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CERTAIN PROPERTIES ON BELL-BASED APOSTOL-TYPE FROBENIUS-GENOCCHI POLYNOMIALS AND ITS APPLICATIONS*

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Abstract. In this paper new class of Bell-based Apostol-type Frobenius-Genocchi polynomials and derive some explicit and implicit summation formulas and symmetric identities is proved by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Apostol-type Frobenius-Genocchi polynomials and Bell-based Apostol-type Frobenius-Genocchi polynomials. Then, we determine the first few zero values of Bell-based Apostol-type Frobenius-Genocchi polynomials and draw the graphical representations for these zero values.

Keywords: Bell polynomials, Apostol-type Frobenius-Genocchi polynomials, Apostol-type Frobenius-Genocchi polynomials.

AMS Subject Classification: Primary 11B68, 33C45, 11Y16.

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1 Introduction

Recently, many authors (see Alam et al. (2023, 2022); Khan (2022); Muhiuddin et al. (2021); Pathan & Khan (2021a)) have introduced and constructed generating functions for new families of special polynomials including two parametric kinds of polynomials as Bernoulli, Euler, Genocchi, etc. They have given fundamental properties of these polynomials. Also, they have established more identities, and relations among trigonometric functions, two parametric kinds of special polynomials by using generating functions. Applying the partial derivative operator to these generating functions, some derivative formulae, and finite combinatorial sums involving the aforementioned polynomials and numbers are obtained. In addition, these special polynomials allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of special polynomials. The Apostol-type Frobenius-Genocchi polynomials appear in combinatorial mathematics and play an important role in the theory and applications of mathematics, thus many number theory and combinatorics experts have extensively studied

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their properties and obtained series of interesting results (see Khan & Srivastava (2019, 2021); Kang & Khan (2020); Khan et al. (2022); Muhiuddin et al. (2021); Pathan & Khan (2021b)).

The Apostol-type Frobenius-Euler polynomials $\mathbb{H}_j^{(\alpha)}(\xi;u;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by (see Kurt & Simsek (2013)):

$$\left(\frac{1-u}{\lambda e^z - u}\right)^\alpha e^{\xi z} = \sum_{j=0}^{\infty} \mathbb{H}_j^{(\alpha)}(\xi;u;\lambda)\frac{z^j}{j!}, \tag{1}$$

where $u \in \mathbb{C}\setminus\{1\}$, $\xi \in \mathbb{R}$ and $|z| < \left|\log\left(\frac{\lambda}{u}\right)\right|$.

At the point $\xi = 0$, $\mathbb{H}_j^{(\alpha)}(u;\lambda) = \mathbb{H}_j^{(\alpha)}(0;u;\lambda)$ are called the Apostol-type Frobenius-Euler numbers of order α. From (1), we find

$$\mathbb{H}_j^{(\alpha)}(\xi;u;\lambda) = \sum_{\nu=0}^{j} \left(\begin{array}{c} j \\ \nu \end{array}\right) \mathbb{H}_\nu^{(\alpha)}(u;\lambda)\xi^{j-\nu}, \tag{2}$$

and

$$\mathbb{H}_j^{(\alpha)}(\xi;-1;\lambda) = \mathbb{E}_j^{(\alpha)}(\xi;\lambda), \tag{3}$$

where $\mathbb{E}_j^{(\alpha)}(\xi;\lambda)$ are the j^{th} Apostol-Euler polynomial of order α.

The Apostol-type Frobenius-Genocchi polynomials are defined by means of the following generating relation (see Yaşar & Özarslan (2015)):

$$\frac{(1-u)z}{\lambda e^z - u} e^{\xi z} = \sum_{j=0}^{\infty} G_j^F(\xi;u;\lambda)\frac{z^j}{j!}, \tag{4}$$

where $u \in \mathbb{C}\setminus\{1\}$, $\xi \in \mathbb{R}$ and $|z| < \left|\log\left(\frac{\lambda}{u}\right)\right|$.

Note that

$$G_j^F(\xi;-1;\lambda) = G_j(\xi;\lambda),$$

where $G_j(\xi;\lambda)$ are called the Apostol-Genocchi polynomials.

For $j \geq 0$, the Stirling numbers of the first kind are defined by (see Alam et al. (2023, 2022); Bell (1934); Border (1984); Carlitz (1960); Duran et al. (2021); Khan (2022); Khan et al. (2022, 2019, 2021))

$$(\xi)_j = \sum_{p=0}^{j} S_1(j,p)\xi^p, \tag{5}$$

where $(\xi)_0 = 1$, and $(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1), (j \geq 1)$. From (5), we get

$$\frac{1}{r!}(\log(1+z))^r = \sum_{j=r}^{\infty} S_1(j,r)\frac{z^j}{j!}, \quad (r \geq 0). \tag{6}$$

For $j \geq 0$, the Stirling numbers of the second kind are defined by (see Khan & Srivastava (2019, 2021); Kang & Khan (2020); Khan & Khan (2020); Khan et al. (2022); Muhiuddin et al. (2021); Nisar & Khan (2019); Pathan & Khan (2021a, 2022, 2020); Yaşar & Özarslan (2015))

$$\xi^j = \sum_{q=0}^{j} S_2(j,q)(\xi)_q. \tag{7}$$

From (7), we see that (see Muhiuddin et al. (2021); Pathan & Khan (2021b))

$$\frac{1}{r!}(e^z-1)^r=\sum_{j=r}^{\infty}S_2(j,r)\frac{z^j}{j!}. \tag{8}$$

For any nonnegative integer r, the r-Stirling numbers $S_r(j,k)$ of the second kind are defined by (see Border (1984))

$$\frac{1}{k!}e^{rz}(e^z-1)^k=\sum_{j=k}^{\infty}S_r(j+r,k+r)\frac{z^j}{j!}. \tag{9}$$

The Apostol-type Bernoulli polynomials $\mathbb{B}_j^{(\alpha)}(\xi;\lambda)$ of order α, the Apostol-type Euler polynomials $\mathbb{E}_j^{(\alpha)}(\xi;\lambda)$ of order α and the Apostol-type Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi;\lambda)$ of order α are defined by (see Duran et al. (2021); Muhiuddin et al. (2021)):

$$\left(\frac{z}{\lambda e^z-1}\right)^{\alpha}e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{B}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!}\quad (\mid z+\log\lambda\mid)<2\pi, \tag{10}$$

$$\left(\frac{2}{\lambda e^z+1}\right)^{\alpha}e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!}\quad (\mid z+\log\lambda\mid)<\pi, \tag{11}$$

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha}e^{\xi z}=\sum_{j=0}^{\infty}\mathbb{G}_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!},(\mid z+\log\lambda\mid<\pi), \tag{12}$$

respectively.

Clearly, we have

$$\mathbb{B}_j^{(\alpha)}(\lambda)=\mathbb{B}_j^{(\alpha)}(0;\lambda),\mathbb{E}_j^{(\alpha)}(\lambda)=\mathbb{E}_j^{(\alpha)}(0;\lambda),\mathbb{G}_j^{(\alpha)}(\lambda)=\mathbb{G}_j^{(\alpha)}(0;\lambda).$$

The Bell polynomials $Bel_j(\xi)$ are defined by the generating function (see Bell (1934))

$$e^{\xi(e^z-1)}=\sum_{j=0}^{\infty}Bel_j(\xi)\frac{z^j}{j!}. \tag{13}$$

When $\xi=1$, $Bel_j=Bel_j(1),(j\geq 0)$ are called the Bell numbers. From (7) and (13), we note that

$$Bel_j(\xi)=\sum_{k=0}^{j}S_2(j,k)\xi^k\quad (j\geq 0). \tag{14}$$

The generalized Bell-based Bernoulli polynomials of two variables ${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi;\eta)$ are defined by (see Duran et al. (2021))

$$\left(\frac{z}{e^z-1}\right)^{\alpha}e^{\xi z+\eta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi;\eta)\frac{z^j}{j!}, \tag{15}$$

so that

$${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi;\eta)=\sum_{r=0}^{j}\binom{j}{r}\mathbb{B}_{j-r}^{(\alpha)}(\xi)Bel_r(\eta). \tag{16}$$

The paper is arranged as follows: In Section 2, we prove Bell-based Apostol-type Frobenius-Genocchi numbers and polynomials and investigate some properties of these numbers and polynomials. In Section 3, we derive summation formulas of Apostol-type Frobenius-Genocchi numbers and polynomials, connected with Apostol-type Bernoulli, Euler, and Genocchi polynomials.

In Section 4, we prove several identities of Bell-based Apostol-type Frobenius-Genocchi polynomials by using different analytical means and applying generating functions. In the last Section 5, we find some related beautiful zeros values and draw graphical representations of Bell-based Apostol-type Frobenius-Genocchi polynomials.

2 Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$

In this section, we define Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;\lambda)$ and explicit formula for the Apostol-type Frobenius-Genocchi polynomials and investigate its properties. First, we start with the following definition.

Definition 1. *For $u,\lambda \in \mathbb{C}$ with $u \neq 1$, and $\xi,\eta \in \mathbb{R}$, the generalized Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ of order α are defined by means of the following generating function:*

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z+\eta(e^z-1)} = \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!},\ |z| < \left|\log\left(\frac{\lambda}{u}\right)\right|. \tag{17}$$

Remark 1. *On taking $\xi = 0$ in (17), we get new type Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\eta;u;\lambda)$ of order α as follows:*

$$\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\eta(e^z-1)} = \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\eta;u;\lambda)\frac{z^j}{j!}. \tag{18}$$

Remark 2. *Upon setting $\eta = 0$ in (17), the Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ of order α reduces to familiar Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi;u;\lambda)$ of order α in (4).*

Remark 3. *When $\eta = 0$ and $\alpha = 1$, the polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ reduce to the usual Frobenius-Genocchi polynomials $\mathbb{G}_j(\xi;u;\lambda)$.*

We note that

$$ {}_{Bel}\mathbb{G}_j^{(1)}(\xi,\eta;u;\lambda) = {}_{Bel}\mathbb{G}_j(\xi,\eta;u;\lambda). \tag{19}$$

Theorem 1. *For $j \geq 0$, we have*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \sum_{s=0}^{j} \binom{j}{s} \mathbb{G}_s^{(\alpha)}(u;\lambda) Bel_{j-s}(\xi;\eta), \tag{20}$$

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \sum_{s=0}^{j} \binom{j}{s} \mathbb{G}_s^{(\alpha)}(\xi;u;\lambda) Bel_{j-s}(\eta), \tag{21}$$

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \sum_{s=0}^{j} \binom{j}{s} {}_{Bel}\mathbb{G}_s^{(\alpha)}(\eta;u;\lambda)\xi^{j-s}. \tag{22}$$

Proof. Using (4), (13), (15) and (17), we get representation (20)-(22). □

Theorem 2. *For $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha+\beta)}(\xi+w,\eta+v;u;\lambda) = \sum_{s=0}^{j} \binom{j}{s} {}_{Bel}\mathbb{G}_s^{(\beta)}(v,w;u;\lambda)\, {}_{Bel}\mathbb{G}_{j-s}^{(\alpha)}(\xi,\eta;u;\lambda), \tag{23} $$

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+\zeta,\eta;u;\lambda) = \sum_{s=0}^{j} \binom{j}{s} \mathbb{G}_{j-s}^{(\alpha)}(\xi;u;\lambda) Bel_s(\eta;\zeta). \tag{24} $$

Proof. In view of (15) and (17), we have

$$ \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha+\beta)}(\xi+\eta,\zeta+w;u;\lambda)\frac{z^j}{j!} = \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha+\beta} e^{(\xi+w)z+(\eta+v)(e^z-1)} $$

$$ = \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} \sum_{s=0}^{\infty} {}_{Bel}\mathbb{G}_s^{(\beta)}(v,w;u;\lambda)\frac{z^s}{s!} $$

$$ = \sum_{j=0}^{\infty}\sum_{s=0}^{j} \binom{j}{s} {}_{Bel}\mathbb{G}_s^{(\beta)}(v,w;u;\lambda)\, {}_{Bel}\mathbb{G}_{j-s}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!}. $$

Now equating the coefficients of the like powers of z in the above equation, we get the result (23). Again by using (15) and (17), we have

$$ \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{(\xi+\zeta)z+\eta(e^z-1)} = \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+\zeta,\eta;u;\lambda)\frac{z^j}{j!} \tag{25} $$

$$ \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} e^{\xi z} e^{\zeta z+\eta(e^z-1)} = \sum_{j=0}^{\infty} \mathbb{G}_j^{(\alpha)}(\xi;u;\lambda)\frac{z^j}{j!} \sum_{s=0}^{\infty} Bel_s(\zeta;\eta)\frac{z^s}{s!}. \tag{26} $$

$$ \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+\zeta,\eta;u;\lambda)\frac{z^j}{j!} = \sum_{j=0}^{\infty}\sum_{s=0}^{j} \binom{j}{s} \mathbb{G}_{j-s}^{(\alpha)}(\xi;u;\lambda) Bel_s(\zeta;\eta)\frac{z^j}{j!}, $$

yields formula (24). □

Theorem 3. *The following differentiation formulas for the Bell-based Apostol-type Frobenius-Genocchi polynomials of order α hold true:*

$$ \frac{\partial\, {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)}{\partial \xi} = j\, {}_{Bel}\mathbb{G}_{j-1}^{(\alpha)}(\xi,\eta;u;\lambda), \tag{27} $$

$$ \frac{\partial\, {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)}{\partial \eta} = {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi+1,\eta;u;\lambda) - {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda). \tag{28} $$

Proof. The proof follows from (17), we have

$$ \sum_{j=1}^{\infty} \frac{\partial}{\partial \xi} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} = \frac{\partial}{\partial \xi}\left(\frac{(1-u)t}{\lambda e^z-u}\right)^{\alpha} e^{\xi z+\eta(e^z-1)} $$

$$ = \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} \frac{\partial}{\partial \xi} e^{\xi z+\eta(e^z-1)} $$

$$ = \left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha} z e^{\xi z+\eta(e^z-1)} $$

$$= \sum_{j=1}^{\infty} {}_{Bel}\mathbb{G}_{j-1}^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^j}{j!},$$

the proof is completed. Again, using (17), we note that

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial \eta} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^j}{j!} = \frac{\partial}{\partial \eta} \left(\frac{(1-u)z}{\lambda e^z - u} \right)^{\alpha} e^{\xi z + \eta(e^z - 1)}$$

$$= \left(\frac{(1-u)z}{\lambda e^z - u} \right)^{\alpha} \frac{\partial}{\partial \eta} e^{\xi z + \eta(e^z - 1)}$$

$$= \left(\frac{(1-u)z}{\lambda e^z - u} \right)^{\alpha} e^{\xi z + \eta(e^z - 1)} (e^z - 1)$$

$$= \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi + 1, \eta; u; \lambda) \frac{z^j}{j!} - \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^j}{j!}.$$

Equating the coefficients of z, we get (28). □

Theorem 4. *Let $j \geq 0$. Then*

$$(2u-1) \sum_{k=0}^{j} \binom{j}{k} \mathbb{H}_k(\xi; u; \lambda) {}_{Bel}\mathbb{G}_{j-k}(\xi, \eta; 1-u; \lambda)$$

$$= u {}_{Bel}\mathbb{G}_j(\xi, \eta; u; \lambda) - (1-u) {}_{Bel}\mathbb{G}_j(\xi, \eta; 1-u; \lambda). \tag{29}$$

Proof. We set

$$\frac{(2u-1)}{(\lambda e^z - \lambda)(\lambda e^z - (1-u))} = \frac{1}{\lambda e^z - u} - \frac{1}{\lambda e^z - (1-u)}.$$

We see that

$$(2u-1) \frac{(1-u)ze^{\xi z}(1-(1-u))e^{\eta(e^z-1)}}{(\lambda e^z - u)(\lambda e^z - (1-u))} = \frac{(1-u)ze^{\eta(e^z-1)}ue^{\xi z}}{\lambda e^z - u} - \frac{(1-u)ze^{\eta(e^z-1)}ue^{\xi z}(1-(1-u))}{\lambda e^z - (1-u)},$$

$$(2u-1) \left(\sum_{k=0}^{\infty} \mathbb{H}_k(\xi; u; \lambda) \frac{z^k}{k!} \right) \left(\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j(\eta; 1-u; \lambda) \frac{z^j}{j!} \right)$$

$$= u \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j(\xi, \eta; u; \lambda) \frac{z^j}{j!} - (1-u) \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j(\xi, \eta; 1-u; \lambda) \frac{z^j}{j!}.$$

In view of the above equation, we obtain (29). □

Theorem 5. *For $j \geq 0$, we have*

$$u {}_{Bel}\mathbb{G}_j(\xi, \eta; u; \lambda) = \sum_{k=0}^{j} \binom{j}{k} \lambda {}_{Bel}\mathbb{G}_{j-k}(\xi, \eta; u; \lambda) - (1-u) Bel_{j-1}(\xi; \eta). \tag{30}$$

Proof. Consider

$$\frac{u}{\lambda(\lambda e^z - u)e^z} = \frac{1}{\lambda e^z - u} - \frac{1}{\lambda e^z}.$$

We find

$$\frac{u(1-u)ze^{\xi z + \eta(e^z-1)}}{\lambda(\lambda e^z - u)e^z} = \frac{(1-u)ze^{\xi z + \eta(e^z-1)}}{\lambda e^z - u} - \frac{(1-u)ze^{\xi z + \eta(e^z-1)}}{\lambda e^z}$$

$$u\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j(\xi,\eta;u;\lambda)\frac{z^j}{j!}=\lambda\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j(\xi,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\frac{z^k}{k!}-(1-u)\sum_{j=0}^{\infty}Bel_{j-1}(\xi;\eta)\frac{z^j}{j!}. \quad (31)$$

Therefore, by above equation, we get the result. □

Theorem 6. *Let $j \geq 0$. Then*

$$_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$$

$$=\frac{1}{1-u}\sum_{k=0}^{j}\binom{j}{k}\left[\lambda\mathbb{H}_{j-k}(u;\lambda)_{Bel}\mathbb{G}_k^{(\alpha)}(\xi+1,\eta;u;\lambda)-u\mathbb{H}_{j-k}(u;\lambda)_{Bel}\mathbb{G}_k^{(\alpha)}(\xi,\eta;u;\lambda)\right]. \quad (32)$$

Proof. In (17), we have

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!}=\left(\frac{(1-u)z}{\lambda e^z-u}\right)\left(\frac{\lambda e^z-u}{(1-u)z}\right)\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z+\eta(e^z-1)}$$

$$=\frac{1}{(1-u)z}\left[\lambda\left(\frac{(1-u)z}{\lambda e^z-u}\right)e^z\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z+\eta(e^z-1)}\right.$$
$$\left.-u\left(\frac{(1-u)z}{\lambda e^z-u}\right)\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\xi z+\eta(e^z-1)}\right]$$

$$=\frac{1}{1-u}\left[\sum_{j=0}^{\infty}\lambda\mathbb{H}_j(u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}{}_{Bel}\mathbb{G}_k^{(\alpha)}(\xi+1,\eta;u;\lambda)\frac{z^k}{k!}\right.$$
$$\left.-u\sum_{j=0}^{\infty}\mathbb{H}_j(u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}{}_{Bel}\mathbb{G}_k^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^k}{k!}\right].$$

By the above equation, we obtain statement of the theorem. □

Theorem 7. *Let $j \geq 0$. Then*

$$_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)=\sum_{s=0}^{j}\sum_{k=0}^{s}\binom{j}{s}(\xi)_k S_2(s,k)_{Bel}\mathbb{G}_j^{(\alpha)}(\eta;u;\lambda). \quad (33)$$

Proof. By (17), we note that

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!}=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\eta(e^z-1)}[e^z-1+1]^{\xi}$$

$$=\left(\frac{(1-u)z}{\lambda e^z-u}\right)^{\alpha}e^{\eta(e^z-1)}\sum_{k=0}^{\infty}(\xi)_k\frac{(e^z-1)^k}{k!}$$

$$=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(\eta;u;\lambda)\frac{z^j}{j!}\sum_{s=0}^{\infty}\sum_{k=0}^{s}(\xi)_k S_2(s,k)\frac{z^s}{s!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\sum_{k=0}^{s}\binom{j}{s}(\xi)_k S_2(s,k)_{Bel}\mathbb{G}_j^{(\alpha)}(\eta;u;\lambda)\right)\frac{z^j}{j!}.$$

This proves the theorem. □

3 Summation formulas for Bell-based Apostol-type Frobenius-Genocchi polynomials

In this section, we derive some implicit formulas for Bell-based Apostol-type Frobenius-Genocchi polynomials of order α related to Apostol-type Bernoulli polynomials, Apostol-type Euler polynomials, Apostol-type Genocchi polynomials and Stirling numbers of the second kind. Now, we begin with the following theorem.

Theorem 8. *The following formula holds true:*

$$ {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda) = \sum_{j,s=0}^{h,f} \binom{f}{s}\binom{h}{j} (\xi-\zeta)^{j+s}\, {}_{Bel}\mathbb{G}_{h+f-j-s}^{(\alpha)}(\zeta,\eta;u;\lambda). \tag{34}$$

Proof. By changing z with $z+w$ in (17), we have

$$\left(\frac{(1-u)(z+w)}{\lambda e^{(z+w)}-u}\right)^{\alpha} e^{\eta(e^{z+w}-1)} = e^{-\xi(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!}. \tag{35}$$

Again changing ξ with ζ in the above equation, we get

$$e^{-\zeta(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\zeta,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!} = \left(\frac{(1-u)(z+w)}{\lambda e^{(z+w)}-u}\right)^{\alpha} e^{\eta(e^{z+w}-1)}. \tag{36}$$

By the last equations, we obtain

$$e^{(\xi-\zeta)(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\zeta,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!} = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!}. \tag{37}$$

$$\begin{aligned}\sum_{N=0}^{\infty} \frac{[(\xi-\zeta)(z+w)]^N}{N!} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\zeta,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!} \\ = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!},\end{aligned} \tag{38}$$

which on using formula Pathan & Khan (2021a)

$$\sum_{N=0}^{\infty} f(N)\frac{(\zeta+\eta)^N}{N!} = \sum_{j,s=0}^{\infty} f(j+s)\frac{\zeta^j}{j!}\frac{\eta^s}{s!}, \tag{39}$$

$$\begin{aligned}\sum_{j,s=0}^{\infty} \frac{(\xi-\zeta)^{j+s} z^j w^s}{j!s!} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\zeta,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!} \\ = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!}.\end{aligned} \tag{40}$$

$$\begin{aligned}\sum_{h,f=0}^{\infty}\sum_{j,s=0}^{h,f} \frac{(\xi-\zeta)^{j+s}}{j!s!} {}_{Bel}\mathbb{G}_{h+f-j-s}^{(\alpha)}(\zeta,\eta;u;\lambda)\frac{z^h}{(h-j)!}\frac{w^f}{(f-s)!} \\ = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^h}{h!}\frac{w^f}{f!}.\end{aligned} \tag{41}$$

In view of above equation, we get the required result. □

Remark 4. *Letting $f = 0$ in (34), we get.*

$$ {}_{Bel}\mathbb{G}_h^{(\alpha)}(\xi, \eta; u; \lambda) = \sum_{j=0}^{h} \binom{h}{j} (\xi - \zeta)^j {}_{Bel}\mathbb{G}_{h-j}^{(\alpha)}(\zeta, \eta; u; \lambda), (j \geq 0). \tag{42} $$

Remark 5. *On changing ξ with $\xi + \zeta$ and setting $\eta = 0$ in (34), we get*

$$ {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\xi + \zeta; u; \lambda) = \sum_{j,s=0}^{h,f} \binom{f}{s} \binom{h}{j} \xi^{j+s} {}_{Bel}\mathbb{G}_{h+f-j-s}^{(\alpha)}(\zeta; u; \lambda), \tag{43} $$

whereas by setting $\xi = 0$ in (34), we get another result involving Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi, \eta; u; \lambda)$ of one and two variables

$$ {}_{Bel}\mathbb{G}_{h+f}^{(\alpha)}(\eta; u; \lambda) = \sum_{j,s=0}^{h,f} \binom{f}{s} \binom{h}{j} (-\zeta)^{j+s} {}_{Bel}\mathbb{G}_{h+f-j-s}^{(\alpha)}(\zeta, \eta; u; \lambda). $$

Theorem 9. *Let $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha+1)}(\xi, \eta; u; \lambda) = \sum_{d=0}^{j} \binom{j}{d} \mathbb{G}_{j-d}(u; \lambda) {}_{Bel}\mathbb{G}_d^{(\alpha)}(\xi, \eta; u; \lambda). \tag{44} $$

Proof. By (17), we have

$$ \frac{(1-u)z}{\lambda e^z - u} \left(\frac{(1-u)z}{\lambda e^z - u} \right)^{\alpha} e^{\xi z + \eta(e^z - 1)} = \frac{(1-u)z}{e^z - u} \sum_{d=0}^{\infty} {}_{Bel}\mathbb{G}_d^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^d}{d!} $$

$$ \left(\frac{(1-u)}{\lambda e^z - u} \right)^{\alpha+1} e^{\xi z + \eta(e^z - 1)} = \frac{(1-u)z}{\lambda e^z - u} \sum_{d=0}^{\infty} {}_{Bel}\mathbb{G}_d^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^d}{d!} $$

$$ = \sum_{j=0}^{\infty} \mathbb{G}_j(u; \lambda) \frac{z^j}{j!} \sum_{d=0}^{\infty} {}_{Bel}\mathbb{G}_d^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^d}{d!} $$

$$ = \sum_{j=0}^{\infty} \left(\sum_{d=0}^{j} \binom{j}{d} \mathbb{G}_{j-d}(u; \lambda) {}_{Bel}\mathbb{G}_d^{(\alpha)}(\xi, \eta; u; \lambda) \right) \frac{z^j}{j!}. $$

This proves the theorem.

□

Theorem 10. *Let $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi + 1, \eta; u; \lambda) = \sum_{p=0}^{j} \binom{j}{p} {}_{Bel}\mathbb{G}_p^{(\alpha)}(\xi, \eta; u; \lambda). \tag{45} $$

Proof. Using definition (17), we have

$$ \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi + 1, \eta; u; \lambda) \frac{z^j}{j!} - \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi, \eta; u; \lambda) \frac{z^j}{j!} $$

$$ = \left(\frac{(1-u)z}{\lambda e^z - u} \right)^{\alpha} e^{\xi z + \eta(e^z - 1)} (e^z - 1) $$

$$= \left(\sum_{p=0}^{\infty} {}_{Bel}\mathbb{G}_p^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^p}{p!}\right)\left(\sum_{j=0}^{\infty}\frac{z^j}{j!}\right) - \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\zeta;u;\lambda)\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{j}\binom{j}{p} {}_{Bel}\mathbb{G}_p^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} - \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!}.$$

This proves the theorem. □

Theorem 11. *Let $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \sum_{k=0}^{j+1}\binom{j+1}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}\mathbb{B}_{k-r}(\xi;\lambda) - \mathbb{B}_k(\xi;\lambda)\right) {}_{Bel}\mathbb{G}_{j-k+1}^{(\alpha)}(0,\eta;u;\lambda). \tag{46}$$

Proof. In view of (10) and (17), we have

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} = \left(\frac{(1-u)z}{\lambda e^z - u}\right)^{\alpha} e^{\xi z+\eta(e^z-1)}\left(\frac{z}{\lambda e^z - 1}\right)\left(\frac{\lambda e^z - 1}{z}\right)$$

$$= \frac{1}{z}\left(\lambda\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{B}_k(\xi;\lambda)\frac{z^k}{k!}\sum_{r=0}^{\infty}\frac{z^r}{r!} - \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{B}_k(\xi;\lambda)\frac{z^k}{k!}\right).$$

On equating the coefficients of same powers of z after using Cauchy product rule in above equation, we get the result. □

Theorem 12. *Let $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \frac{1}{2}\sum_{k=0}^{j}\binom{j}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}\mathbb{E}_{k-r}(\xi;\lambda) + \mathbb{E}_k(\xi;\lambda)\right) {}_{Bel}\mathbb{G}_{j-k}^{(\alpha)}(0,\eta;u;\lambda). \tag{47}$$

Proof. From (11) and (17), we have

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} = \left(\frac{(1-u)z}{\lambda e^z - u}\right)^{\alpha} e^{\xi z+\eta(e^z-1)}\left(\frac{2}{\lambda e^z + 1}\right)\left(\frac{\lambda e^z + 1}{2}\right)$$

$$= \frac{1}{2}\left(\lambda\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{E}_k(\xi;\lambda)\frac{z^k}{k!}\sum_{r=0}^{\infty}\frac{z^r}{r!} + \sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{E}_k(\xi;\lambda)\frac{z^k}{k!}\right).$$

On equating the coefficients of same powers of z after using Cauchy product rule in above equation, we get the result. □

Theorem 13. *Let $j \geq 0$. Then*

$$ {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda) = \frac{1}{2}\sum_{k=0}^{j+1}\binom{j+1}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}\mathbb{G}_{k-r}(\xi;\lambda) + \mathbb{G}_k(\xi;\lambda)\right) {}_{Bel}\mathbb{G}_{j-k+1}^{(\alpha)}(0,\eta;u;\lambda). \tag{48}$$

Proof. By (12) and (17) , we have

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)\frac{z^j}{j!} = \left(\frac{(1-u)z}{\lambda e^z - u}\right)^{\alpha} e^{\xi z+\eta(e^z-1)}\left(\frac{2z}{\lambda e^z + 1}\right)\left(\frac{\lambda e^z + 1}{2z}\right)$$

$$= \frac{1}{2z}\left(\lambda\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{G}_k(\xi;\lambda)\frac{z^k}{k!}\sum_{r=0}^{\infty}\frac{z^r}{r!}\right.$$

$$\left.+\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(0,\eta;u;\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\mathbb{G}_k(\xi;\lambda)\frac{z^k}{k!}\right).$$

On equating the coefficients of same powers of z after using Cauchy product rule in above equation, we finish the proof. □

4 Identities for Bell-based Apostol-type Frobenius-Genocchi polynomials

In this section, we give general symmetry identities for the Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ and generalized Apostol-type Frobenius-Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi;u;\lambda)$ by applying the generating functions (5) and (17).

Theorem 14. *Let $a,b,>0$ with $a\neq b$ and $j\geq 0$. Then*

$$\sum_{k=0}^{j}\binom{j}{k}b^k a^{j-k}{}_{Bel}\mathbb{G}j-k^{(\alpha)}(b\xi,\eta;u;\lambda){}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi,\eta;u;\lambda)$$

$$=\sum_{k=0}^{j}\binom{j}{k}a^k b^{j-k}{}_{Bel}\mathbb{G}_{j-k}^{(\alpha)}(a\xi,\eta;u;\lambda){}_{Bel}\mathbb{G}_k^{(\alpha)}(b\xi,\eta;u;\lambda). \tag{49}$$

Proof. Let

$$A(z)=\left(\frac{(1-u)^2z^2}{(\lambda e^{az}-u)(\lambda e^{bz}-u)}\right)^{\alpha}e^{2ab\xi z+\eta(e^{az}-1)+\eta(e^{bz}-1)}.$$

Then the expression for $A(z)$ is symmetric in a and b, we obtain

$$A(z)=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(b\xi,\eta;u;\lambda)\frac{(az)^j}{j!}\sum_{k=0}^{\infty}{}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi,\eta;u;\lambda)\frac{(bz)^k}{k!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}b^k a^{j-k}{}_{Bel}\mathbb{G}_{j-k}^{(\alpha)}(b\xi,\eta;u;\lambda){}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi,\eta;u;\lambda)\right)\frac{z^j}{j!}.$$

Similarly, we can show that

$$A(z)=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{G}_j^{(\alpha)}(a\xi,\eta;u;\lambda)\frac{(bz)^j}{j!}\sum_{k=0}^{\infty}{}_{Bel}\mathbb{G}_k^{(\alpha)}(b\xi,\eta;u;\lambda)\frac{(az)^k}{k!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}a^k b^{j-k}{}_{Bel}\mathbb{G}_{j-k}^{(\alpha)}(a\xi,\eta;u;\lambda){}_{Bel}\mathbb{G}_k^{(\alpha)}(b\xi,\eta;u;\lambda)\right)\frac{z^j}{j!}.$$

On comparing the coefficients of z^j on the right hand sides of the last two equations, we arrive at the desired result. □

Remark 6. *For $\alpha = 1$ in (49), the result reduces to*

$$\sum_{k=0}^{j} \binom{j}{k} b^k a^{j-k}{}_{Bel}\mathbb{G}_{j-k}(b\xi, \eta; u; \lambda)_{Bel}\mathbb{G}_k(a\xi, \eta; u; \lambda)$$

$$= \sum_{k=0}^{j} \binom{j}{k} a^k b^{j-k}{}_{Bel}\mathbb{G}_{j-k}(a\xi, \eta; u; \lambda)_{Bel}\mathbb{G}_k(b\xi, \eta; u; \lambda). \tag{50}$$

Theorem 15. *Let $a, b, > 0$ with $a \neq b$ and $s \geq 0$. Then*

$$\sum_{k=0}^{s} \binom{s}{k} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} a^{s-k} b^k {}_{Bel}\mathbb{G}_{s-k}^{(\alpha)}\left(b\xi + \frac{b}{a}i + j, \eta; u; \lambda\right) {}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi, \eta; u; \lambda)$$

$$= \sum_{k=0}^{s} \binom{s}{k} \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j} b^{s-k} a^k {}_{Bel}\mathbb{G}_{s-k}^{(\alpha)}\left(a\xi + \frac{a}{b}i + j, \eta; u; \lambda\right) {}_{Bel}\mathbb{G}_k^{(\alpha)}(b\xi, \eta; u; \lambda). \tag{51}$$

Proof. Consider the identity

$$B(z) = \left(\frac{(1-u)^2 z^2}{(\lambda e^{az} - u)(\lambda e^{bz} - u)}\right)^{\alpha} \frac{1 + \lambda(-1)^{a+1} e^{abz}}{(\lambda e^{az} + 1)(\lambda e^{bz} + 1)} e^{2ab\xi z + \eta(e^{az}-1) + \eta(e^{bz}-1)}$$

$$B(z) = \left(\frac{(1-u)z}{\lambda e^{az} - u}\right)^{\alpha} e^{ab\xi z + \eta(e^{az}-1)} \left(\frac{1 - \lambda(-e^{-bz})^a}{\lambda e^{bz} + 1}\right) \left(\frac{(1-u)z}{\lambda e^{bz} - u}\right)^{\alpha}$$
$$\times \left(\frac{1 - \lambda(-e^{-az})^b}{\lambda e^{az} + 1}\right) e^{ab\xi z + \eta(e^{bz}-1)}$$

$$= \left(\frac{(1-u)z}{\lambda e^{az} - u}\right)^{\alpha} e^{ab\xi z + \eta(e^{az}-1)} \sum_{i=0}^{a-1}(-\lambda)^i e^{bzi} \left(\frac{(1-u)z}{\lambda e^{bz} - u}\right)^{\alpha} e^{ab\xi z + \eta(e^{bz}-1)} \sum_{j=0}^{b-1}(-\lambda)^j e^{azj}$$

$$= \left(\frac{1-u}{\lambda e^{az} - u}\right)^{\alpha} e^{\eta(e^{az}-1)} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} e^{(b\xi + \frac{b}{a}i + j)at} \sum_{k=0}^{\infty} {}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi, \eta; u; \lambda) \frac{(bz)^k}{k!}$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} {}_{Bel}\mathbb{G}_j^{(\alpha)}\left(b\xi + \frac{b}{a}i + j, \eta; u; \lambda\right) \frac{(az)^s}{s!} \sum_{k=0}^{\infty} {}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi, \eta; u; \lambda) \frac{(bz)^k}{(k)!}$$

$$= \sum_{s=0}^{\infty}\sum_{k=0}^{s} \binom{s}{k} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j} a^{s-k} b^k {}_{Bel}\mathbb{G}_{s-k}^{(\alpha)}\left(b\xi + \frac{b}{a}i + j, \eta; u; \lambda\right)$$
$$\times {}_{Bel}\mathbb{G}_k^{(\alpha)}(a\xi, \eta; u; \lambda) \frac{z^j}{j!}.$$

On the other hand, we have

$$B(z) = \sum_{s=0}^{\infty}\sum_{k=0}^{s} \binom{s}{k} \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j} b^{s-k} a^k {}_{Bel}\mathbb{G}_{s-k}^{(\alpha)}\left(a\xi + \frac{a}{b}i + j, \eta; u; \lambda\right)$$
$$\times {}_{Bel}\mathbb{G}_k^{(\alpha)}(b\xi, \eta; u; \lambda) \frac{z^j}{j!}.$$

On comparing the coefficients of z^j on the right hand sides of the last two equations, we arrive at the desired result. □

5 Computational values and graphical representations of Bell-based Apostol-type Frobenius-Genocchi polynomials

In this section, computational values and graphical representations of Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ are shown.

A few of them are

$$
\begin{aligned}
{}_{Bel}\mathbb{G}_0^{(2)}(\xi,\eta;u;\lambda) &= 0,\\
{}_{Bel}\mathbb{G}_1^{(2)}(\xi,\eta;u;\lambda) &= 0,\\
{}_{Bel}\mathbb{G}_2^{(2)}(\xi,\eta;u;\lambda) &= \frac{2(-1+u)^2}{(u-\lambda)^2},\\
{}_{Bel}\mathbb{G}_3^{(2)}(\xi,\eta;u;\lambda) &= \frac{6(-1+u)^2(-\lambda(-2+\eta+\xi)+u(\eta+\xi))}{(u-\lambda)^3},\\
{}_{Bel}\mathbb{G}_4^{(2)}(\xi,\eta;u;\lambda) &= \frac{12(-1+u)^2(2\lambda(u+2\lambda)-4\lambda(-u+\lambda)(\eta+\xi))}{(u-\lambda)^4}\\
&\quad+\frac{12(-1+u)^2(u-\lambda)^2(\eta+(\eta+\xi)^2)}{(u-\lambda)^4},\\
{}_{Bel}\mathbb{G}_5^{(2)}(\xi,\eta;u;\lambda) &= -\frac{40(-1+u)^2(u^2\lambda+7u\lambda^2+4\lambda^3-3\lambda(-u+\lambda)(u+2\lambda)(\eta+\xi))}{(-u+\lambda)^5}\\
&\quad-\frac{120(-1+u)^2(u-\lambda)^2\lambda(\eta+(\eta+\xi)^2)}{(-u+\lambda)^5}\\
&\quad+\frac{20(-1+u)^2(-u+\lambda)^3(\eta+2\eta(\eta+\xi)+(\eta+\xi)(\eta+(\eta+\xi)^2))}{(-u+\lambda)^5}.
\end{aligned}
$$

We investigate the beautiful zeros of the Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(\alpha)}(\xi,\eta;u;\lambda)$ by using a computer. We plot the zeros of the Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(2)}(\xi,\eta;u;\lambda)=0$ for $j=20$ (Figure 1).

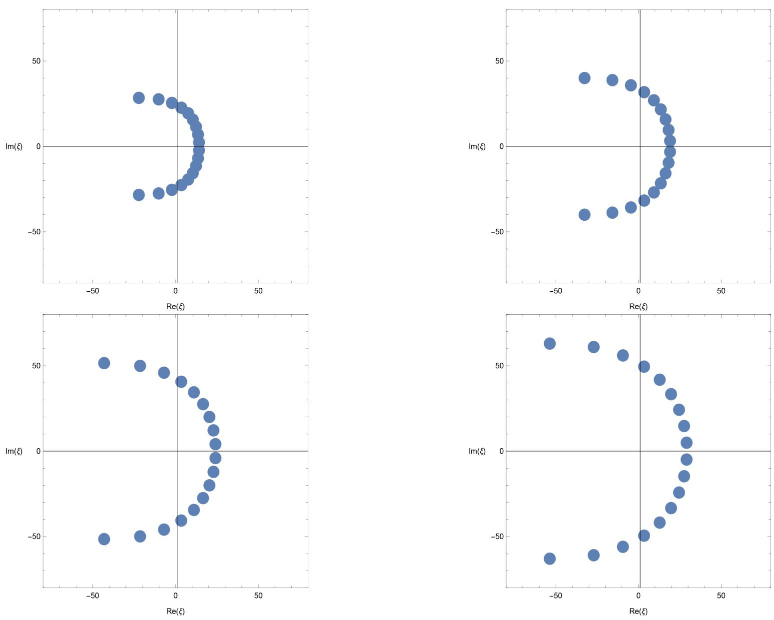

Figure 1: Zeros of ${}_{Bel}\mathbb{G}_j^{(2)}(\xi,\eta;u;\lambda)=0$

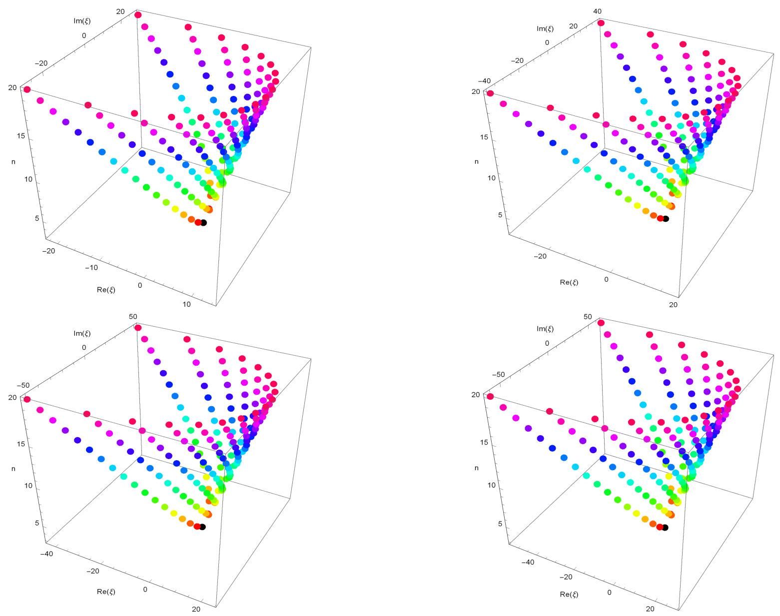

Figure 2: Zeros of ${}_{Bel}\mathbb{G}_j^{(2)}(\xi, \eta; u; \lambda) = 0$

Table 1. Approximate solutions of ${}_{Bel}\mathbb{G}_j^{(2)}(\xi, 3; 2; 3) = 0$

degree j	ξ
3	3.0000
4	$3.0000 - 3.8730i \quad 3.0000 + 3.8730i$
5	$2.3871 - 6.7917i, \quad 2.3871 + 6.7917i, \quad 4.2257$
6	$1.4526 - 9.1777i, \quad 1.4526 + 9.1777i,$ $4.5474 - 3.2493i, \quad 4.5474 + 3.2493i$
7	$0.2913 - 11.2248i, \quad 0.2913 + 11.2248i,$ $4.4567 - 6.0495i, \quad 4.4567 + 6.0495i, \quad 5.5040$
8	$-1.0419 - 13.0467i, \quad -1.0419 + 13.0467i, \quad 4.0983 - 8.5046i,$ $4.0983 + 8.5046i, \quad 5.9436 - 2.9433i, \quad 5.9436 + 2.9433i$
9	$-2.504 - 14.712i, \quad -2.504 + 14.712i, \quad 3.5279 - 10.6883i,$ $3.5279 + 10.6883i, \quad 6.0710 - 5.6194i, \quad 6.0710 + 5.6194i, \quad 6.8112,$
10	$-4.064 - 16.262i, \quad -4.064 + 16.262i, \quad 2.7747 - 12.6634i,$ $2.7747 + 12.6634i, \quad 5.9768 - 8.0629i, \quad 5.9768 + 8.0629i,$ $7.3124 - 2.7542i, \quad 7.3124 + 2.7542i$
11	$-5.698 - 17.720i, \quad -5.698 + 17.720i, \quad 1.864 - 14.482i,$ $1.864 + 14.482i, \quad 5.7048 - 10.3017i, \quad 5.7048 + 10.3017i,$ $7.5614 - 5.3280i, \quad 7.5614 + 5.3280i, \quad 8.1364$

In Figure 1(top-left), we choose $\eta = 3, u = 2$, and $\lambda = 3$, In Figure 1(top-right), we choose $\eta = 5, u = 3$, and $\lambda = 4$, In Figure 1(bottom-left), we choose $\eta = 5, u = 4$, and $\lambda = 5$, In Figure 1(bottom-right), we choose $\eta = 9, u = 5$, and $\lambda = 6$,

Stacks of zeros of the Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(2)}(\xi, \eta; u; \lambda) = 0$ for $3 \leq j \leq 20$, forming a 3D structure, are presented (Figure 2).

In Figure 2(top-left), we choose $\eta = 3, u = 2$, and $\lambda = 3$, In Figure 2(top-right), we choose $\eta = 5, u = 3$, and $\lambda = 4$, In Figure 2(bottom-left), we choose $\eta = 7, u = 4$, and $\lambda = 5$, In Figure 2(bottom-right), we choose $\eta = 9, u = 5$, and $\lambda = 6$,

Next, we calculated an approximate solution satisfying the Bell-based Apostol-type Frobenius-Genocchi polynomials ${}_{Bel}\mathbb{G}_j^{(2)}(\xi, 3; 2; 3) = 0$. The results are given in Table 1.

6 Conclusion

In this paper, we have proved Bell-based Apostol-type Frobenius-Genocchi numbers and polynomials and investigated some properties of these numbers and polynomials. We derived summation formulas of Bell-based Apostol-type Frobenius-Genocchi numbers and polynomials, connected with Apostol-type Bernoulli, Euler, and Genocchi polynomials. Also we proved several identities of Bell-based Apostol-type Frobenius-Genocchi polynomials by using different analytical means and applying generating functions. Moreover, we have derived the first few zero values of sine and cosine Bell-based Apostol-type Frobenius-Genocchi polynomials and have drawn the graphical representations for these zero values. Consequently, the results of this article may potentially be used in mathematics, in mathematical physics and in engineering.

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