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NUMERICAL MODELING OF NON-CAPACITY MODEL FOR SEDIMENT TRANSPORT IN OPEN CHANNEL HYDRAULICS BY ROE SCHEME WITH A NEW DISCRETIZATION OF THE SOURCE TERM

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Abstract. This work presents sediment transport in open channel hydraulics over mobile bed. The mathematical model is a combination of the shallow water equations for water-sediment mixture, the sediment transport diffusion and the bed morphology change equations. The system is solved by the finite volume Roe scheme, associated with an original treatment of the source term. In order to show the performance of the non-capacity model and the numerical scheme on problems with low sediment entrainment, the method is applied on open channel hydraulic. The numerical scheme treat different case of channels using different size of sediments. Through the obtained results the scheme proved a high level of performance, stability and accuracy.

Keywords: Finite volume method, open channel hydraulic, sediment transport, non-capacity model, erodible bed, Roe scheme, discretization of source term.

AMS Subject Classification: 65M08, 76M12, 37M05, 93A30.

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1 Introduction

Open channel hydraulics has always been a very interesting domain for researchers, specially by being a source of hydro-power generation, water reserve, irrigation, peach, etc. Different mathematical models has been developed to study this phenomenon, which are well cited in Jelti et al. (2017); Cao et al. (2002) developed a new mathematical model called "the coupled model", considering the strong interaction between flow, sediment transport and morphological evolution of the bed. The coupled model uses the shallow water equations for sediment-water mixture instead of the simple shallow water equations to model the sediment transport. The coupled model links all conservation equations and provides a synchronous resolution procedure, also, it treats entrainment and deposition sediment as independent processes (this property is called non-capacity model) (Jelti et al., 2017; Cao et al., 2004; Wu, 2008).

This work uses a 1D noncapicity model for open channel hydraulics, sediment transport and mobile bed. The mathematical model consists of four equations; the mass and the momentum conservation equation for the water-sediment mixture, the transport diffusion equation for sediment particles and bed morphology change equation, together with empirical formulations for bed friction and sediment exchange between the water column and the bed (Jelti et al., 2017; Wu, 2008).

The governing equations are solved numerically using the finite volume Roe scheme formulation. An original formulation to discretize the source term which satisfies the C-property is introduced. The MUSCL method with generalized minmod limiter and the Runge-Kutta are used to achieve a second order accuracy. The numerical scheme well resolves several tests with no-spurious observed oscillations even in the most complicated problems as dam-break flow (Jelti et al., 2017).

In order to show the capacity of the model to detect small bed change even in problems with low sediment entrainment, in this paper the application of the model is restricted on open channel hydraulics, and attention is given to the evolution of the flow, sediment transport and bed morphological development. Many results are interpreted, such as the effect of the sediment size on the bed mobility and velocity profiles, etc.

This work is an application of partial differential equations which requires a robust numerical methods (Nachaoui et al., 2021; Gasimov et al., 2019; Rasheed et al., 2021). In order to obtain the best approximation of the real physical problem, the mathematical model has been developed several time during this decades. The main value of this work is the mathematical modelization of the open channel hydraulics over mobile bed using the coupled model and the non-capacity model. The obtained mathematical model is resolved by the well known Roe scheme with a new discretization of the source term developed in (Jelti et al., 2017).

This work is organized as follows. Section 2 presents the mathematical model for open channel hydraulic over erodible sediment bed, as well as the empirical functions considered. In Section 3, the Roe scheme is formulated and discretized. The new discretization of the source term is introduced in Section 4. Section 5 treats the tests and the numerical results. Finally concluding remarks are summarized in Section 6.

2 The mathematical model

There are many mathematical models developed in the literature, in this study we apply the mathematical model used in Jelti et al. (2017) on open channel hydraulic over different types of bottoms.

We consider in this work, a one-dimensional non-capacity model in a channel with rectangular cross section of constant width, over a mobile bed composed of uniform and noncohesive sediment particles. The governing equations are written as Benkhaldoun et al. (2009); Simpson & Castelltort (2006); Jelti et al. (2017); Wu (2008); Cao et al. (2002):

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = \frac{E-D}{1-p} \tag{1}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} = B \tag{2}$$

$$\frac{\partial (hc)}{\partial t} + \frac{\partial (huc)}{\partial x} = E - D \tag{3}$$

$$\frac{\partial z}{\partial t} = -\frac{E-D}{1-p} \tag{4}$$

where B is the source term defined by :

$$B = -gh\frac{\partial z}{\partial x} - \frac{\rho_s - \rho_w}{2\rho} gh^2 \frac{\partial c}{\partial x} - ghS_f - \frac{\rho_0 - \rho}{\rho}\frac{E-D}{1-p}\, u \tag{5}$$

t is the time, x the streamwise coordinate, h the flow depth, u the depth-averaged streamwise velocity, z the bed elevation, c the flux-averaged volumetric sediment concentration, g the gravitational acceleration, p the bed sediment porosity. D and E are the sediment deposition and entrainment fluxes across the bottom boundary of flow, they represent the exchange between water column and bed. S_f is the friction slope, $\rho = \rho_w(1-c)+\rho_s c$ is the density of water-sediment mixture,

$\rho_0 = \rho_w p + \rho_s(1-p)$ is the density of the saturated bed, ρ_w and ρ_s are the densities of water and sediment, respectively.

Equation (1) represents the mass conservation equation for the water-sediment mixture.

Equation (2)represents the momentum conservation equation for the water-sediment mixture.

The mass conservation equation for sediment is represented by Equation (3), in which suspended and bed load are considered in a single mode indicated by the total sediment load.

Equation (4) indicates the bed change rate.

To complete the governing equations above, the same empirical functions are taken as (Jelti et al., 2017).

3 Numerical scheme

Several numerical schemes exist, in this study the hyperbolic system (1-4) is solved numerically using Roe scheme (Roe, 1981). Knowing that, all equations form one system to reach a synchronous solution. The same procedure is followed in Jelti et al. (2017). Equations (1-4) arranged in the conservative form:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = S + Q \tag{6}$$

or non-conservative form:

$$\frac{\partial U}{\partial t} + \mathcal{A}(U)\frac{\partial U}{\partial x} = Q \tag{7}$$

where

$$U = \begin{pmatrix} h \\ hu \\ hc \\ z \end{pmatrix}, \; F = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huc \\ 0 \end{pmatrix}, \; S = \begin{pmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{(\rho_s-\rho_w)}{2\rho}gh^2\frac{\partial c}{\partial x} \\ 0 \\ 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} \frac{E-D}{1-p} \\ -ghS_f - \frac{\rho_0-\rho}{\rho}\frac{E-D}{1-p}u \\ E-D \\ -\frac{E-D}{1-p} \end{pmatrix}$$

the matrix $\mathcal{A}(U)$ is given by:

$$\mathcal{A}(U) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ gh - u^2 - \frac{\rho_s-\rho_w}{2\rho}ghc & 2u & \frac{\rho_s-\rho_w}{2\rho}gh & gh \\ -uc & c & u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{A}(U)$ has the four following distinct real eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = u, \quad \lambda_3 = u - \sqrt{gh}, \quad \text{and} \quad \lambda_4 = u + \sqrt{gh}$$

The spatial domain is discretized into control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ with the same length Δx. The time interval is divided into subintervals $[t_n, t_{n+1}]$ with uniform size Δt. We integrate system (6) using the finite volume method formulation, we obtain the following discrete equation:

$$\frac{\partial U_i}{\partial t} = -\frac{1}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) + S^n_i + Q^n_i \tag{8}$$

The Euler scheme is used to discritize the time:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + \Delta t S_i^n + \Delta t Q_i^n \tag{9}$$

In formulations (8)-(9), U_i^n, S_i^n, Q_i^n are the space averages of U, S, Q on the control volume$[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time t_n.

$F_{i\pm 1/2}^n$ are the numerical fluxes at the interfaces $x = x_{i\pm 1/2}$, they are approximated using Roe scheme as follows:

$$F_{i+1/2}^n = \frac{1}{2}\left(F_{i+\frac{1}{2},R}^n + F_{i+\frac{1}{2},L}^n\right) - \frac{1}{2}\left|\mathcal{A}\left(\widetilde{U}_{i+\frac{1}{2}}^n\right)\right|\left(U_{i+\frac{1}{2},R}^n - U_{i+\frac{1}{2},L}^n\right) \tag{10}$$

where $F_{i+\frac{1}{2},R}^n = F(U_{i+\frac{1}{2},R}^n)$, $F_{i+\frac{1}{2},L}^n = F(U_{i+\frac{1}{2},L}^n)$ and $U_{i+\frac{1}{2},R}^n$, $U_{i+\frac{1}{2},L}^n$ are the right and left approximations of the solution U at the interface $x = i + \frac{1}{2}$. The matrix $\left|\mathcal{A}(\widetilde{U}_{i+\frac{1}{2}}^n)\right|$ uses the well known Roe average $\widetilde{U}_{i+\frac{1}{2}}^n$ defined by Roe (1981).

To reach a second order finite volume accuracy, we use a monotone upstream-centered scheme for conservation laws method incorporating a slope limiters in the spatial approximation and two-step Runge-Kutta TVD method for time integrating as reported in Jelti et al. (2017).

4 Decomposition and discretization of the source term

The source term has a greet effect on the resolution of the system, knowing that a simple central discretization does not preserve the well known C-property therefore numerical waves can appear. In this paper, we apply the original discrestization satisfying the C-property developed in Jelti et al. (2017). Consequently, the source terms given in Equation (6) are decomposed in the following form

$$S_i^n = \frac{1}{2}(S_{i,R}^n + S_{i,L}^n) \text{ and } Q_i^n = \frac{1}{2}(Q_{i,R}^n + Q_{i,L}^n) \tag{11}$$

where

$$S_{i,R}^n = \begin{pmatrix} 0 \\ -g\frac{z_{i+\frac{1}{2},R}-z_{i-\frac{1}{2},R}}{\Delta x}\frac{h_{i+\frac{1}{2},R}+h_{i-\frac{1}{2},R}}{2} - \frac{(\rho_s-\rho_w)g}{2\rho}\frac{(h_{i+\frac{1}{2},R}+h_{i-\frac{1}{2},R})^2}{4}\frac{c_{i+\frac{1}{2},R}-c_{i-\frac{1}{2},R}}{\Delta x} \\ 0 \\ 0 \end{pmatrix}$$

$$S_{i,L}^n = \begin{pmatrix} 0 \\ -g\frac{z_{i+\frac{1}{2},L}-z_{i-\frac{1}{2},L}}{\Delta x}\frac{h_{i+\frac{1}{2},L}+h_{i-\frac{1}{2},L}}{2} - \frac{(\rho_s-\rho_w)g}{2\rho}\frac{(h_{i+\frac{1}{2},L}+h_{i-\frac{1}{2},L})^2}{4}\frac{c_{i+\frac{1}{2},L}-c_{i-\frac{1}{2},L}}{\Delta x} \\ 0 \\ 0 \end{pmatrix}$$

$$Q_{i,R}^n = \begin{pmatrix} \frac{E-D}{1-p} \\ -g\frac{h_{i+\frac{1}{2},R}+h_{i-\frac{1}{2},R}}{2}S_f - \frac{(\rho_0-\rho)}{\rho}\frac{(E-D)}{(1-p)}\frac{u_{i+\frac{1}{2},R}+u_{i-\frac{1}{2},R}}{2} \\ E-D \\ -\frac{E-D}{1-p} \end{pmatrix}$$

and

$$Q_{i,L}^n = \begin{pmatrix} \frac{E-D}{1-p} \\ -g\frac{h_{i+\frac{1}{2},L}+h_{i-\frac{1}{2},L}}{2}S_f - \frac{(\rho_0-\rho)}{\rho}\frac{(E-D)}{(1-p)}\frac{u_{i+\frac{1}{2},L}+u_{i-\frac{1}{2},L}}{2} \\ E-D \\ -\frac{E-D}{1-p} \end{pmatrix}$$

5 Numerical results

In this section, we resolve the numerical scheme presented in the previous chapter. This system is already applied on several complicated test problem such as dam-break in Jelti et al. (2017), and approved its capacity to well capture shocks with high accuracy and without producing any nonphysical oscillations while maintaining the exact conservation property. The principal goal in this paper is to show the scheme capacity to detect the bed change even in problems with low sediment entrainment. The first test problem is an open channel hydraulic over mobile and smooth bed. The second and the third tests problems are open channels over mobile bed containing different form of bump.

The channel is supposed to be horizontal with rectangular cross-section composed of non-cohesive and uniform sediments. Step size space is $\Delta x = 10$m and Δt is computed according to a specified value of C_{FL} number equal to 0.85.

5.1 Open channel hydraulic over mobile and horizontal bed

Attention in given to the behavior of the flow over movable bed. The channel length is $1,000$ m with the following initial conditions:

$$h(x) = 10\text{m},\ u(0,x) = 0\text{m/s},\ c(0,x) = 0.001 \tag{12}$$

The objective of this test is to explore the behavior of the bed and the water free surface in such fluvial process with low sediment transport over horizontal bed. Knowing that, the numerical scheme in use is well tested and proves its capacity to model problems involving high concentration of sediment. In order to show the influence of the diameter, we compute this test using different sediment diameters $d = 1$ mm, $d = 4$ mm and $d = 8$ mm.
Results of water free surface, bed, concentration and velocity profiles are shown in Figure 1 for $t = 10$ min, $t = 20$ min, and $t = 1$ hour.

- Ten minutes after the water flow, a remarkable erosion occurred for the bottom composed of sediments of $d = 4$ mm and $d = 4$ mm contrariwise to that one composed of sediments of $d = 8$ mm, and this is due to the capacity of the flow to erode the smaller grains easily than the larger ones. After twenty min of the flow the channel's bottom try to find his initial level, this is in accordance with the experiments carried out by Capart & Young (1998);

- As we can see the water free surface stayed stable for both sediments sizes at $t = 10$ min, at $t = 20$ min and $t = 1$ hour. The variation of this profile is negligible comparing to the evolution of the water free surface after dam-break flow Jelti et al. (2017), and this is due the bed rate change;

- As it is shown the values of the concentration profiles increase after ten minutes of flow, and the highest value back to the the smaller sediment size. The concentration profiles decrease after twenty minutes;

- The highest value of velocities back to the sediment of medium size. The same for the velocity profiles, it increase ten minute after the flow and decrease twenty minute afterward.

5.2 Open channel hydraulic over mobile bed containing a bump

This problem is of length $1,000$m with the following initial conditions:

$$z(x) = \begin{cases} \sin^2(\frac{\pi(x-300)}{200}), & 300 < x \leq 500 \\ & 0, \text{Otherwise} \end{cases},$$
$$Q(0,x) = 10\text{m3/s},\ c(0,x) = 0.001,\ h(x) = 10 - z(x)$$

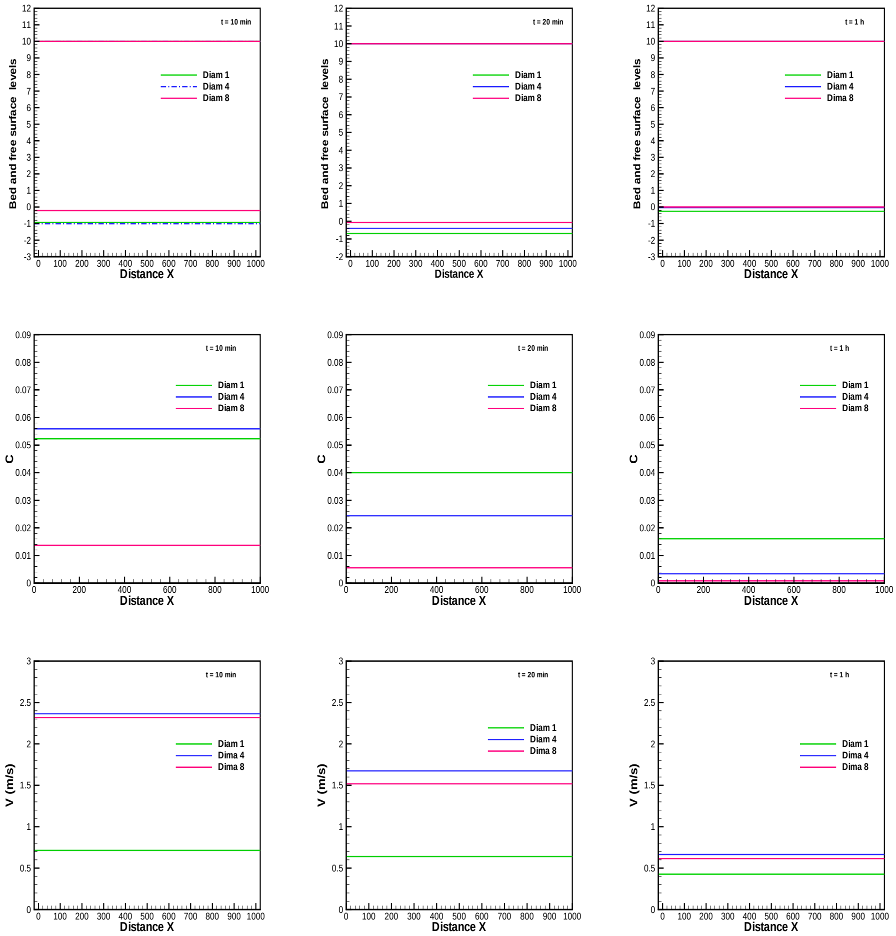

Figure 1: Bed and water levels , concentration and velocity profiles using different diameters and different computational time.

The same test was studied in Benkhaldoun et al. (2009); Hudson and Sweby (2003) among others, but computed with different mathematical models and different numerical schemes. We remember that suspended sediment and bed load are treated in single mode. In this test we use sediments of 1 mm of diameter.

Noting that IC represents the initial condition.

- As can be seen in figure 2 Roe scheme with the new discretization locates the correct bump location during the time;
- The water free surface remain stable except for small lowering level below the bump;
- We remark that, one minute after the flow the bottom is eroded by -1 m and the level of the bump is lowered. After 20 min the bump is definitely crushed, and the phenomena of aggregation appeared which justifies the raising of the level of the bottom after 50 hours .
- As it is obvious in figure 2 the velocity value increase one minute after the flow (after being null as initial condition), and then it decrease after 20 minute arriving at zero after 10 hours;
- The concentration value increase specially around the bump in first 20 min after the flow, and decrease until zero after 10 hours.

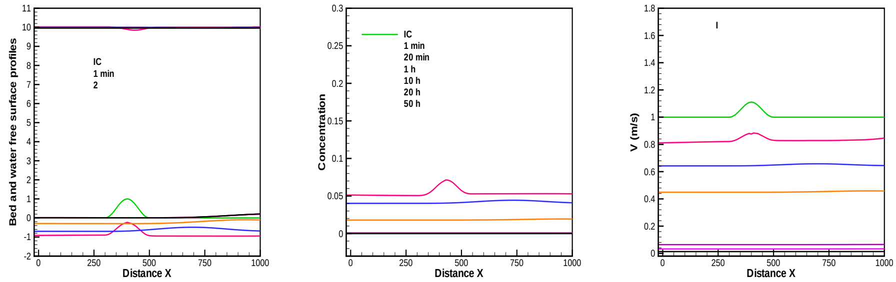

Figure 2: Bed and water levels , concentration and velocity profiles at different computational time.

5.3 Open channel hydraulic over mobile bed containing a rectangular bump

Results in the previous test were very satisfying, so we decided to test the flow over a bottom with rectangular bump where only sediments of 1 mm are used. This problem is of length 1, 500m with the following initial conditions:

$$z(x) = \begin{cases} 8, & |x - 750| \leq 1500/8 \\ & 0, \text{Otherwise} \end{cases},$$

$$Q(0, x) = 10\text{m3/s},\ c(0, x) = 0.001,\ h(x) = 15 - z(x)$$

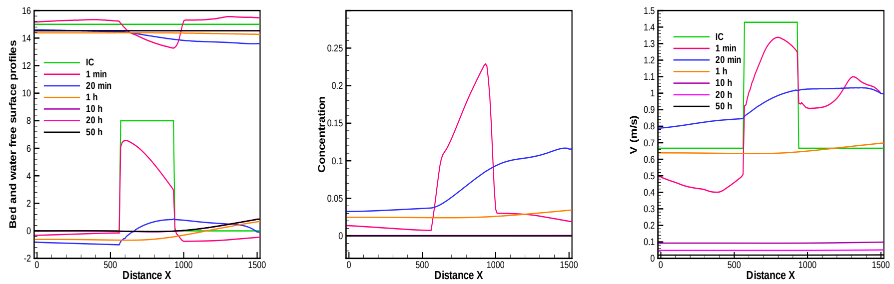

Figure 3: Bed and water levels , concentration and velocity profiles at different time using sediments of 1 mm of size.

Roe scheme with the new discretization well simulates the rectangular bump and locates the changes during the time in figure 3, and remarks are reported as follows

- Contrary to the previous test, the water free surface sustain remarkable changes and this is due to the height of the rectangular bump. After 1 hour it becomes stable;
- We attend a bump destruction after 1 min of the flow. Erosion reach -1 m in upstream, the level of the bottom becomes stable after 1 hour;
- After being null, the concentration increase 1 min after the flow. After that it decrease until zero;
- The velocity profiles undergoes several fluctuations, generally it decrease after 1 min arriving at a very low level.

6 Conclusion

The mathematical model associated to the finite volume Roe scheme with the new discretization of the source term treated very well the dam-break problem as it is presented in Jelti et al.

(2017). In order to test and to show the ability of this numerical scheme to detect the effect of the transport sediment on the flow in problems with low sediment entrainment, we applied it in this paper on open channel hydraulics.

This work is an application of Roe scheme with an original discretization of the source term on a problem of open channel hydraulics of different types: channel over mobile and horizontal bed, and on channel over mobile bed with a bump. Through the obtained results, the numerical scheme detected the bed rate change and changes on velocity, concentration and free water-surface profiles.

As we know that the effect of sediment transport on the flow is less pronounced in problems as channel hydraulics than dam-breaks ones, even this the Roe scheme with the new discretization detected the bed rate change in figures 1, 2 and 3. We conclude with the remarks brought by the obtained results:

- When the sediment size is finer, the bed mobility is greater and vice versa;
- Sediment transport interact with flow even in problems with low sediment concentration;
- The numerical scheme in use capt very well changes related to bed and water free surface.

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