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CONTROL PROBLEM OF VARIABLE ORDER FRACTIONAL SYSTEMS WITH TYPE I RIEMANN–LIOUVILLE DERIVATIVE

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Abstract. In this paper, the fractional optimal control problem for variable order differential system is considered. The considered fractional time derivative is Type I Riemann–Liouville. We first study the problem by using the Lax–Milgram Theorem, the existence and the uniqueness of the solution of the variable order fractional differential system is proved in a Hilbert space. Then we show that the considered optimal control problem has a unique solution.

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1 Introduction

System analysis consists of studying a set of concepts allowing a better knowledge of its properties. Among these notions, there are controllability, observability, stability, stabilization, detectability, spreadability, and so forth. For further details about these concepts, see for example Lions (1988); Russel & Weiss (1994); El Jai & Kassara (1994); Zuazua et al. (1993) and references therein. This analysis can be done through the operators appearing in the model and through the spatial distribution of the action or of the measure in the geometrical domain where the system is evolving.

Any study concerning the analysis of a dynamic system is generally followed by a control step, which consists of determining a command which makes it possible to lead the studied system to a certain objective. By way of illustration, we have the possibility to bring a system from an initial to a desired state at a finite instant T, which defines the concept of exact controllability. This last is hard to achieve in real problems, so as an alternative there is the notion of approximate controllability where we target a neighborhood of the desired state. The notion of controllability is one of the most extensively studied subject in systems analysis, since any system can't be left behave freely we need to control it.

Lately the fractional calculus has attracted considerable attention and interest the same as classical calculus. From a mathematical point of view, it is a generalization of classical integer order calculus to integrals and derivatives of non-integer (arbitrary) real or complex order. Generally speaking, there are three well known derivatives that are used frequently, Grunwald–

Letnikov fractional derivative, Riemann–Liouville fractional derivative, and Caputo fractional derivative. The former two derivatives are often used by pure mathematicians, while the last one is adopted by applied scientists, since it is more convenient in engineering applications.

In fact, the fractional calculus has been acknowledged as a promising mathematical tool to efficiently characterize the historical memory and global correlation of complex dynamic systems, phenomena or structures. However, various literature indicated that the memory and/or nonlocality of the system may change with time, space or other conditions Sun et al. (2011); Lorenzo et al. (2002). The variable-order fractional operators depending on their non-stationary power-law kernel can describe the memory and hereditary properties of many physical phenomena and processes. Therefore, to accurately characterize complex physical systems and processes, variable-order fractional calculus was availably employed as a potential candidate to provide an effective mathematical framework Zhang et al. (2013). Subsequently, variable-order fractional differential equations have attracted more and more attention, ascribing to its suitability in modeling along with a large variety of phenomena, ranging from many fields of science and engineering, including anomalous diffusion Chechkin et al. (2005); Sun et al. (2009), viscoelastic mechanics Coimbra (2003); Smit et al. (1970), control system Kumar et al. (2017); Xue (2017), petroleum engineering Obembe et al. (2017), and other branches of physics and engineering, just to mention a few Cai et al. (2018); Jiang et al. (2017); Zhang & Li (2015); Sun et al. (2019); Li et al. (2017); Patnaik et al. (2020). Many researchers were interested in presenting some approaches for solving fractional variable order linear differential equations either analytically or numerically Malesza et al. (2019); Katsikadelis (2018); Derakhshan & Aminataei (2020); Ghomanjani (2020).

A lot was done in the area of calculus of variations and optimal control of fractional differential equations with constant order integer time derivatives Debbouche et al. (2017); Bahaa (2016); Karite et al. (2018, 2020), and we are here interested in the fractional control problem with variable order differential equations with Riemann–Liouville derivative. The work is an extension of what was done by G.M. Bahaa Bahaa (2017) for fractional variable-order systems in Caputo sense.

The remaining of this paper is organized as follows. In Section 2, we introduce some basic definitions for variable-order fractional operators. We formulate the variable-order fractional Dirichlet problem with Riemann–Liouville derivative in section 3. In Section 4, the minimization problem is formulated and we prove the uniqueness of its solution. Finally, we conclude by some conclusions and remarks.

2 Preliminaries on variable order fractional operators

Many researchers have been interested in investigating fractional differential equations in the last decades. The notion of a variable-order operator is much more recent, several authors have presented various definitions of variable-order differential operators. Our goal is to consider fractional derivatives of variable-order, with α depending on time t. In fact, some phenomena in physics are better described when the order of the fractional operator is not constant, for example, in the diffusion process in an inhomogeneous or heterogeneous medium, or processes where the changes in the environment modify the dynamic of the particle Chechkin et al. (2005); Sun et al. (2009); Santamaria et al. (2006). Motivated by the above considerations, we introduce some variable order fractional integrals and variable order fractional derivatives of function in the Riemann-Liouville and Caputo sense.

Let $\Omega \subset I\!R^n$ be bounded with a smooth boundary $\partial\Omega$. For $T > 0$, denote $Q = \Omega \times [0, T]$ and $\Sigma = \partial\Omega \times [0, T]$.

Definition 1. *Podlubny (1999) Let* $0 < \alpha(t) < 1$ *for all* $t \in [0, T]$, $f \in L^1([0, T])$, *and* Γ *be the*

Euler's gamma function, which is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \mathrm{d}t.$$

This function is a generalization of the factorial, if $n \in I\!N$, *then* $\Gamma(n) = (n-1)!$.

Now, we present the fundamental notions of the fractional calculus of variable order. We consider the fractional order of the derivative and of the integral to be a continuous function of two variables, $\alpha(\cdot,\cdot)$ with domain $[0,T]^2$, taking values on the open interval $(0,1)$. Let $z : [0,T] \to \mathbb{R}$ be a function. We first recall the generalization of fractional integrals for a variable-order $\alpha(\cdot,\cdot)$.

Definition 2. *Almeida et al. (2019) The left and right Riemann-Liouville fractional integrals of order* $\alpha(\cdot,\cdot)$ *are defined by*

$${}_0I_t^{\alpha(\cdot,\cdot)} z(t) = \int_0^t \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} z(\tau) d\tau, \quad t > 0,$$

and

$${}_tI_T^{\alpha(\cdot,\cdot)} z(t) = \int_t^T \frac{1}{\Gamma(\alpha(\tau,t))} (\tau-t)^{\alpha(\tau,t)-1} z(\tau) d\tau, \quad t < T.$$

respectively.

For fractional derivatives, we consider two types: the Riemann–Liouville and the Caputo fractional derivatives.

Definition 3. *Almeida et al. (2019) The left and right Riemann–Liouville fractional derivatives of order* $\alpha(\cdot,\cdot)$ *are defined by*

$$\begin{aligned} {}_0D_t^{\alpha(\cdot,\cdot)} z(t) &= \frac{d}{dt} {}_0I_t^{\alpha(\cdot,\cdot)} z(t) \\ &= \frac{d}{dt} \int_0^t \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} z(\tau) d\tau, \quad t > 0, \end{aligned}$$

and

$$\begin{aligned} {}_tD_T^{\alpha(\cdot,\cdot)} z(t) &= -\frac{d}{dt} {}_tI_T^{\alpha(\cdot,\cdot)} z(t) \\ &= \frac{d}{dt} \int_t^T \frac{-1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} z(\tau) d\tau, \quad t < T. \end{aligned}$$

respectively.

Definition 4. *Almeida et al. (2019) The left and right Caputo fractional derivatives of order* $\alpha(\cdot,\cdot)$ *are defined by*

$${}_0^C D_t^{\alpha(\cdot,\cdot)} z(t) = \int_0^t \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} z^{(1)}(\tau) d\tau, \quad t > 0,$$

and

$${}_t^C D_T^{\alpha(\cdot,\cdot)} z(t) = \int_t^T \frac{-1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} z^{(1)}(\tau) d\tau, \quad t < T.$$

respectively.

In this paper we are interested in type I Riemann-Liouville fractional derivative defined below.

Definition 5. *Almeida et al. (2019) Given a function* $z : [0,T] \to \mathbb{R}$,

1. *the type I left Riemann–Liouville fractional derivative of order* $\alpha(t)$ *is defined by*

$$ {}_0D_t^{\alpha(t)}z(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha(t)}z(\tau)d\tau, $$

2. *the type I right Riemann–Liouville fractional derivative of order* $\alpha(t)$ *is defined by*

$$ {}_tD_b^{\alpha(t)}z(t) = \frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^T (\tau-t)^{-\alpha(t)}z(\tau)d\tau. $$

The Caputo type I derivatives are given using the previous Riemann–Liouville fractional derivatives.

Definition 6. *Given a function* $z : [0,T] \to \mathbb{R}$,

1. *the type I left Caputo derivative of order* $\alpha(t)$ *is defined by*

$$ \begin{aligned} {}_0^CD_t^{\alpha(t)}z(t) &= {}_0D_t^{\alpha(t)}(z(t)-z(0)) \\ &= \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha(t)}[z(\tau)-z(0)]d\tau, \end{aligned} $$

2. *the type I right Caputo derivative of order* $\alpha(t)$ *is defined by*

$$ \begin{aligned} {}_t^CD_T^{\alpha(t)}z(t) &= {}_tD_T^{\alpha(t)}(z(t)-z(T)) \\ &= \frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^T (\tau-t)^{-\alpha(t)}[z(\tau)-z(T)]d\tau. \end{aligned} $$

Remark 1. *The fractional derivatives just defined are linear operators.*

3 Dirichlet problem with variable order fractional derivative

Let us consider the variable-order fractional partial differential system. We consider the following abstract fractional sub-diffusion system of variable order

$$ \begin{cases} {}_0^{RL}D_t^{\alpha(t)}y(x,t) + \mathcal{A}y(x,t) = \mathcal{B}u(t), & \text{in } \ Q, \\ y(\xi,t) = 0 & \text{on } \ \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha(t)}y(x,t) = y_0(x) & \text{in } \ \Omega, \end{cases} \tag{1} $$

where $\frac{1}{n} < \alpha(t) < 1$, $n \in I\!N$ and ${}_0^{RL}D_t^{\alpha(t)}$ denotes the Riemann–Liouville type I derivative of variable-order $\alpha(\cdot)$ (for details on variable-order fractional derivatives, see, e.g., Almeida et al. (2019)). The second order operator $\mathcal{A} \in \mathcal{L}\left(H_0^1(\Omega), H_0^{-1}(\Omega)\right)$, the state y belongs to $L^2(0,T;H_0^1(\Omega))$. And the control u belongs to $U = L^2(0,T;I\!R^m)$, where m is the number of actuators. The initial datum y_0 is in $H^2(\Omega)\cap H_0^1(\Omega)$, $\mathcal{B} : I\!R^m \longrightarrow L^2(Q)$ is the control operator, which is linear, possibly unbounded, and depending on the number and structure of actuators El Jai et al. (1993).

We define the following bilinear form in relation with the second order operator $\mathcal{A}$.

Definition 7. *For each fixed* $t \in]0,T[$, *we define a family of bilinear forms* $h(t,y;\phi)$ *on* $H_0^1(\Omega)$ *by*

$$ h(t,y;\phi) = \langle \mathcal{A}y, \phi \rangle, \qquad y,\phi \in H_0^1(\Omega). \tag{2} $$

The following assumptions are needed to prove our main results.

(H1) *The bilinear form $h(t;y,\phi)$ is coercive on $H_0^1(\Omega)$, that is*

$$h(t,y;y) \geq \lambda \parallel y \parallel^2, \qquad \lambda > 0. \tag{3}$$

(H2) *We also assume that the function $t \to h(t;y,\phi)$ is continuously differentiable in $]0,T[$ and the bilinear form $h(t;y,\phi)$ is symmetric.*

Lemma 1 (Lax–Milgram)**.** *Assume that h is bounded and coercive with constant $\lambda > 0$. Then, there exists a unique $u \in H_0^1(\Omega)$ solution of (2).*

We also need the following lemmas which assure the integration by parts for a fractional diffusion equation with a Riemann–Liouville derivative.

Lemma 2 (Fractional Green's formula)**.** *Let $0 < \alpha < 1$ and y be the solution of system (1). Then for any $\varphi \in C^\infty(\overline{Q})$, we have*

$$\begin{aligned}\int_0^T \int_\Omega \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t &= \int_\Gamma y(x,T)_t I_T^{1-\alpha(t)}\varphi(x,T)d\Gamma - \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)d\Gamma \\ &+ \int_0^T \int_\Gamma \varphi(x,t)\frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\mathrm{d}\Gamma\mathrm{d}t - \int_0^T \int_\Gamma \frac{\partial \varphi(x,t)}{\partial \nu_{\mathcal{A}}} y(x,t)\mathrm{d}\Gamma\mathrm{d}t \\ &+ \int_0^T \int_\Omega y(x,t)\left({}_t^C D_T^{\alpha(t)}\varphi(x,t) + \mathcal{A}^*\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Proof. Let $\varphi \in C^\infty(\overline{Q})$, we have

$$\begin{aligned}\int_0^T \int_\Omega \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega \varphi(x,t){}_0^{RL}D_t^{\alpha(t)} y(x,t)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega \varphi(x,t)\mathcal{A}y(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4}$$

Moreover,

$$\begin{aligned}\int_0^T \int_\Omega \mathcal{A}y(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t &= -\int_0^T \int_\Gamma y(x,t)\frac{\partial \varphi(x,t)}{\partial \nu_{\mathcal{A}}}\mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\varphi(x,t)\mathrm{d}\Gamma\mathrm{d}t \\ &+ \int_0^T \int_\Omega y(x,t)\mathcal{A}^*\varphi(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

And,

$$\begin{aligned}\int_0^T \int_\Omega \varphi(x,t){}_0^{RL}D_t^{\alpha(t)} y(x,t)\mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega y(x,t){}_t^C D_T^{\alpha(t)}\varphi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Gamma y(x,t)_t I_T^{1-\alpha(t)}\varphi(x,t)\mathrm{d}\Gamma\mathrm{d}t \\ &= \int_0^T \int_\Omega y(x,t){}_t^C D_T^{\alpha(t)}\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_\Gamma y(x,T)_t I_T^{1-\alpha(t)}\varphi(x,T)\mathrm{d}\Gamma \\ &- \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma.\end{aligned}$$

We then substitute in (4), we get

$$\begin{aligned}\int_0^T \int_\Omega \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t &= \int_\Gamma y(x,T)_t I_T^{1-\alpha(t)}\varphi(x,T)\mathrm{d}\Gamma - \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma \\ &+ \int_0^T \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\varphi(x,t)\mathrm{d}\Gamma\mathrm{d}t - \int_0^T \int_\Gamma y(x,t)\frac{\partial \varphi(x,t)}{\partial \nu_{\mathcal{A}}}\mathrm{d}\Gamma\mathrm{d}t \\ &+ \int_0^T \int_\Omega y(x,t)\left({}_t^C D_T^{\alpha(t)}\varphi(x,t) + \mathcal{A}^*\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

□

From the previous lemma, we deduce the following result

Lemma 3. *Let $0 < \alpha < 1$. Then for any $\varphi \in C^{\infty}(\overline{Q})$ such that $\varphi(x,T) = 0$ in Ω and $\varphi = 0$ on Σ, we have*

$$\begin{aligned}\int_0^T \int_\Omega \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t &= -\int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)d\Gamma + \int_0^T \int_\Gamma \varphi(x,t)\frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\mathrm{d}\Gamma\mathrm{d}t \\ &- \int_0^T \int_\Gamma \frac{\partial \varphi(x,t)}{\partial \nu_{\mathcal{A}}} y(x,t)\mathrm{d}\Gamma\mathrm{d}t \\ &+ \int_0^T \int_\Omega y(x,t)\left({}_t^{C}D_T^{\alpha(t)}\varphi(x,t) + \mathcal{A}^*\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t\end{aligned}$$

Lemma 4. *If **(H1)** and **(H2)** are verified, then system* (1) *admits a unique solution $y \in \mathcal{Z}(0,T)$, where $\mathcal{Z}(0,T) := \left\{ y \,|\, y(x,t) \in L^2(0,T;H_0^1(\Omega)),\ {}_0^{RL}D_t^{\alpha(t)} y(x,t) \in L^2(0,T;H_0^{-1}(\Omega)) \right\}$.*

Proof. From **(H1)** and using lemma 1, there exists a unique element $y(x,t) \in H_0^1(\Omega)$ such that

$$\left({}_0^{RL}D_t^{\alpha(t)} y(x,t), \varphi\right)_{L^2(Q)} + h(t;y,\varphi) = L(\varphi), \qquad \forall\, \varphi \in H_0^1(\Omega),$$

where $L(\varphi)$ is a continuous linear form on $H_0^1(\Omega)$.
i.e.

$$\left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t), \varphi\right)_{L^2(Q)} = L(\varphi), \qquad \forall\, \varphi \in H_0^1(\Omega),$$

which can be written as

$$\int_Q \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x)\mathrm{d}x\mathrm{d}t = L(\varphi), \qquad \forall\, \varphi \in H_0^1(\Omega). \tag{5}$$

$L(\varphi)$ takes the following form

$$L(\varphi) = \int_Q \mathcal{B}u(t)\varphi(x)\mathrm{d}x\mathrm{d}t - \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma. \tag{6}$$

Then equation (5) is equivalent to

$$\int_Q \left({}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(x,t)\right)\varphi(x)\mathrm{d}x\mathrm{d}t = \int_Q \mathcal{B}u(t)\varphi(x)\mathrm{d}x\mathrm{d}t - \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma, \quad \forall\, \varphi \in H_0^1(\Omega) \tag{7}$$

Taking $y(x,0) = y_0(x) = 0$, we get the fractional differential equation

$${}_0^{RL}D_t^{\alpha(t)} y(x,t) + \mathcal{A}y(t) = \mathcal{B}u(t).$$

By applying Fractional Green's Formula to (7), we have

$$\begin{aligned}&-\int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma - \int_0^T \int_\Gamma \frac{\partial \varphi(x,t)}{\partial \nu_{\mathcal{A}}} y(x,t)\mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_\Omega y(x,t)\left({}_t^{C}D_T^{\alpha(t)}\varphi(x,t) + \mathcal{A}^*\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t \\ &= \int_Q \mathcal{B}u(t)\varphi(x)\mathrm{d}x\mathrm{d}t - \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma + \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma \\ &= \int_\Gamma y(x,0)_t I_T^{1-\alpha(t)}\varphi(x,0)\mathrm{d}\Gamma.\end{aligned}$$

Then for any $\varphi \in C^{\infty}(\overline{Q})$ such that $\varphi(x,T) = 0$ in Ω and $\varphi = 0$ on Σ, we deduce the two conditions of system (1). □

4 Control problem of a variable-order fractional system

For a control $u \in U = L^2(0,T;\mathbb{R}^m)$, the state $y(u;x,t)$ (which could be denoted for simplification by $y_u(x,t)$) of the system is given by

$$\begin{cases} {}_0^{RL}D_t^{\alpha(t)} y_u(x,t) + \mathcal{A} y_u(x,t) = \mathcal{B}u(t), & \text{in} \quad Q, \\ y_u(\xi,t) = 0 & \text{on} \quad \Sigma, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha(t)} y_u(x,t) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{8}$$

Let us consider the following cost function $\mathcal{J}(u)$ given by

$$\begin{cases} \mathcal{J}(u) = \displaystyle\int_Q (y_u(x,t) - z_d)^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \|u\|_U^2 \mathrm{d}t, \\ u \in U_{ad}. \end{cases} \tag{9}$$

where U_{ad} is the set of admissible controls and z_d is the desired final state.

Problem: The objective is to minimize the following optimization problem

$$\begin{cases} \inf \mathcal{J}(u), \\ u \in U_{ad}. \end{cases} \tag{10}$$

We define the backward system by

$$\begin{cases} {}_t^{RL}D_T^{\alpha(t)} \phi(x,t) + \mathcal{A}^* \phi(x,t) = y_u(x,t) - z_d, & \text{in} \quad Q, \\ \phi(\xi,t) = 0 & \text{on} \quad \Sigma, \\ \lim\limits_{t\to 0^+} {}_tI_T^{1-\alpha(t)} \phi(x,t) = 0 & \text{in} \quad \Omega. \end{cases} \tag{11}$$

We multiply the left member of the first equation of system (11) by $y(x,t)$, we get

$$\begin{aligned} \int_0^T\int_\Omega y(x,t)\left({}_t^{RL}D_T^{\alpha(t)}\phi(x,t) + \mathcal{A}^*\phi(x,t)\right)\mathrm{d}x\mathrm{d}t &= \int_\Gamma y(x,T)\,{}_tI_T^{1-\alpha(t)}\phi(x,T)\mathrm{d}\Gamma - \int_\Gamma y(x,0)\,{}_tI_T^{1-\alpha(t)}\phi(x,0)\mathrm{d}\Gamma \\ &- \int_0^T\int_\Gamma y(x,t)\frac{\partial\phi(x,t)}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma\mathrm{d}t + \int_0^T\int_\Gamma \frac{\partial y(x,t)}{\partial\nu_{\mathcal{A}}}\phi(x,t)\mathrm{d}\Gamma\mathrm{d}t \\ &+ \int_0^T\int_\Omega \phi(x,t)\left[{}_t^{C}D_T^{\alpha(t)}y(x,t) + \mathcal{A}^*y(x,t)\right]\mathrm{d}x\mathrm{d}t. \end{aligned}$$

We can state the following result.

Theorem 1. *Problem* (10) *admits a unique solution given by* (8) *and*

$$\int_0^T\int_\Omega (\phi(x,t)\mathcal{B} + u)(v-u)\mathrm{d}x\mathrm{d}t \geq 0, \tag{12}$$

where ϕ is the adjoint state.

Proof. The control $u \in U_{ad}$ is optimal if and only if

$$\mathcal{J}'(u)(v-u) \geq 0, \qquad \forall\, v \in U_{ad}.$$

It gives

$$(y_u(x,t) - z_d, y_v(x,t) - y_u(x,t))_{L^2(Q)} + (u, v-u)_U \geq 0,$$

i.e.

$$\int_Q (y_u(x,t) - z_d)(y_v(x,t) - y_u(x,t))\mathrm{d}x\mathrm{d}t + (u,v-u)_U \geq 0. \tag{13}$$

By multiplying the first equation of system (11) by $(y_v(x) - y_u(x))$ and applying Fractional Green's Formula similarly to the previous step, we obtain

$$\int_0^T \int_\Omega \left({}_t^{RL}D_T^{\alpha(t)} \phi(x,t) + \mathcal{A}^* \phi(x,t) \right) (y_v(x,t) - y_u(x,t)) \mathrm{d}x\mathrm{d}t =$$

$$= \int_0^T \int_\Omega (y_u(x,t) - z_d)(y_v(x,t) - y_u(x,t)) \mathrm{d}x\mathrm{d}t$$

For the left member of the equation, we obtain

$$\int_0^T \int_\Omega \left({}_t^{RL}D_T^{\alpha(t)} \phi(x,t) + \mathcal{A}^* \phi(x,t) \right) (y_v(x,t) - y_u(x,t)) \mathrm{d}x\mathrm{d}t =$$

$$= \int_\Gamma (y_v(x,0) - y_u(x,0)) \, {}_tI_T^{1-\alpha(t)} \phi(x,0) \mathrm{d}\Gamma$$
$$+ \int_0^T \int_\Gamma (y_v(x,t) - y_u(x,t)) \frac{\partial \phi(x,t)}{\partial \nu_{\mathcal{A}^*}} \mathrm{d}\Gamma\mathrm{d}t$$
$$- \int_0^T \int_\Gamma \left(\frac{\partial y_v(x,t)}{\partial \nu_{\mathcal{A}^*}} - \frac{\partial y_u(x,t)}{\partial \nu_{\mathcal{A}^*}} \right) \phi(x,t) \mathrm{d}\Gamma\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \phi(x,t) \left({}_t^{C}D_T^{\alpha(t)} + \mathcal{A} \right) (y_v(x,t) - y_u(x,t)) \mathrm{d}x\mathrm{d}t$$

Since from (1) we have

$$\begin{cases} \left({}_0^{RL}D_t^{\alpha(t)} + \mathcal{A} \right) (y_u(x,t) - y_u(x,t)) = \mathcal{B}(v-u), & \text{in} \quad Q, \\ y_u(x)|_\Sigma = 0, \\ \phi(x)|_\Sigma = 0. \end{cases}$$

Taking into account, the initial and boundary conditions, then we obtain

$$\int_0^T \int_\Omega (y_u(x) - z_d)(y_v(x) - y_u(x)) \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \phi(x,t) \mathcal{B}(v-u) \mathrm{d}x\mathrm{d}t.$$

Hence, (13) becomes

$$\int_0^T \int_\Omega \phi(x,t) \mathcal{B}(v-u) \mathrm{d}x\mathrm{d}t + (u, v-u)_U \geq 0.$$

i.e.

$$\int_0^T \int_\Omega (\phi(x,t)\mathcal{B} + u)(v-u) \mathrm{d}x\mathrm{d}t \geq 0.$$

Which concludes the proof. □

5 Conclusion

An optimal control problem for variable-order differential system was considered. We opted for the Type I Riemann–Liouville derivative, where α contains one parameter depending on time t. The existence and uniqueness of the solution is proved for this type of systems and the optimal control is computed. The optimal problem is a generalization of the one of parabolic systems with Dirichlet conditions considered by Lions. The work done here could be extended to the case of semilinear fractional systems as well as to systems expressed with other types of fractional derivatives.

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Dedication

This work is dedicated to the soul of our Professor and Godfather Prof. Ali Boutoulout. A man with such kindness and warmhearted that helped and guided so many young researchers in finding their path in the world of research. You will be always remembered. May his soul rest in peace.

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