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CONTROLLABILITY OF MILD SOLUTIONS FOR A NONLOCAL FRACTIONAL CONFORMABLE CAUCHY PROBLEM OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. In the present paper, we are concerned with the controllability of mild solutions for a Cauchy problem governed by a conformable fractional derivative of positive and inferior than one order , an infinitesimal generator of a cosine family on a Banach space and a bounded linear operator. The elements x_0 and x_1 are two fixed vectors in X, and f, g, h are given functions. For certain conditions on the given data, the main result is obtained by means of the Banach contraction principle combined with the cosine family of linear operators.

Keywords: Fractional differential equations, Cosine family of linear operators, Conformable fractional derivative, Nonlocal conditions; Controllability of mild solutions.

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1 Introduction

Fractional calculus attracts the attention of many researchers due to their applications in modeling of different phenomena in physics, engineering, optimization, data science, biology, finance, chemistry, and so on Hamlet & Tagiev (2021); Laghrib (2020); Nachaoui (2020); Nachaoui et al. (2021). For example in the work Khalil et al. (2014), the authors have proposed a new definition of fractional derivative named conformable fractional derivative. This novel fractional derivative quickly became the subject of many research works. Including the works Atraoui & Bouaouid (2021); Bouaouid et al. (2019) in which the authors have proved the existence of mild solution for the following nonlocal conformable fractional Cauchy problem:

$$\frac{d^\alpha}{dt^\alpha}[\frac{d^\alpha x(t)}{dt^\alpha}] = Ax(t) + f(t, x(t)), \quad x(0) = x_0 + g(x), \quad \frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x), \quad t \in [0, \tau], \tag{1}$$

where τ is a positive real number, $\frac{d^\alpha(.)}{dt^\alpha}$ represents the conformable fractional derivative of order $\alpha \in]0, 1]$ taking in vector sense and A is the infinitesimal generator of a cosine family $(\{C(t), S(t)\})_{t\in\mathbb{R}}$ on a Banach space $(X, \|\ .\ \|)$ (see Travis & Webb (1978)). The elements x_0 and x_1 are two fixed vectors in X, and $f : [0, \tau] \times X \longrightarrow X$, $g, h : \mathcal{C} \longrightarrow X$ are given functions, with $\mathcal{C}$ is the Banach space of continuous functions $x(.)$ defined from $[0, \tau]$ into X equipped with

the norm $\mid x \mid_c = \sup_{t\in[0,\tau]} \parallel x(t) \parallel$. The expressions $x(0) = x_0 + g(x)$ and $\frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x)$ mean the so-called nonlocal conditions, which can be applied in physics with better effects than the classical initial condition Byszewski (1991).
Recently, the study of control problems with conformable fractional derivative has attracted the attention of many mathematicians and physicists in various fields of science Jneid (2019); Jneid & Awadalla (2020).
Motivated by the fact that the controllability is a most important qualitative behavior of a dynamical system, in this work, we are interested with the study of the controllability of Cauchy problem (1). Precisely, we will be prove a controllability result for the following Cauchy problem

$$\frac{d^\alpha}{dt^\alpha}[\frac{d^\alpha x(t)}{dt^\alpha}] = Ax(t) + f(t, x(t)) + Bu(t), \;\; x(0) = x_0 + g(x), \;\; \frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x), \;\; t \in [0, \tau], \tag{2}$$

where the control function $u(.)$ is an element of $L^2([0, \tau], U)$ with U is a Banach space and B is a bounded linear operator from U into X.
The rest of this paper is organized as follows. In section 2, we briefly recall some tools related to the conformable fractional calculus and theory of cosine families. In section 3, we present the main result.

2 Preliminaries

We first recall some preliminary facts on the conformable fractional calculus.

Definition 1. *(Khalil et al. (2014)) For $\alpha \in]0, 1]$ the conformable fractional derivative of order α of a function $x(.) : \;\; [0, +\infty[\longrightarrow \mathbb{R}$ is defined as*

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon} \;\; for \;\; t > 0 \;\; and \;\; \frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha},$$

provided that the limits exist.

The conformable fractional integral $I^\alpha(.)$ of a function $x(.)$ is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds \;\; \text{for} \;\; t > 0.$$

Theorem 1. *(Khalil et al. (2014)) If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{d^\alpha(I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

Theorem 2. *(Abdeljawad (2015)) If $x(.)$ is a differentiable function, then we have*

$$I^\alpha(\frac{d^\alpha x(.)}{dt^\alpha})(t) = x(t) - x(0).$$

Definition 2. *(Abdeljawad (2015)) The conformable fractional Laplace transform of order $\alpha \in]0, 1]$ of a function $x(.)$ is defined as follows*

$$\mathcal{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\frac{\lambda t^\alpha}{\alpha}} x(t) dt, \quad \lambda > 0.$$

The following proposition gives us the action of the conformable fractional Laplace transform on the conformable fractional derivative.

Proposition 1. *(Abdeljawad (2015)) If $x(.)$ is a differentiable function, then we have the following result*

$$\mathcal{L}_\alpha(\frac{d^\alpha x(t)}{dt^\alpha})(\lambda) = \lambda\mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

According to Bouaouid et al. (2018), we have the following remark.

Remark 1. *For two functions $x(.)$ and $y(.)$, we have*

$$\mathcal{L}_\alpha(x(\frac{t^\alpha}{\alpha}))(\lambda) = \mathcal{L}_1(x(t))(\lambda),$$

$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1}x(\frac{t^\alpha - s^\alpha}{\alpha})y(s)ds\right)(\lambda) = \mathcal{L}_1(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda),$$

provided that the both terms of each equality exist.

Now, we present some definitions concerning the cosine family of linear operators.

Definition 3. *(Travis & Webb (1978)) A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on a Banach space X is called a strongly continuous cosine family if and only if:*

1. *$C(0) = I$, where I is the identity operator in the space X.*
2. *$C(s+t) + C(s-t) = 2C(s)C(t)$ for all $t, s \in \mathbb{R}$.*
3. *The function $t \longmapsto C(t)x$ is strongly continuous for each $x \in X$.*

We also define the sine family $(S(t))_{t\in\mathbb{R}}$ associated with the cosine family $(C(t))_{t\in\mathbb{R}}$ as follows

$$S(t)x = \int_0^t C(s)xds, \;\; x \in X.$$

The infinitesimal generator A of a strongly continuous cosine family $(\{C(t), S(t)\})_{t\in\mathbb{R}}$ on X is defined by

$$D(A) = \{x \in X, \; t \longmapsto C(t)x \text{ is a twice continuously differentiable function}\}$$

and

$$Ax = \frac{d^2C(0)x}{dt^2}, \;\; x \in D(A).$$

Proposition 2. *(Travis & Webb (1978)) For $\lambda \in \mathbb{C}$ such that $Re(\lambda) > \omega$, we have*

1. *$\lambda^2 \in \rho(A)$, ($\rho(A)$ is the resolvent set of A).*
2. *$\lambda(\lambda^2 I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}C(t)xdt$ for all $x \in X$.*
3. *$(\lambda^2 I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}S(t)xdt$ for all $x \in X$.*

3 Main results

Lemma 1. *If $x \in \mathcal{C}$ is a solution of Cauchy problem (2), then the function $x(.)$ satisfies the following integral equation*

$$x(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1}S(\frac{t^\alpha - s^\alpha}{\alpha})\Big(f(s, x(s)) + Bu(s)\Big)ds.$$

The proof of this result is essentially based on the conformable fractional Laplace transform. For the complete proof one can see the work Bouaouid et al. (2019).

Definition 4. *(Bouaouid et al. (2019)) A function $x \in \mathcal{C}$ is called a mild solution of Cauchy problem (2) if*

$$x(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})\Big(f(s, x(s)) + Bu(s)\Big)ds.$$

Now, we deal with the controllability of mild solutions of Cauchy problem (2).

Definition 5. *The Cauchy problem (2) is said to be controllable on $[0, \tau]$, if for every $x^* \in X$ there is exists a control function $u \in L^2([0, \tau], U)$ such that the mild solution $x(.)$ of (2) satisfies $x(\tau) = x^*$.*

In the sequel of this paper, we denote by $|.|_{\mathcal{L}(X)}$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined from X into itself. We also introduce the following notations:

$$M = \sup_{t\in[0,\tau]} |C(\frac{t^\alpha}{\alpha})|_{\mathcal{L}(X)} \text{ and } N = \sup_{t\in[0,\tau]} |S(\frac{t^\alpha}{\alpha})|_{\mathcal{L}(X)}.$$

To obtain the main result, we will need the following assumptions:

(H_1) The function $f(., x) : [0, \tau] \longrightarrow X$ is continuous for all $x \in X$.

(H_2) The function $f(t, .) : X \longrightarrow X$ is continuous and there exist positive constants L_1, K_1 such that
$\| f(t, y) - f(t, x) \| \leq L_1 \| y - x \|$ and $\| f(t, x) \| \leq K_1 \| x \|$ for all x, $y \in X$.

(H_3) The function $g : \mathcal{C} \longrightarrow X$ is continuous.

(H_4) There exist positive constants L_2 and K_2 such that
$\| g(y) - g(x) \| \leq L_2 \mid y - x \mid_c$ and $\| g(x) \| \leq K_2 \mid x \mid_c$ and for all x, $y \in \mathcal{C}$.

(H_5) The function $h : \mathcal{C} \longrightarrow X$ is continuous.

(H_6) There exist positive constants L_3 and K_3 such that
$\| h(y) - h(x) \| \leq L_3 \mid y - x \mid_c$ and $\| h(x) \| \leq K_3 \mid x \mid_c$ for all x, $y \in \mathcal{C}$.

(H_7) The bounded linear operator $W : L^2([0, \tau], U) \longrightarrow X$ defined by

$$W(u) = \int_0^\tau s^{\alpha-1} S(\frac{\tau^\alpha - s^\alpha}{\alpha}) Bu(s)ds$$

has an induced inverse operator $\tilde{W}^{-1}$, which takes values in $L^2([0, \tau], U)/Ker(W)$ and there exist positive constants R_1, R_2 such that
$\| B \|_{\mathcal{L}(U,X)} \leq R_1$ and $\| \tilde{W}^{-1} \|_{\mathcal{L}(X, L^2([0,\tau],U)/Ker(W))} \leq R_2$.

Theorem 3. *Assume that $(H_1) - (H_7)$ hold, then Cauchy problem (2) is controllable on $[0, \tau]$, provided that*

$$\max(\theta_1, \theta_2) < 1,$$

where

$$\theta_1 = MK_2 + NK_3 + N\bigg(K_1 + MR_1R_2K_2\bigg)\frac{\tau^\alpha}{\alpha} + N^2\bigg(K_3 + K_1\frac{\tau^\alpha}{\alpha}\bigg)\frac{\tau^\alpha}{\alpha},$$

$$\theta_2 = MK_2 + NK_3 + N\bigg[\frac{\tau^\alpha}{\alpha}\bigg(K_1 + R_1R_2\bigg\{MK_2 + N\bigg[K_3 + K_1\frac{\tau^\alpha}{\alpha}\bigg]\bigg\}\bigg)\bigg].$$

Proof. By using hypothesis (H_7) for an arbitrary function x, we can define a control u_x as follows

$$u_x = \tilde{W}^{-1}\Big(x^* - C(\frac{\tau^\alpha}{\alpha})[x_0 + g(x)] - S(\frac{\tau^\alpha}{\alpha})[x_1 + h(x)] - \int_0^\tau s^{\alpha-1} S(\frac{\tau^\alpha - s^\alpha}{\alpha}) f(s, x(s)) ds\Big).$$

For this control, we define the operator $\Psi : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\Psi(x)(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})\Big(f(s, x(s)) + Bu_x(s)\Big)ds.$$

We also introduce for a radius $r > 0$ the ball $B_r := \{x \in \mathcal{C},\ | x |_c \leq r\}$.
We will show that the operator Ψ has a fixed point, which is a controllable mild solution of Cauchy problem (2). To do so, we will give the proof in tree steps.

Step 1: Prove that there exists a radius $\delta > 0$ such that $\Psi : B_\delta \longrightarrow B_\delta$.

For $x \in \mathcal{C}$ and $t \in [0, \tau]$, we have

$$\Psi(x)(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})\Big(f(s, x(s)) + Bu_x(s)\Big)ds.$$

Then one has

$$\| \Psi(x)(t) \| \leq M \| x_0 + g(x) \| + N\Bigg[\| x_1 + h(x) \| + \int_0^t s^{\alpha-1} \| f(s, x(s)) + Bu_x(s) \| ds\Bigg].$$

By using hypothesis (H_2), (H_4), (H_6) and (H_7), we obtain

$$\| \Psi(x)(t) \| \leq M\Bigg[\| x_0 \| + K_2 \mid x |_c \Bigg] + N\Bigg[\| x_1 \| + K_3 \mid x |_c + \Bigg(K_1 \mid x |_c + R_1 \| u_x \|_2 \Bigg)\frac{\tau^\alpha}{\alpha}\Bigg].\ (*)$$

On the other hand, we known that

$$u_x = \tilde{W}^{-1}\Big(x_1 - C(\frac{\tau^\alpha}{\alpha})[x_0 + g(x)] - S(\frac{\tau^\alpha}{\alpha})[x_1 + h(x)] - \int_0^\tau s^{\alpha-1} S(\frac{\tau^\alpha - s^\alpha}{\alpha}) f(s, x(s)) ds\Big).$$

In view of assumptions (H_2), (H_4), (H_6) and (H_7), we obtain

$$\begin{aligned}\| u_x \|_2 &\leq R_2 \| x^* - C(\frac{\tau^\alpha}{\alpha})[x_0 + g(x)] - S(\frac{\tau^\alpha}{\alpha})[x_1 + h(x)] - \int_0^\tau s^{\alpha-1} T(\frac{\tau^\alpha - s^\alpha}{\alpha}) f(s, x(s)) ds \| \\ &\leq R_2\Bigg(\| x^* \| + M\Bigg[\| x_0 \| + K_2 \mid x |_c \Bigg] + N\Bigg[\| x_1 \| + K_3 \mid x |_c + \frac{K_1 \tau^\alpha}{\alpha} \mid x |_c \Bigg]\Bigg).\end{aligned}$$

By replacing this estimate in ($*$), we get

$$\begin{aligned}&\| \Psi(x)(t) \| \leq M\Bigg[\| x_0 \| + K_2 \mid x |_c \Bigg] + N\Bigg[\| x_1 \| + K_3 \mid x |_c \Bigg] \\ &+ N\Bigg(K_1 \mid x |_c + R_1 R_2\Bigg\{ \| x^* \| + M\Bigg[\| x_0 \| + K_2 \mid x |_c \Bigg] + N\Bigg[\| x_1 \| + K_3 \mid x |_c + \frac{K_1 \tau^\alpha}{\alpha} \mid x |_c \Bigg]\Bigg\}\Bigg)\frac{\tau^\alpha}{\alpha}.\end{aligned}$$

Separating the terms containing the expression $| x |_c$, one has

$$\| \Psi(x)(t) \| \leq \theta_0 + \theta_1 \mid x |_c,$$

where

$$\theta_0 = M \| x_0 \| + N \| x_1 \| + N R_1 R_2\Bigg(\| x^* \| + M \| x_0 \| + N \| x_1 \| \Bigg)\frac{\tau^\alpha}{\alpha}$$

$$\theta_1 = MK_2 + NK_3 + N\Big(K_1 + MR_1R_2K_2\Big)\frac{\tau^\alpha}{\alpha} + N^2\Big(K_3 + K_1\frac{\tau^\alpha}{\alpha}\Big)\frac{\tau^\alpha}{\alpha},$$

Hence, it suffices to choose δ such that $\delta \geq \frac{\theta_0}{1-\theta_1}$.
Step 2: We show that Ψ is a contraction operator on B_δ.
For y, $x \in \mathcal{C}$, we have

$$\Psi(y)(t) - \Psi(x)(t) = C(\frac{t^\alpha}{\alpha})[g(y) - g(x)] + S(\frac{t^\alpha}{\alpha})[h(y) - h(x)] + \int_0^t s^{\alpha-1}S(\frac{t^\alpha - s^\alpha}{\alpha})\Big(f(s, y(s)) - f(s, x(s)) + B(u_y - u_x)(s)\Big)ds.$$

According to (H_2), (H_4), (H_6) and (H_7), we obtain

$$\begin{aligned}\| \Psi(y)(t) - \Psi(x)(t) \| &\leq M \| g(y) - g(x) \| + N \| h(y) - h(x) \| \\ &\quad + N\int_0^t s^{\alpha-1} \| f(s, y(s)) - f(s, x(s)) + B(u_y - u_x)(s) \| ds \\ &\leq K_2M \mid y - x \mid_c + N\Big[K_3 \mid y - x \mid_c + \frac{\tau^\alpha}{\alpha}\Big(K_1 \mid y - x \mid_c + R_1 \| u_y - u_x \|_2\Big)\Big]. \quad (**)\end{aligned}$$

In the other hand, we know that

$$u_y - u_x = \tilde{W}^{-1}\Big(- C(\frac{\tau^\alpha}{\alpha})[g(y) - g(x)] - S(\frac{\tau^\alpha}{\alpha})[h(y) - h(x)] - \int_0^\tau s^{\alpha-1}S(\frac{\tau^\alpha - s^\alpha}{\alpha})(f(s, y(s)) - f(s, x(s)))ds\Big).$$

Then one has

$$\begin{aligned}\| u_y - u_x \|_2 &\leq R_2\Big(M \| g(y) - g(x) \| + N\Big[\| h(y) - h(x) \| + \int_0^\tau s^{\alpha-1} \| f(s, y(s)) - f(s, x(s))) \| ds\Big]\Big) \\ &\leq R_2\Big(MK_2 \mid y - x \mid_c + N\Big[K_3 \mid y - x \mid_c + \frac{K_1\tau^\alpha}{\alpha} \mid y - x \mid_c\Big]\Big).\end{aligned}$$

By replacing this estimate in $(**)$, we obtain

$$\begin{aligned}&\| \Psi(y)(t) - \Psi(x)(t) \| \leq MK_2 \mid y - x \mid_c + K_3N \mid y - x \mid_c \\ &+N\Big[\frac{\tau^\alpha}{\alpha}\Big(K_1 \mid y - x \mid_c + R_1R_2\Big\{K_2M \mid y - x \mid_c + N\Big[K_3 \mid y - x \mid_c + K_1\frac{\tau^\alpha}{\alpha} \mid y - x \mid_c\Big]\Big\}\Big)\Big] \\ &\qquad\qquad \leq \theta_2 \mid y - x \mid_c .\end{aligned}$$

Where

$$\theta_2 = K_2M + K_3N + N\Big[\frac{\tau^\alpha}{\alpha}\Big(K_1 + R_1R_2\Big\{K_2M + N\Big[K_3 + K_1\frac{\tau^\alpha}{\alpha}\Big]\Big\}\Big)\Big].$$

Taking the supremum, we get

$$\mid \Psi(y) - \Psi(x) \mid_c \leq \theta_2 \mid y - x \mid_c .$$

Since $\theta_2 < 1$, then Ψ is a contraction operator on B_δ. Hence there exists an unique element $x_\delta(.) \in B_\delta$ such that $\Psi(x_\delta)(t) = x_\delta(t)$ for all $t \in [0, \tau]$.
Step 3: Prove that x_δ is controllable.

For $x_\delta(.)$, we have

$$
\begin{aligned}
x_\delta(\tau) &= \Psi(x_\delta)(\tau)\\
&:= C(\frac{\tau^\alpha}{\alpha})[x_0+g(x_\delta)] + S(\frac{\tau^\alpha}{\alpha})[x_1+h(x_\delta)] + \int_0^\tau s^{\alpha-1}S(\frac{\tau^\alpha-s^\alpha}{\alpha})\Big(f(s,x_\delta(s)) + Bu_{x_\delta}(s)\Big)ds\\
&= C(\frac{\tau^\alpha}{\alpha})[x_0+g(x_\delta)] + S(\frac{\tau^\alpha}{\alpha})[x_1+h(x_\delta)]\\
&+ \int_0^\tau s^{\alpha-1}S(\frac{\tau^\alpha-s^\alpha}{\alpha})f(s,x_\delta(s))ds + \int_0^\tau s^{\alpha-1}S(\frac{\tau^\alpha-s^\alpha}{\alpha})Bu_{x_\delta}(s)ds\\
&= -W(u_{x_\delta}) + x^* + \int_0^\tau s^{\alpha-1}S(\frac{\tau^\alpha-s^\alpha}{\alpha})Bu_{x_\delta}(s)ds\\
&= -W(u_{x_\delta}) + x^* + W(u_{x_\delta})\\
&= x^*.
\end{aligned}
$$

In conclusion, Cauchy problem (2) has a controllable mild solution on $[0,\tau]$. □

4 Conclusion

The existence of mild solutions of a Cauchy problem of nonlocal differential equations of second order with conformable fractional derivative is studied in the work Atraoui & Bouaouid (2021); Bouaouid et al. (2019). Our contribution in this present work is the study of the controllability of mild solutions for such Cauchy problem by means of the Banach fixed point theorem combined with theory of cosine family of linear operators.

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