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MODIFIED CHIOS-LIKE METHOD FOR RECTANGULAR DETERMINANT CALCULATIONS

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Abstract. In this paper, we present an optimization of the Chio's-like approach for calculating the $m \times n$ order rectangular determinants. Since the Chio's-like method reduces the order of determinant for 1, the newly approach reduces the order of determinant for 2. Also, we developed the computer algorithm and compared to the algorithm based on Cullis/Radic definition as well as Chio's-like method. A significant improvement is noted on the execution time. Compared to the Cullis/Radic definition an improvement of 33% is noted, while compared to the Chio's-like method an improvement of 20% is noted. Furthermore, we performed the time complexity analysis of the newly presented algorithm and is found that the asymptotic time complexity of newly presented algorithm is $O(m^2 \cdot n \cdot (n-m))$.

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1 Introduction

The determinant of rectangular matrixes, initially it was presented by Cullis (1913) in his book Matrices and Determinandoids, and improved by Radic (1966), and several rectangular determinant properties were presented.

Related to the rectangular determinants the permutation method is defined as follows:

$$\det(A_{m\times n}) = |A_{m\times n}| \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} \tag{1}$$

$$= \sum_{j_1<j_2<\cdots<j_m} (-1)^{r+s} \cdot \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix},$$

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where $r = 1 + \cdots + m, s = j_1 + \cdots + j_m$, which was later defined by Stojakovic (1952) as follows:

$$det_n A = \sum_{(j)}^{\binom{r}{n}} \left\{ \sum_{(l)}^{\binom{s}{n}} \left[\sum_{(\sigma)}^{n!} (-1)^{J(\sigma)} \cdot a_{\rho_1\sigma_1} \cdots a_{\rho_n\sigma_n} \right]_{(i)} \right\}_{(j)} . \tag{2}$$

Related to the rectangular determinants another definition is given by Joshi (1980), which provides the same result as the Cullis/Radic definition.

$$\sum_{d=1}^{n-m+1} \sum_{p=1}^{N_d} \det A_p^d, \tag{3}$$

where the N_d is cardinal number of S_d, and A_p^d is a submatrix of determinant A of order $m \times n$, and S_d is defined as follows:

$$S_d = \{e_d^p = (d, K_{p2}, \cdots, K_{pm})\}. \tag{4}$$

Another definition of rectangular determinants is provided in 1997 by Stanimirovic & Stankovic (1997):

$$det_{(\varepsilon,p)} A = \sum_{\alpha_1 < \cdots < \alpha_p \beta_1 < \cdots < \beta_p} \epsilon^{(\alpha_1 + \cdots + \alpha_p) + (\beta_1 + \cdots + \beta_p)} A \begin{pmatrix} \alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_p \end{pmatrix} \tag{5}$$

Bayat (2020), expanded the definition of the rectangular determinant as follows:

Determinant of $A \in \mathbb{C}^{m \times n}$ is a function $det_{(\overrightarrow{\varepsilon},p)} : \mathbb{C}^{m \times n} \to \mathbb{C}$ defined by:

$$\underset{(\overrightarrow{\varepsilon},p)}{det}(A) = \begin{cases} \sum_{\substack{I \in Q_{p,m} \\ J \in Q_{p,n}}} \overrightarrow{\varepsilon} det(A[I,J]), & \text{if } 1 \le p \le min\{m,n\} \\ 1 & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

scalars $\overrightarrow{\varepsilon}_{I,J}$ are components of vector $\overrightarrow{\varepsilon} \in \mathbb{C}^k$ for $k = \binom{m}{p}\binom{n}{p}$.

Different definitions are provided which are depended on the value of $\overrightarrow{\varepsilon}_{I,J}$. The determinant is square n case that $n = m = p$ and $\overrightarrow{\varepsilon} = 1$. In case of $\overrightarrow{\varepsilon}_{I,J} = 1$ and $I \in Q_{p,m}$ and $J \in Q_{p,n}$ we have the Stojakovic definition. In case of $\varepsilon = -1$, we have the Radic definition, and in case of $\overrightarrow{\varepsilon}_{I,J} = \varepsilon^{\sum_{l=1}^{p}(i_l + j_l)}$ for $I \in Q_{p,m}$ and $J \in Q_{p,n}$, we have the definition of Stanimirovic and Stankovic.

In following is provide another methodology of implementation in the rectangular determinants of the Dodgson's method provided by Amiri et al. (2010).

$$det\left(A_{\substack{1\le i\le m \\ 1\le j\le n}}\right) \cdot det\left(A_{\substack{i\ne m-1,m \\ j\ne n-1,n}}\right) = det\left(A_{\substack{i\ne m \\ j\ne n}}\right) \cdot det\left(A_{\substack{i\ne m-1 \\ j\ne n-1}}\right) -$$

$$det\left(A_{\substack{i\ne m \\ j\ne n-1}}\right) \cdot det\left(A_{\substack{i\ne m-1 \\ j\ne n}}\right) + det\left(A_{i\ne m-1,m}\right) \cdot det\left(A_{j\ne n-1,n}\right). \tag{7}$$

Considering as pivot block the inner determinant of original matrix is modified the Amiri's method later by Bayat (2020). An A of order $m \times n$ a rectangular matrix. Then for $p = min(m,n) \ge 2$, we have:

$$det\left(A_{\substack{1\le i\le m \\ 1\le j\le n}}\right) \cdot det\left(A_{\substack{1<i<m \\ 1<j<n}}\right) = \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\le i<m \\ 1\le j<n}}\right) \cdot \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i\le m \\ 1<j\le n}}\right) -$$

$$\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\le i<m \\ 1<j\le n}}\right) \cdot \underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i\le m \\ 1\le j<n}}\right) + \underset{(\varepsilon,p)}{det}\left(A_{\substack{1\le i\le m \\ 1<j<n}}\right) \cdot \underset{(\varepsilon,p-2)}{det}\left(A_{\substack{1<i<m \\ 1\le j\le n}}\right) \tag{8}$$

which later was improved where we have identified 9 different cases creation of pivot block based on Dodgson's method used in rectangular determinant calculation, as follows Salihu et al. (2022a):

Theorem 1. *The pivot block* $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1<i<m\\1<j<n}}\right)$ *of Bayat's formula can be any block of order* $(m-2)\times(n-2)$ *from the given determinant, and the following cases are:*

Case 1: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\1\leq j\leq n-2}}\right)$.

Case 2: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\2\leq j\leq n-1}}\right)$.

Case 3: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{1\leq i\leq m-2\\3\leq j\leq n}}\right)$.

Case 4: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\1\leq j\leq n-2}}\right)$.

Case 5: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\2\leq j\leq n-1}}\right)$.

Case 6: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{2\leq i\leq m-1\\3\leq j\leq n}}\right)$.

Case 7: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\1\leq j\leq n-2}}\right)$.

Case 8: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\2\leq j\leq n-1}}\right)$.

Case 9: Pivot block is: $\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{3\leq i\leq m\\3\leq j\leq n}}\right)$.

Proof. See Theorem 3 in Salihu et al. (2022a). □

We have further improved the above-mentioned theorem by creating pivot block during the removal of any two rows and any two columns, as follows Salihu et al. (2022b):

Theorem 2. *Suppose that A is rectangular matrix of order $m\times n, m>2$ and $m<n-1$, its determinant can be calculated using formula below:*

$$\underset{(\varepsilon,p)}{det}\left(A_{\substack{1\leq i\leq m\\1\leq j\leq n}}\right)\cdot\underset{(\varepsilon,p-2)}{det}\left(A_{\substack{i\neq k,l\\j\neq r,s}}\right)=\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq l\\j\neq s}}\right)\cdot\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq k\\j\neq r}}\right)-$$

$$\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq l\\j\neq r}}\right)\cdot\underset{(\varepsilon,p-1)}{det}\left(A_{\substack{i\neq k\\j\neq s}}\right)+\underset{(\varepsilon,p)}{det}\left(A_{\substack{1\leq i\leq m\\j\neq r,s}}\right)\cdot\underset{(\varepsilon,p-2)}{det}\left(A_{\substack{i\neq k,l\\1\leq j\leq n}}\right),\qquad(9)$$

where k,l are any two rows of the matrix A and $k\neq l$. While r,s are any two columns of the matrix A and $r\neq s$.

For $m\leq 2$ and $m\geq n-1$, the Theorem 3 does not hold.

Proof. See Theorem 3 in Salihu et al. (2022b). □

For which we have also provided the time complexity analysis Salihu et al. (2022c), also provided the comparison of number of operations for the different methods for rectangular determinant calculations Salihu et al. (2023)

In Salihu & Marevci (2021), we have implemented the Chio's condensation method used in square determinants to be applied also in square determinants (see Section 2 for details of this implementation). Which is further improved in this paper, which is based on the concept of changing order of determinants Salihu & Marevci (2019).

2 Chios-like method for rectangular determinant calculations

Theorem 3. *(Chio's-like method for rectangular determinants): For rectangular determinants of the order $m \times n$, in cases for 2×3, 2×4 and 3×4, the following formula holds:*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_c|}{a_{11}^{m-2}} + (-1)^m \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}, \tag{10}$$

where

$$|A_c| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-1)\times(n-1)} \tag{11}$$

and $a_{11} \neq 0$.

Proof. See Theorem 2.2 in Salihu & Marevci (2021). □

The following is presented pseudocode of computer algorithm (*det_Chio*) for Theorem 3.

P 1: Pseudocode of *det_Chio* algorithm for Theorem 3 (Chio's-like) method to calculate rectangular determinants of order $m \times n$ (Salihu & Marevci, 2021)

Step 1: Checking if A(1,1) is equal to 0
 if $A(1,1) = 0$
  Exchange rows to find nonzero element
Step 2: Calculating sub matrices
 Initialize $B = zeros(m-1, n-1)$;
 Create Loop for i from 1 to m-1
  Create Loop for j from 1 to n-1
   $B(i,j) = A(1,1) * A(i+1, j+1) - A(1, j+1) * A(i+1, 1)$
  end
 end
Step 3: Calculate the final result of rectangular determinant
 $d = 1/A(1,1)\hat{}(m-2) * det\_Chio(B) + (-1)\hat{}m * det\_Chio(A(1:m, 2:n))$;

Lemma 1. *For the second order block determinant the following formula holds:*

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{11} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{12}$$

Lemma 2. *$det(B) = k \cdot det(A)$ if a rectangular matrix B is produced from multiplying elements of a row of a rectangular matrix A with a scalar k (**?**).*

Lemma 3. *Suppose that two rectangular determinants that have identical columns except the first column, then the following formula holds:*

$$A_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}, \text{and } A_2 = \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix},$$

then

$$A_1 - A_2 = \begin{vmatrix} a_{11}-b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}-b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} + (-1)^{m-1} \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m2} & \cdots & a_{mn} \end{vmatrix} \tag{13}$$

Proof.

$$A_1 - A_2 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} - \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$= \sum_{j_1<j_2<\cdots<j_m} (-1)^{r+s} \cdot \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix} - \sum_{j_1<j_2<\cdots<j_m} (-1)^{r+s} \cdot \begin{vmatrix} b_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ b_{2j_1} & a_{2j_2} & \cdots & a_{2j_m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix}$$

$$= \left((-1)^{(1+\cdots+m)+(1+\cdots+m)} \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} \right.$$

$$+(-1)^{(1+\cdots+m)+(1+\cdots+m-1+m+1)} \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,m-1} & a_{1,m+1} \\ a_{21} & a_{22} & \cdots & a_{2,m-1} & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m-1} & a_{m,m+1} \end{vmatrix}$$

$$\left. + \cdots + (-1)^{(1+\cdots+m)+(n-m+\cdots+n)} \cdot \begin{vmatrix} a_{1,n-m} & a_{1,n-m+1} & \cdots & a_{1n} \\ a_{2,n-m} & a_{2,n-m+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,n-m} & a_{m,n-m+1} & \cdots & a_{mn} \end{vmatrix} \right)$$

$$- \left((-1)^{(1+\cdots+m)+(1+\cdots+m)} \cdot \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1m} \\ b_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} \right.$$

$$+(-1)^{(1+\cdots+m)+(1+\cdots+m-1+m+1)} \cdot \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1,m-1} & a_{1,m+1} \\ b_{21} & a_{22} & \cdots & a_{2,m-1} & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{m,m-1} & a_{m,m+1} \end{vmatrix}$$

$$\left. + \cdots + (-1)^{(1+\cdots+m)+(n-m+\cdots+n)} \cdot \begin{vmatrix} a_{1,n-m} & a_{1,n-m+1} & \cdots & a_{1n} \\ a_{2,n-m} & a_{2,n-m+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,n-m} & a_{m,n-m+1} & \cdots & a_{mn} \end{vmatrix} \right).$$

If we include all square blocks of second determinants that have first column of original matrix A2, as well as and based on the following square determinant property:

$$A1_{Square} - A2_{Square} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} - \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1m} \\ b_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} - b_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} - b_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix},$$

we have:

$$A1 - A2 = \begin{vmatrix} a_{11} - b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} - b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$= \left((-1)^{(1+\cdots+m)+(2+\cdots+m+m+1)} \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,m+1} \\ a_{21} & a_{22} & \cdots & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m+1} \end{vmatrix} \right.$$

$$+(-1)^{(1+\cdots+m)+(2+\cdots+m+m+2)} \cdot \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1m} & a_{1,m+2} \\ a_{22} & a_{23} & \cdots & a_{2m} & a_{2,m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mm} & a_{m,m+2} \end{vmatrix}$$

$$\left. + \cdots + (-1)^{(1+\cdots+m)+(n-m+\cdots+n)} \cdot \begin{vmatrix} a_{1,n-m} & a_{1,n-m+1} & \cdots & a_{1n} \\ a_{2,n-m} & a_{2,n-m+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,n-m} & a_{m,n-m+1} & \cdots & a_{mn} \end{vmatrix} \right).$$

Since second part fullfils definition of determinant, and from above equation it is obvious that sign is vareid from number of rows, we have:

$$A1 - A2 = \begin{vmatrix} a_{11} - b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} - b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} - \left((-1)^m \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m2} & \cdots & a_{mn} \end{vmatrix} \right)$$

$$= \begin{vmatrix} a_{11} - b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} - b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} + (-1)^{m+1} \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m2} & \cdots & a_{mn} \end{vmatrix}.$$

The proof is complete. □

Lemma 4. *If two rows from matrix A are exchanged to produce the rectangular matrix B, then $det(B) = -det(A)$.*

3 Main Results

Theorem 4. *Suppose that A is rectangular matrix of order $m \times n, m > 3$ and $m \leq n-1$, its determinant can be calculated using formula below:*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n}$$

$$= \frac{|A_c|}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{m-3}} + (-1)^m \cdot \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{mm} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{14}$$

where:

$$|A_c| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times(n-2)} \tag{15}$$

and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$

Proof. Implementing recursively Theorem 3 in Theorem 3 we have:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{1}{a_{11}^{m-2}} \cdot \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-1)\times(n-1)}$$

$$+(-1)^m \cdot \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{16}$$

$$= \frac{1}{a_{11}^{m-2} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{m-3}} \cdot \begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} & \cdots & \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{m1} & a_{m3} \end{vmatrix} \end{vmatrix} & \cdots & \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{vmatrix} \end{vmatrix} \end{vmatrix}_{(m-2)\times(n-2)}$$

$$+\frac{(-1)^{m-2}}{a_{11}^{m-2}}\cdot\begin{vmatrix}\begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{1n}\\ a_{21} & a_{2n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}a_{11} & a_{13}\\ a_{m1} & a_{m3}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{1n}\\ a_{m1} & a_{mn}\end{vmatrix}\end{vmatrix}_{(m-1)\times(n-2)} \quad +(-1)^m\cdot\begin{vmatrix}a_{12} & a_{13} & \cdots & a_{1n}\\ a_{22} & a_{23} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m2} & a_{m3} & \cdots & a_{mn}\end{vmatrix}_{m\times(n-1)}$$

Based on Lemma 1 and Lemma 2, we have:

$$\begin{vmatrix}\begin{vmatrix}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ \begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}\end{vmatrix} & \cdots & \begin{vmatrix}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{1n}\\ a_{21} & a_{2n}\end{vmatrix}\\ \begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{1n}\\ a_{31} & a_{3n}\end{vmatrix}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ \begin{vmatrix}a_{11} & a_{12}\\ a_{m1} & a_{m2}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{m1} & a_{m3}\end{vmatrix}\end{vmatrix} & \cdots & \begin{vmatrix}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{1n}\\ a_{21} & a_{2n}\end{vmatrix}\\ \begin{vmatrix}a_{11} & a_{12}\\ a_{m1} & a_{m2}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{1n}\\ a_{m1} & a_{mn}\end{vmatrix}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)}$$

$$=\begin{vmatrix}a_{11}\cdot\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \cdots & a_{11}\cdot\begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ a_{11}\cdot\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \cdots & a_{11}\cdot\begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{m1} & a_{m2} & a_{mn}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)} \tag{17}$$

$$=a_{11}^{m-2}\cdot\begin{vmatrix}\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{m1} & a_{m2} & a_{mn}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)}$$

Implementing formula (17) in formula (16), we have:

$$\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{vmatrix}_{m\times n}$$

$$=\frac{a_{11}^{m-2}}{a_{11}^{m-2}\cdot\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}^{m-3}}\cdot\begin{vmatrix}\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \cdots & \begin{vmatrix}a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{m1} & a_{m2} & a_{mn}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)} \tag{18}$$

$$+\frac{(-1)^{m-2}}{a_{11}^{m-2}}\cdot\begin{vmatrix}\begin{vmatrix}a_{11}&a_{13}\\a_{21}&a_{23}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{1n}\\a_{21}&a_{2n}\end{vmatrix}\\\vdots&\ddots&\vdots\\\begin{vmatrix}a_{11}&a_{13}\\a_{m1}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{1n}\\a_{m1}&a_{mn}\end{vmatrix}\end{vmatrix}_{(m-1)\times(n-2)}+(-1)^m\cdot\begin{vmatrix}a_{12}&a_{13}&\cdots&a_{1n}\\a_{22}&a_{23}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m2}&a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-1)}$$

From Theorem 3, we know that:

$$\frac{1}{a_{11}^{m-2}}\cdot\begin{vmatrix}\begin{vmatrix}a_{11}&a_{13}\\a_{21}&a_{23}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{1n}\\a_{21}&a_{2n}\end{vmatrix}\\\vdots&\ddots&\vdots\\\begin{vmatrix}a_{11}&a_{13}\\a_{m1}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{1n}\\a_{m1}&a_{mn}\end{vmatrix}\end{vmatrix}_{(m-1)\times(n-2)} \tag{19}$$

$$=\begin{vmatrix}a_{11}&a_{13}&\cdots&a_{1n}\\a_{21}&a_{23}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-1)}-(-1)^m\cdot\begin{vmatrix}a_{13}&\cdots&a_{1n}\\a_{23}&\cdots&a_{2n}\\\vdots&\ddots&\vdots\\a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-2)}$$

Implementing formula (19) in formula (18), we get:

$$\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{vmatrix}_{m\times n}=\frac{1}{\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^{m-3}} \tag{20}$$

$$\cdot\begin{vmatrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{31}&a_{32}&a_{3n}\end{vmatrix}\\\vdots&\ddots&\vdots\\\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{m1}&a_{m2}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{m1}&a_{m2}&a_{mn}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)}+(-1)^{m-1}\cdot\begin{vmatrix}a_{11}&a_{13}&\cdots&a_{1n}\\a_{21}&a_{23}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-1)}$$

$$-(-1)^{m-1}\cdot(-1)^m\cdot\begin{vmatrix}a_{13}&\cdots&a_{1n}\\a_{23}&\cdots&a_{2n}\\\vdots&\ddots&\vdots\\a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-2)}+(-1)^m\cdot\begin{vmatrix}a_{12}&a_{13}&\cdots&a_{1n}\\a_{22}&a_{23}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m2}&a_{m3}&\cdots&a_{mn}\end{vmatrix}_{m\times(n-1)}$$

$$=\frac{1}{\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^{m-3}}\cdot\begin{vmatrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{31}&a_{32}&a_{3n}\end{vmatrix}\\\vdots&\ddots&\vdots\\\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{m1}&a_{m2}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{m1}&a_{m2}&a_{mn}\end{vmatrix}\end{vmatrix}_{(m-2)\times(n-2)}$$

$$+(-1)^m \cdot \left(\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} + \frac{(-1)^{m-1}}{(-1)^m} \cdot \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \right.$$

$$\left. -(-1)^m \cdot \begin{vmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)} \right)$$

$$= \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{m-3}} \cdot \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times(n-2)}$$

$$+(-1)^m \cdot \left(\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} - \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \right.$$

$$\left. -(-1)^m \cdot \begin{vmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)} \right)$$

From Lemma 3, we know that:

$$\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} - \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{21}$$

$$= \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} + (-1)^{m-1} \cdot \begin{vmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)}$$

Hence:

$$\begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{22}$$

$$-\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} - (-1)^{m-1}\cdot \begin{vmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}$$

Implementing formula (22) in formula (20), we get:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{m-3}} \cdot \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{Vmatrix}_{(m-2)\times(n-2)}$$

$$+(-1)^m \cdot \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{23}$$

The proof is complete.

□

The pseudocode of Theorem 4 is like pseudocode presented in P 2, as following:

P 2: Pseudocode of *det_Chio2* algorithm for Theorem 4 (Modified Chio's-like) method to calculate rectangular determinants of order $m \times n$

```
Step 1: Checking for conditions:
    if m > n
        d = 0;
        return
    end
    if m == n
        Calculate determinant with Radic Definition
        return
    end
    if m < 3
        Calculate determinant with Radic Definition
        return
    end
Step 2: Calculate pivot block:
    Pivot = det(A(1 : 2, 1 : 2));
    if Pivot = 0
        Interchange rows to have another nonzero pivot block
    end
Step 2: Calculating sub matrices
    Initialize B = zeros(m − 2, n − 2);
    Create Loop for i from 3 to m
        Create Loop for 3 from 1 to n
```

$B(i-2, j-2) = det(A([12i], [12j]));$
end
end
Step 3: Calculate the final result of rectangular determinant
$d = det\_Chio2(B)/Pivot^\wedge(m-3) + (-1)^m * det\_Chio2([bsxfun$
$(@minus, A(1:m, 2), A(1:m, 1))A(1:m, 3:n)]);$

Corollary 1. *The pivot block can be determined any block created from two first columns and two consecutive rows, and the following formula holds:*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_c|}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{i+1,1} & a_{i+1,2} \end{vmatrix}^{m-3}} + (-1)^m \cdot \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{24}$$

where:

$$|A_c| = \begin{vmatrix} \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} & \cdots & \begin{vmatrix} a_{i1} & a_{i2} & a_{in} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,n} \\ a_{11} & a_{12} & a_{1n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{i1} & a_{i2} & a_{in} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times(n-2)} \tag{25}$$

and $\begin{vmatrix} a_{i1} & a_{i2} \\ a_{i+1,1} & a_{i+1,2} \end{vmatrix} \neq 0.$

Proof. Based on the Llemma 4, since we must interchange two rows even times, then we have the same sign before the determinant.

□

The pseudocode presented on P 2 changes on step 2 and 3 as follows, to represent the algorithm of Corollary 1.

P 3: Pseudocode of $det\_Chio2\_Corollary$ algorithm for Ccorollary 1 to calculate rectangular determinants of order $m \times n$

Step 1: Checking for conditions:
if $m > n$
$d = 0;$
return
end
if $m == n$
Calculate determinant with Radic Definition
return
end
if $m < 3$
Calculate determinant with Radic Definition
return
end

Step 2: Calculate pivot block:
  $Pivot = det(A(i : i + 1, 1 : 2));$
  if $Pivot = 0$
    Interchange rows to have another nonzero pivot block
  end
Step 3: Calculating sub matrices
  Initialize $B = zeros(m - 2, n - 2);$
  Create Loop for k *from* 1 *to* $i - 1$ *and* k *from* $i + 2$ *to* m
    Create Loop for 3 from 1 to n
      $B(k - 2, l - 2) = det(A([ii + 1k], [12l]));$
    end
  end
Step 4: Calculate the final result of rectangular determinant
  $d = det\_Chio\_Corollary(B)/Pivot^\wedge(m - 3) + (-1)^m * det\_Chio\_Corollary$ $([bsxfun(@minus, A(1 : m, 2), A(1 : m, 1))A(1 : m, 3 : n)]);$

3.1 Time complexity analysis and execution time simulation

Time complexity analysis of the above-mentioned algorithm which is presented in following function is calculated in Table 1.

Steps:	$Function : det\_Chio2$
1:	[m,n] = size(A);
2:	if m>n d=0; return end
3:	if m==n d=det(A); return end
4:	if m<3 d=det_A(A); return end
5:	Pivot = det(A(1:2,1:2));

6:	if Pivot == 0 //select next pivot block
	A=det_Chio2(A([3 1 2 4:m],1:n));
	return
	end

7:	B=zeros(m-2,n-2);

8:
```
for i = 3:m
    for j = 3:n
        B(i-2, j-2) = det(A([1 2 i], [1 2 j]));
    end
end
```

9:
```
d= det_Chio2(B)/Pivot^(m-3)+(-1)^m
*det_Chio2([bsxfun(@minus, A(1:m,2), A(1:m,1)) A(1:m,3:n)]);
```

Table 1: Time complexity analysis of *det_Chio*2 algorithm.

Step:	Execution time
1:	Is executed one time and is a constant: $T_1 = Const_1$
2:	Is executed one time and is a constant: $T_2 = Const_2$
3:	Is executed one time and since it is a square determinant the time complexity is: $T_3 = n^3$
4:	Is executed one time and since Radic definition time complexity (det_A) is: $T_4(3,n) = C\binom{n}{3} \cdot 3^3 = \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{3!\cdot(n-3)!} \cdot 3^3$ $= n \cdot (n-1) \cdot (n-2) \cdot 4.5 \approx n^3$
5:	Is executed one time and is a constant: $T_5 = Const_5$
6:	Is executed one time and the cost for this operation is the same as the cost of *det_chio*2 algorithm since it is the same algorithm just in another order of rows
7:	Is executed one time and is a constant: $T_6 = Const_6$
8:	Due to nested loop, we have: $(m-2)\cdot(n-2)$ times and cost is: $T_7 = 3^3$
9:	Based on above step each recursive function have $(m-2)\cdot(n-2)$ operations, hence: $T_8 = T_{8-1}(m-2, n-2) \cdot T_{8-2}(m, n-1) + c$ Since first part $T_{8-1}(m-2, n-2)$ goes up to third order the time complexity is $T_{8-1} = (\frac{m}{2})$, while second part goes up to $T_{8-2} = n-m$ recursive calls we have time complexity of $(n-m)$, hence $T_8 = (\frac{m}{2}) \cdot (n-m)$

$$\text{Total Cost} = 1 \cdot T_1 + Max(1 \cdot T_2, 1 \cdot T_3, 1 \cdot T_4, (1 \cdot T_5 + 1 \cdot T_6 + 1 \cdot T_7 + 1 \cdot T_8)$$

$$= 1 \cdot const_1 + Max(1 \cdot const_2, 1 \cdot n^3, 1 \cdot n^3,$$

$$(1 \cdot const_5 + 1 \cdot const_6 + (m-2)\cdot(n-2)\cdot 3^3 + (m-2)\cdot(n-2)\cdot(m/2)\cdot(n-m)))$$

Hence, the highest order is $(m-2)\cdot(n-2)\cdot(m/2)\cdot(n-m)$. After eliminating constants and other lower grades, we can summarize the asymptotic time complexity as $O(m^2 \cdot n \cdot (n-m))$.

For the execution time simulation, the environment in Table 2 is used. We evaluated the execution time of algorithm based on Cullis/Radic definition, algorithm based on Chio's-like theorem and algorithm based on modified Chio's-like method. Are generated random determinants of order 5×6 to 5×30, 10×11 to 10×30, 15×16 to 15×25, 20×21 to 20×25, 25×26 to 25×30, 30×31 to 30×35, 35×36 to 35×40, 40×45 to 40×50, and 45×46 to 45×50. Some of the results are presented in Table 3, while graphically are presented on Figure 1.

Table 2: Simulation environment.

Name:	Lenovo e15-gen 1
CPU:	Intel Core i7-1051U 1.8Ghz
GPU:	16 GB DDR4
HDD:	256 GB SSD
OS:	Windows 11 pro 64-bit
Software:	MATLAB, Version 9.0.0321247 (R2016a), 64-bit

Table 3: Testing results.

Order	Cullis/Radic	Theorem 1 Chio's like	Theorem 2 Modified Chio's Like
	1	2	3
5×10	0.00353	0.00322	0.00268
5×15	0.01427	0.01293	0.01079
5×20	0.05975	0.05383	0.04493
5×25	0.17206	0.15629	0.13003
5×30	0.48024	0.43446	0.36061
10×15	0.01869	0.01698	0.01405
10×20	0.67564	0.61346	0.50444
10×25	14.73657	13.32566	11.16421
10×30	133.66395	121.11837	100.15194
15×20	0.06707	0.06094	0.05023
15×25	17.12297	15.52686	12.93031
20×25	0.42062	0.38189	0.31935
20×30	1.45213	1.31980	1.09640
30×35	3.90665	3.55110	2.99780
35×40	11.49126	10.44706	8.70545
40×45	24.65122	22.40539	18.78745
45×50	48.22486	43.55681	36.09037

Order	1-2	1-3	2-3	(1/2-1) %	(1/3-1)%	(2/3-1)%
5×10	0.00032	0.00085	0.00053	9.86%	31.68%	19.87%
5×15	0.00134	0.00348	0.00214	10.34%	32.22%	19.83%
5×20	0.00592	0.01482	0.00890	10.99%	32.99%	19.82%
5×25	0.01577	0.04202	0.02625	10.09%	32.32%	20.19%
5×30	0.04577	0.11963	0.07386	10.54%	33.17%	20.48%
10×15	0.00171	0.00464	0.00293	10.08%	33.03%	20.84%
10×20	0.06218	0.17121	0.10903	10.14%	33.94%	21.61%
10×25	1.41091	3.57236	2.16145	10.59%	32.00%	19.36%
10×30	12.54558	33.51201	20.96642	10.36%	33.46%	20.93%
15×20	0.00613	0.01684	0.01071	10.06%	33.53%	21.32%
15×25	1.59611	4.19266	2.59655	10.28%	32.43%	20.08%
20×25	0.03872	0.10126	0.06254	10.14%	31.71%	19.58%
20×30	0.13232	0.35573	0.22340	10.03%	32.44%	20.38%
30×35	0.35555	0.90885	0.55330	10.01%	30.32%	18.46%
35×40	1.04420	2.78581	1.74161	10.00%	32.00%	20.01%
40×45	2.24583	5.86377	3.61794	10.02%	31.21%	19.26%
45×50	4.66804	12.13449	7.46644	10.72%	33.62%	20.69%

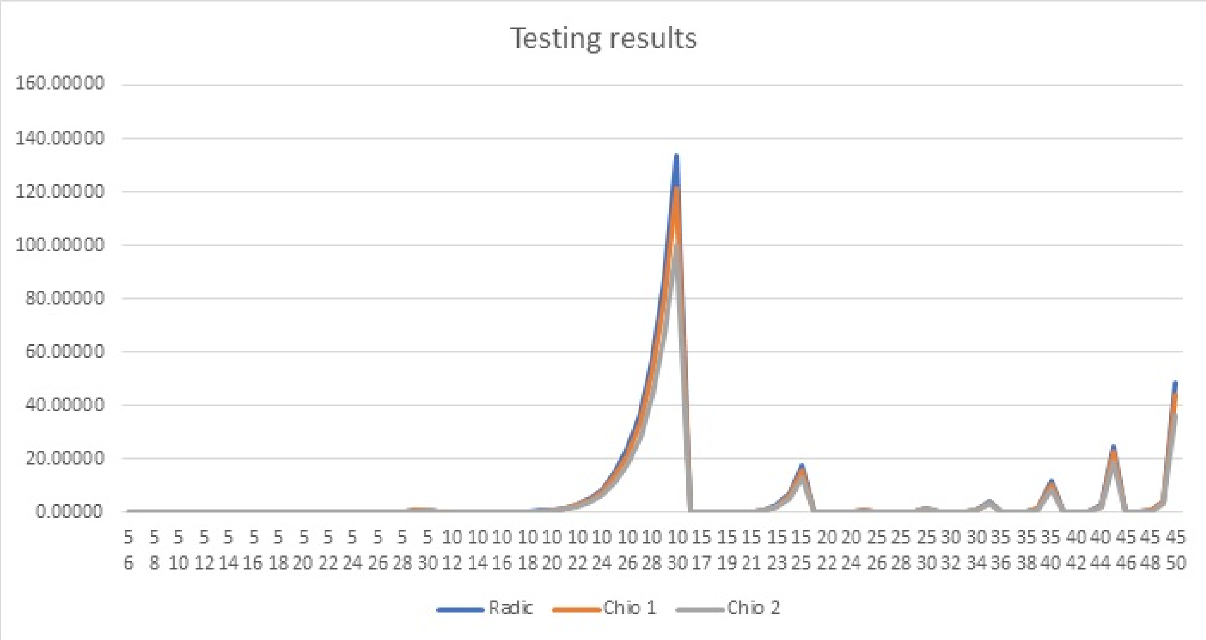

Figure 1: Comparison of execution time of determinant calculation between Cullis/Radic, Theorem 3 and Theorem 4

From the results obtained during the simulation, as it can be noted from the Table 3 as well as from Figure 1, there is an improvement of execution time of algorithm based on newly presented theorem. Compare to the Cullis/Radic definition the improvement is about 33%. Compared to the Chio's-like method, presented in Theorem 3, the improvement is about 20%.

4 Conclusion

In this study, we modified the Chio's-like method used for rectangular determinant calculation. The reduction order is for two rows and two columns, while calculating the third order blocks, and the pivot block is a second order determinant. We also developed a computer algorithm to calculate the rectangular determinant based on Theorem 4. In addition, we compared the execution time of Cullis/Radic definition, Chio's-like method, and newly presented approach. Also, the time complexity analysis of the presented algorithm is performed and resulted that the asymptotic time complexity of newly presented methodology is $O(m^2 \cdot n \cdot (n-m))$.

The testing was performed on MATLAB in Windows 11 environment, installed on Lenovo e15-gen1. We used to generate random determinations and calculate on same conditions with three different algorithms, results are presented on Table 3 and Figure 1. From the simulation is noted an improvement of about 20% compared to the Chio's-like methodology, and an improvement of about 33% compared to the Cullis/Radic definition.

References

Cullis, E.C. (1913). *Matrices and determinoids*. Cambridge: Cambridge: University Press.

Radic, M. (1966). Definition of Determinant of Rectangular Matrix, *Glasnik Matematicki*, 17-22.

Stojakovic, M. (1952). Determinante nekvadratnih matrica, *Vesnik DMNRS*.

Joshi V.N. (1980). A determinant for rectangular matrices, *Bulletin of the Australian Mathematical Society*, *21*(1), 137-146.

Stanimirovic P. & Stankovic, M. (1980). Determinants of rectangular matrices and the Moore-Penrose inverse, *Novi Sad J Math.*, *27*(1), 53-69.

Bayat, M. (2020). A bijective proof of generalized Cauchy–Binet, Laplace, Sylvester and Dodgson formulas, *Linear and Multilinear Algebra*.

Amiri, A., Fathy, M. & Bayat, M. (2010). Generalization of Some Determinantial Identities for Non-Square Matrices Based on Radic's Definition, *TWMS Journal of Pure and Applied Mathematics*, *1*(2), 163-175.

Salihu, A., Snopçe, H., Ajdari, J. & Luma, A. (2022a). Generalization of Dodgson's condensation method for calculating determinant of rectangular matrices, *IEEE International Conference on Electrical, Computer and Energy Technologies (ICECET)*.

Salihu, A., Snopçe, H., Luma, A. & Ajdari, J. (2022b). Optimization of Dodgson's Condesation Method for Rectangular Determinant Calculations, *Advanced Mathematical Models & Applications*, *7*(3), 264-274.

Salihu, A., Snopçe, H., Luma, A. & Ajdari, J. (2022c). Time Complexity Analysis for Cullis/Radic and Dodgson's Generalized/Modified Method for Rectangular Determinants Calculations, *International Journal of Computers and Their Applications*, *29*(4), 236-246.

Salihu, A., Snopçe, H., Luma, A. & Ajdari, J. (2023). Comparison of time complexity growth for different methods/algorithms for rectangular determinant calculations, *IEEE International Conference on Recent Trends in Electronics and Communication (ICRTEC)*.

Salihu, A., Marevci, F. (2021). Chio's-like method for calculating the rectangular (non-square) determinants: Computer algorithm interpretation and comparison, *European Journal of Pure and Applied Mathematics*, *14*(2), 431-450.

Salihu, A., Marevci, F. (2019). Determinants Order Decrease/Increase for k Orders, Interpretation with Computer Algorithms and Comparison, *International Journal of Mathematics and Computer Science*, *14*(2), 501-518.

Makarewicz, A., Pikuta, P. (2020). Cullis–Radic determinant of a rectangular matrix which has a number of identical columns, *Annales Universitatis Mariae Curie-Skłodowska*, *69*(1), 41-60.

Makarewicz, A., Pikuta, P. & Szalkowski, D. (2014). Properties of the determinant of a rectangular matrix, *Annales Universitatis Mariae Curie-Skłodowska*, *68*(1), 31-41.