theorem 3		
Order	P 2	P 3	1 + 2	P 4	P 5	4 + 5
	1	2	3	4	5	6
5×10	0.02069	0.00095	0.02163	0.01889	0.00089	0.01979
5×15	0.13218	0.00010	0.13228	0.12402	0.00010	0.12412
5×20	0.46704	0.00012	0.46715	0.46179	0.00010	0.46188
6×10	0.05891	0.00063	0.05954	0.05523	0.00066	0.05588
6×15	0.59147	0.00024	0.59170	0.58885	0.00018	0.58902
6×20	3.61102	0.00003	3.61105	3.48135	0.00004	3.48139
7×10	0.08799	0.00100	0.08900	0.08566	0.00107	0.08673
7×15	3.10514	0.00084	3.10597	3.06499	0.00092	3.06591
7×20	25.76111	0.00013	25.76124	23.84185	0.00011	23.84196
8×10	0.29595	0.00083	0.29678	0.28296	0.00087	0.28383
8×15	10.91854	0.00009	10.91863	10.61895	0.00009	10.61904
9×15	49.75410	0.00375	49.75785	45.61906	0.00419	45.62326
10×15	95.28347	0.00066	95.28413	91.66535	0.00066	91.66601

While from the second comparison as presented on table 2 and table 3 (some of the results) and graphically on figure 2, it is noted an improvement of 4.37% of newly presented theorem/algorithm over the theorem 2, respectively algorithm P 2.

Table 3: Differences of the second comparison

1-4	3-6	2-4	(1/4-1)%	(3/6-1)%	(2/5-1)%
0.00179	0.00185	0.00005	9.49%	9.34%	6.06%
0.00816	0.00816	0.00001	6.58%	6.58%	7.37%
0.00525	0.00527	0.00002	1.14%	1.14%	19.39%
0.00368	0.00366	-0.00003	6.67%	6.54%	-3.97%
0.00262	0.00268	0.00006	0.44%	0.45%	34.86%
0.12967	0.12966	-0.00001	3.72%	3.72%	-25.58%
0.00233	0.00227	-0.00007	2.72%	2.61%	-6.26%
0.04015	0.04006	-0.00008	1.31%	1.31%	-9.11%
1.91926	1.91929	0.00003	8.05%	8.05%	27.62%
0.01299	0.01295	-0.00003	4.59%	4.56%	-3.92%
0.29959	0.29959	0.00000	2.82%	2.82%	-1.10%
4.13503	4.13459	-0.00044	9.06%	9.06%	-10.59%
3.61812	3.61813	0.00001	3.95%	3.95%	1.37%

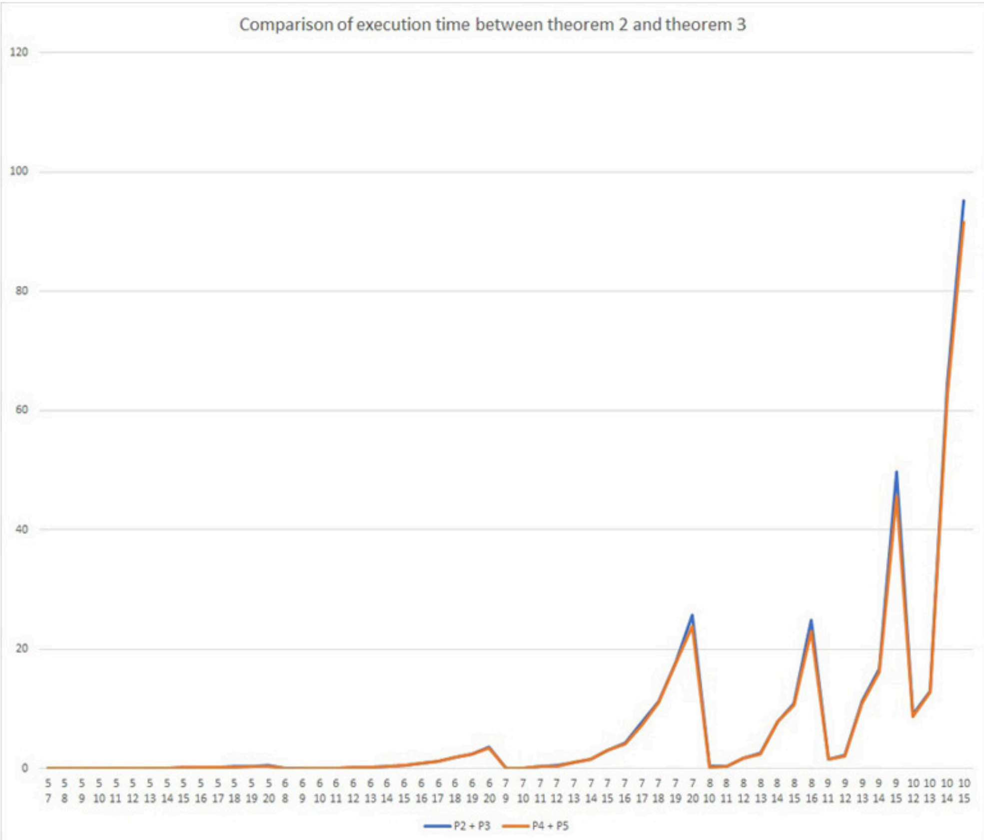

Figure 2: Comparison of execution time of determinant calculation between Theorem 2 and Theorem 3

4 Conclusion

In this study, we optimized Dodgson's condensation approach for computing rectangular matrix determinants. Dodgson's formula was used to convert the rectangular determinant to a second order square determinant, as shown in theorem 1. We've expanded this method/algorithm to calculate rectangular determinants by using nine different cases of determining pivot blocks, as presented in theorem 2.

We have further improved this approach such that any two rows/columns from the original matrix can be excluded from the pivot block. The advantages of newly presented approach are in cases where several elements are equal to zero, since operations with zero elements require less time to process, as well as in cases of determinant of pivot block equals to zero, since there can be generated another pivot block that its determinant is not equal to zero.

We also developed a computer algorithm to calculate the rectangular determinant based on theorem 3. To see the benefit of the above newly method, we developed an algorithm that finds two rows/columns with the highest non-zero elements that are excluded form pivot block with the highest possible zero elements.

In addition, we compared the execution time of different algorithms for computing rectangular determinants to evaluate if the newly presented formula/algorithm improves the calculation of rectangular determinants.

To see how zero elements affect algorithm performance. Initially, we evaluated the algorithms P 1, P 2, and P 4, calculating determinant of rectangular matrices with all non-zero elements and found that they all take about the same time. Then, we evaluated the execution time of all algorithms while generating rectangular determinants with integer elements ranging from 0 to 99999 and an average of 39% of zero elements. Some test results are shown in table 1,2 and 3, while graphically are presented in figures 1 and 2.

The first comparison the newly presented approach is compared with Bayat's formula, which uses the inner determinant of the original matrix as a pivot block. Our average improvement was 9.13%. Second, the newly presented method is compared against the algorithm that finds which of nine blocks holds the most zero elements. A 4.37% improvement was noted.

Regarding the effect of the algorithm to determine the two rows/columns with the highest non-zero elements presented in P 5 and the algorithm to determine the pivot block from 9 different cases with the highest number of zero elements presented in P 3, we tested both algorithms for a random matrix of order 10000×10000 and noticed that both algorithms are executed in about 3 seconds, implying that P 3 and P 5 have very little impact on overall performance of determinant calculation.

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