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AN ITERATIVE METHOD FOR CAUCHY PROBLEMS SUBJECT TO THE CONVECTION-DIFFUSION EQUATION

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Abstract. In this text, we presented the Nachaoui's iterative alternating method for solving the Cauchy problem governed by the convection-diffusion equation. The method is an iterative algorithm that alternates between solving two subproblems of the same type with boundary conditions of the Dirichlet and Neuman type on the inaccessible part of the boundary. The algorithm continues iterating until a convergence criterion is met. We discussed the convergence and computational efficiency of the method. The numerical results show that the method is computationally efficient and that the relaxation parameter can greatly reduce the number of iterations.

Keywords: Inverse problems, Ill-posed problems, convection-diffusion equation.
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1 Introduction

In this work, we consider the convection-diffusion equation (CDE)

$$\mathcal{L}u := -\mu\Delta u + \beta \cdot \nabla u = f \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is open, bounded and connected, $\mu > 0$ is the diffusion coefficient and $\beta \in [W^{1,\infty}(\Omega)]^n$ is the convective velocity field. The coefficients μ and β, and the source term f are assumed to be known.

Problem (1) describes the stationary distribution of a physical quantity u (e.g., temperature or concentration) determined by two basic physical mechanisms, namely the convection and diffusion. The broad interest in solving problem (1) is caused not only by its physical meaning just explained but also (and perhaps mainly) by the fact that it is a simple model problem for convection–diffusion effects which appear in many more complicated problems arising in applications (e.g. in various fluid flow problems).

CDEs can be used to describe the transport process of heat or mass, such as the distribution of pollutants in air, river and groundwater. There have been extensive researches on numerical methods for linear or nonlinear CDEs Hayeck et al. (1990); Fu (2010); Gue & Stynes (1993); Yücel et al. (2015); Rashidinia et al. (2018); Xiao et al. (2019); Darehmiraki et al. (2022); Tian & Lin (2022).

If the convection term dominates the diffusion term in CDEs, then the standard finite difference method (FDM) or finite element method (FEM) will produce nonphysical oscillations. De Figueiredo et al. (1990) and Gupta et al. (1994) have investigated a boundary element formulation for the steady-state convection-diffusion equation with constant and variable velocities, respectively, when applied to convective heat problems. In Nachaoui & Nassif (1996), the authors studied the convergence of an iterative method as well as its finite element discretized form for a drift-diffusion model in semiconductors Nachaoui (1999). Some analytical solutions of the equation and their influence on atmospheric processes have been presented by McKenna McKenna (1997). In solving the Cauchy problems of the elliptic type linear PDEs in the closed walled shells,Lin & Liu (2022) take as example a strong convection diffusion equation.

Despite the apparent simplicity of problem (1), its numerical solution is still a challenge when convection is strongly dominant. The basic difficulty is that, in this case, the solution of (1) typically possesses interior and boundary layers, which are small subregions where the derivatives of the solution are very large. The widths of these layers are usually significantly smaller than the mesh size and hence the layers cannot be resolved properly. This leads to unwanted spurious (nonphysical) oscillations in the numerical solution, the attenuation of which has been the subject of extensive research for more than three decades.

In practical applications, it is often the case that we do not have access to the full set of boundary data in a given domain. Instead, we may only be able to measure certain quantities on a part of the boundary while the rest of the boundary remains inaccessible Jourhmane & Nachaoui (1999); Huang & Chen (2000); Kraus et al. (2001); Yarmukhamedov & Yarmukhamedov (2003); Essaouini et al. (2004); Nachaoui (2004); Regińska & Regiński (2006); Shi et al. (2009); Qian et al. (2010); Gasimov et al. (2019); Kabanikhin (2012); Mukanova B. (2013); Lavrentiev (2013); Huang et al. (2017); Isakov (2017); Qian & Feng (2017); Vasil'ev et al. (2017); Hernandez-Montero et al. (2019); Aboud et al. (2021); Berdawood et al. (2021); Nachaoui & Salih (2021); Reddy et al. (2021). Such problems arise in various areas of science and engineering, such as fluid dynamics Chakib et al. (2019), in structural mechanics Ellabib et al. (2021), in electromagnetism Nachaoui (2004), meteorology, and environmental modeling and in medicine Nachaoui et al. (2023), where it is important to estimate the state of a system based on incomplete or noisy observations. This motivates the study of data assimilation problems governed by (1), where the goal is to reconstruct the unknown solution u using the available data. In this context, the solution of (1) represents the true state of the system, and the data assimilation problem is to find an approximation of the solution that is consistent with the available measurements. The development of efficient and accurate methods for solving such problems is an active area of research with many potential applications.

This paper presents a numerical technique based on the iterative algorithm proposed in Jourhmane & Nachaoui (1999, 2002), in order to solve the Cauchy problem associated to the steady-state convection- diffusion equation. In Section 2 we formulate the mathematical problem under investigation and then, in Section 2.1, we present the methods employed to solve the problem. In Section 3 the numerical results obtained for some test examples related to different geometries, namely circular, annular and rectangular domains, are presented.

2 The inverse Cauchy problem and method of resolution

Inverse problems involve determining unknown parameters in a system based on observations or measurements of the system's response. These problems arise in many areas of science and engineering, including medical imaging, geophysics, and non-destructive testing Gorelick et al. (1983); Chen & Wong (1998); Huang & Chen (2000); Ellabib & Nachaoui (2001); Chakib & Nachaoui (2005); Arsenashvili et al. (2008); Aster et al. (2012); Chakib et al. (2012); Boulkhemair et al. (2013); Dvalishvili et al. (2017); Rasheed et al. (2021); Aboud et al. (2022); Nachaoui et al. (2022); Rashid et al. (2023).

In this paper, we are interested in determining the trace of the solution of the Cauchy problem for the convection-diffusion equation on the part of the boundary where no data is prescribed, i.e. at Γ_u from boundary Cauchy data (Φ, T) on Γ_m.

$$\begin{cases} \text{find } u \text{ such that} & & \\ -\mu\Delta u + \beta\cdot\nabla u & = f & \text{in } \Omega \\ u & = T_1 & \text{on } \Sigma_1 \\ \frac{\partial u}{\partial n} & = g & \text{on } \Sigma_3 \\ \frac{\partial u}{\partial n} & = \Phi & \text{on } \Gamma_m \\ u & = T_2 & \text{on } \Gamma_m \end{cases} \tag{2}$$

Of course, the part of the boundary, Σ_1 and/or Σ_3 can be empty and in this case the corresponding boundary conditions no longer appear in the problem.

There are many applications of this type of inverse problem, including environmental monitoring, combustion modeling, and fluid dynamics. Accurate determination of the u on at Γ_u can lead to improved understanding and control of these systems, which can have significant practical implications.

It is known that these problems are ill-posed in the sense of Hadamard Hadamard (1953); Dvalishvili et al. (2017). The existence and uniqueness of a solution is subject to compatibility conditions. But the continuity in relation to the data is not assured. Small variations in the data can cause very large discrepancies between the corresponding solutions. Therefore, developing efficient and accurate numerical methods for solving this inverse problem is of great importance.

An example of the domain in which the convection-diffusion equation is satisfied is given in figure 1 . This figure is given as an axample but the method proposed is independent of the type of the domain and it is valid whatever the type of applied boundary conditions.

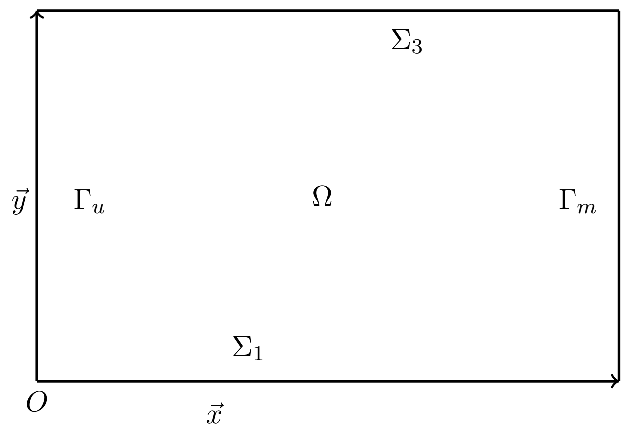


Figure 1: Notation of the domain Ω

2.1 The JN iterative method

The JN iterative alternating method Jourhmane & Nachaoui (1999) provides a computationally efficient way of solving the Cauchy problem for the CDE.

This method is particularly useful for simulating fluid flows Chakib et al. (2019) and various engineering applications Nachaoui (2004); Mukanova B. (2013); Ellabib et al. (2021); Nachaoui et al. (2023). It was also used in problems posed in non-homogeneous media Aboud et al. (2021) and with nonlinearities Essaouini et al. (2004). This motivates its application here in order to study its contribution to solving data assimilation problems governed by the convection-diffusion equation. The method is able to handle complex geometries with ease, and the JN Relaxed alternating iterative approach (Algorithm 1) allows for a better convergence rate and overall computational efficiency. Additionally, the method is easily parallelizable, which makes it ideal for large-scale simulations.

Algorithm 1: Nachaoui's Iterative Alternating Method

1: For a specific τ_0 on Γ_u and given tolerance ε_1,take $k = 0$ and solve:

$$\begin{cases} -\mu\Delta u^{(0)} + \beta \cdot \nabla u^{(0)} = f & \text{in } \Omega \\ u^{(0)} = T_1 & \text{on } \Sigma_1 \\ \partial_\nu u^{(0)} = g & \text{on } \Sigma_3 \\ \partial_\nu u^{(0)} = \Phi & \text{on } \Gamma_m \\ u^{(0)} = \tau_0 & \text{on } \Gamma_u \end{cases}$$

2: Take $\eta = \frac{\partial u^{(2k)}}{\partial n}|_{\Gamma_u}$ and $k = k + 1$

3: Solve:

$$\begin{cases} -\mu\Delta u^{(2k-1)} + \beta \cdot \nabla u^{(2k-1)} = f & \text{in } \Omega \\ u^{(2k-1)} = T_1 & \text{on } \Sigma_1 \\ \partial_\nu u^{(2k-1)} = g & \text{on } \Sigma_3 \\ u^{(2k-1)} = T_2 & \text{on } \Gamma_m \\ \partial_\nu u^{(2k-1)} = \eta & \text{on } \Gamma_u \end{cases} \tag{3}$$

4: If

$$\|u^{(2k-1)} - u^{(2(k-1))}\|_{L^2(\Gamma_u} < \varepsilon_1, \tag{4}$$

stop

5: Take

$$\tau = \theta u^{(2k-1)}_{|\Gamma_u} + (1 - \theta) u^{(2(k-1))}_{|\Gamma_u}, \tag{5}$$

with $\theta \in]0, 2[$ and solve:

$$\begin{cases} -\mu\Delta u^{(2k)} + \beta \cdot u^{(2k)} = f & \text{in } \Omega \\ u^{(2k)} = T_1 & \text{on } \Sigma_1 \\ \partial_\nu u^{(2k)} = g & \text{on } \Sigma_3 \\ \partial_\nu u^{(2k)} = \Phi & \text{on } \Gamma_m \\ u^{(2k)} = \tau & \text{on } \Gamma_u \end{cases} \tag{6}$$

6: Return to step 2

Note that for $\theta = 1$, we obtain the classic KMF algorithm widely used in the literature (see for example Nachaoui et al. (2021b) for a study of this algorithm and its comparison with other algorithms for the Cauchy problem governed by the Poisson equation). We will show through the numerical results that the JN algorithm ($\theta \neq 1$ can be an acceleration of the one where $\theta = 1$.

3 Results and Analysis

In this section, we discuss the performance of the Nachaoui's iterative alternating method for solving the Cauchy problem (2). We first present some numerical results for two-dimensional examples with a known solution with $\theta = 1$ in (5). We then discuss the convergence and computational efficiency of the JN method (Algorithm 1 with relaxation parameter $\theta \neq 1$).

Consider the following domain $\Omega = [-0.5, 0.5] \times [-1, 1]$ with boudary given by $\partial\Omega = \Sigma_1 \cup \Gamma_m \cup \Sigma_3 \cup \Gamma_u$ as in figure 1.

In all the results, the finite element method P_1 was used for the approximation of the two problems (3) and (6) with 125 points in the x direction and 250 points in the y direction.

3.1 Relevance of stopping criteria

In this section, we will focus on the evolution of the number of iterations as a function of the value ε_1 in the stopping criterion (4) as well as the completion error.

Example 1.
We start with the first example where we consider a two-dimensional Cauchy problem where we take diffusion coefficient $\mu = 1$ *and the convection coefficient* $\beta = (1, 2)^T$*. The source term* f *is taken to be*

$$f(x, y) = 3cos(3x) - 4sin(2y) + 4cos(2y) + 9sin(3x)$$

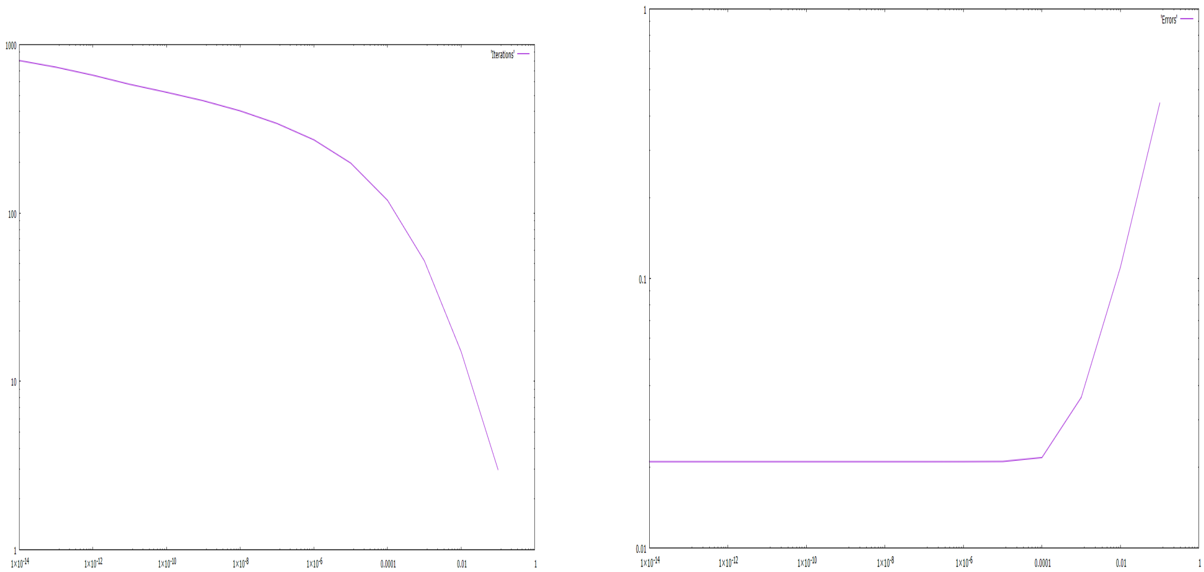
and the boundary conditions are as follows: on Γ_m *we take*

$$T_2(y) = cos(2y) + sin(3/2) \text{ and } \phi = 3cos(3/2)$$

on Σ_1 *we take* $T_1 = cos(2) + sin(3x)$ *and on* Σ_3 *we take* $g = -2sin(2)$*. The exact solution associated with these data is*

$$u_{ex} = cos(2y) + sin(3x)$$

The results for this first example are shown in figure 2



(a) iterations as a function of ε_1 **(b)** Error as a function of ε_1

Figure 2: Tests For Example 1 with ε_1

Figure (2a) shows us the number of iterations required to reac h the stopping criterion (4). We can see that the curve is in two parts. The very first is rapidly decreasing, up to $\varepsilon_1 = 10^{-4}$. From this value, the second part becomes linear.

We have represented in Figure (2b) the error of u on the part of boundary to be completed, as a function of ε_1. The curve, plotted on log-log scales, is again divided into two parts with equivalent behavior. The decrease in error is piecewise linear, and the threshold of the transition is the same as that determined in figure (2a). There therefore appear to be two convergence regimes.

Example 2.
In this example the source term and the boundary conditions are calculated from the exact solution

$$u_{ex} = exp(x^2 + y^2).$$

We store other data as in Example 1.

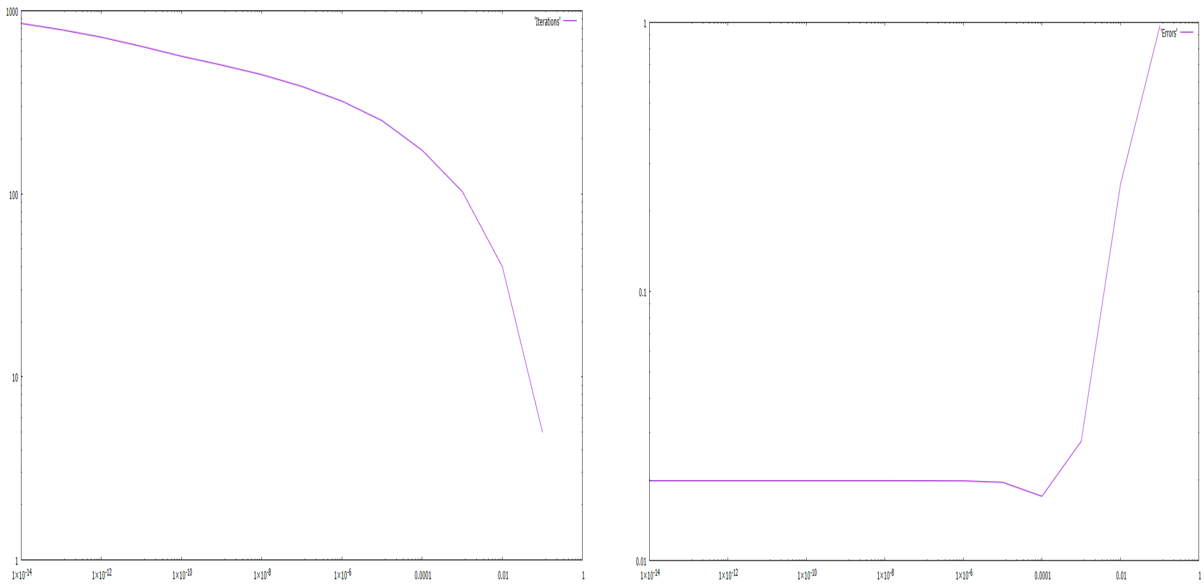
The results for this second example are shown in figure 3.

The results presented in figure 3a for the iterations and in figure 3b for the error confirm the observations already made from figures 2a and 2b.

Here again, two regions can clearly be distinguished in these curves. A first of fast decrease and the second of linear decrease.

We present in the next section results for Algorithm 1 where the stopping condition (4) is replaced by by the one in (7).

We'll look at the number of iterations necessary to satisfy the stopping criterion $1 - \varepsilon_2$ and see what is the associated error rate.



(a) iterations as a function of ε_1

(b) Error as a function of ε_1

Figure 3: Tests For Example 2 with ε_1

3.1.1 Stopping criterion associated with evolution

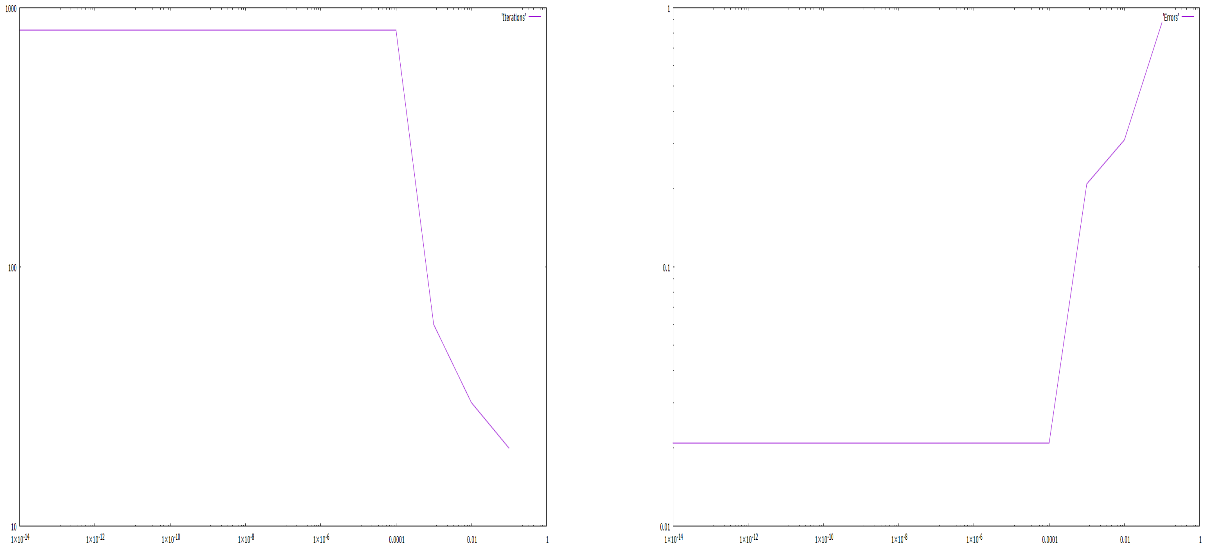
Consider the following stopping criterion

$$\frac{\|u^{(2k+2)} - u^{(2k+3))}\|\|_{\Gamma_u}}{\|u^{(2k)} - u^{(2k+1)}\|\|_{\Gamma_u}} < 1 - \varepsilon_2, \tag{7}$$

Recall that this stopping criterion measures the evolution of the solution during iterations.

If it is large, it means that the solution changes a lot at each iteration, whereas if it is small, it means that it changes very little, and it is therefore not necessary to continue the process. calculation. Indeed, the gain in precision compared to the number of iterations required is not necessarily large enough to justify the extension of the calculation. Indeed, the gain in precision compared to the number of iterations required is not necessarily large enough to justify the extension of the calculation.

Figure (4a) represents the number of iterations necessary to satisfy $1 - \varepsilon_2$ for Example 1 This curve has two parts. The first, for large ε_2, above of 10^{-3}, shows that few iterations are



(a) iterations as a function of ε_1

(b) Error as a function of ε_1

Figure 4: Tests For Example 1 with ε_2

needed to satisfy the criterion. We are in the presence, from this value, of a real stall in the curve, since when $\varepsilon_2 = 10^{-4}$, the number of iterations climbs to 820, whereas it only needed 40 for $\varepsilon_2 = 10^{-3}$.

This implies that the speed of convergence remains almost constant. The error decreases at the beginning, then remains almost constant for the rest of the iterations. The evolution of the error as a function of ε_2 is given by Figure (4b).

Here again, the curve shows a stall, which shows that crossing the threshold previously established in Figure (4a) considerably improves the precision. The error rate which is 20% for $\varepsilon_2 = 10^{-3}$, becomes 10^{-2} for $\varepsilon_2 = 10^{-4}$. As soon as this threshold is crossed, the precision increases, but this is done at the cost of a large number of iterations to be performed. We therefore manage here to distinguish the two parts of the convergence curve of ε_1. The best setting to make for this stop criterion is to reach the threshold, but not to exceed it, since this will increase the calculation time in a reckless way.

The results for Example 2 (which we do not present here because they are similar to those of Example 1) confirm the previous conclusions.

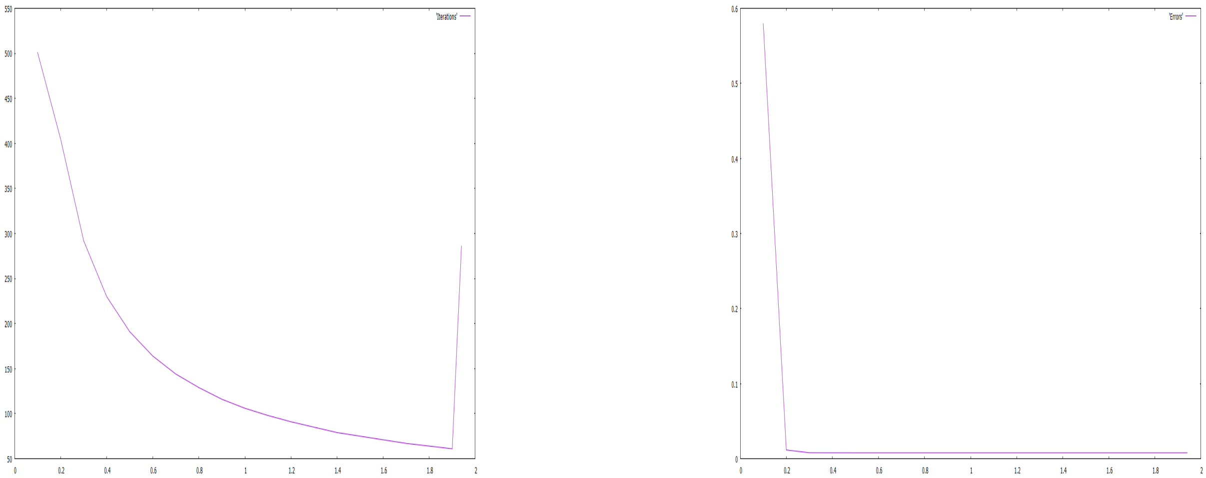
3.2 Accelerating convergence

In practice, we have seen that the KMF algorithm (with $\theta = 1$) converges very slowly, and it was necessary to define stopping criteria to calculate the results in a reasonable time. This is why we opted for Algorithm 1 in order to accelerate the convergence

The relaxation used in our calculation code takes place at each half iteration, that is, between the iteration $2k$ and $2k+1$, when calculating the boundary conditions for passing from one half iteration to the next. We introduced a coefficient θ which is used in the relations (5) during the computation of the boundary conditions on the boubdary to be completed

We observe Figure (5a) that for $\theta \in]0, 1.9[$, the iteration number is very small compared to the iteration number for $\theta = 1$. The error is approximately the same for all values in this interval.

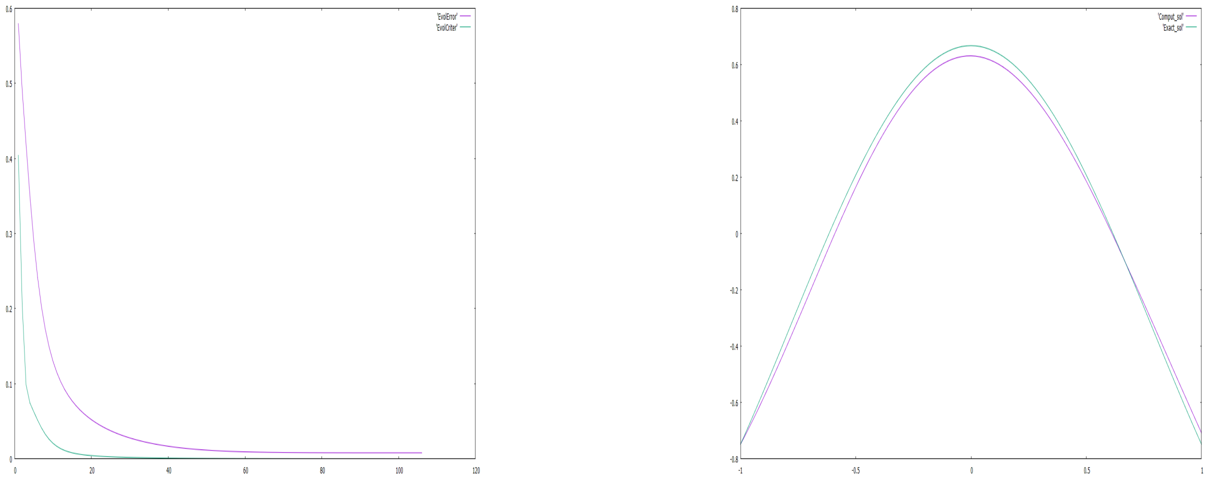
Let $E_C = \|u^{(2k-1)} - u^{(2(k-1))}\|_{L^2(\Gamma_u)}$ and $E_r = \|u_{ex} - u^{(2(k))}\|_{L^2(\Gamma_u)}$



(a) iterations as a function of θ

(b) Error as a function of θ

Figure 5: Example 1, results for various θ values.



(a) Evolution of E_C and E_r as functions of iterations

(b) Exact and approximate solutions on the boundary Γ_u

Figure 6: Example 1, results on Γ_u for $\theta = 1.9$

We observe from figure (6a) that E_C and E_r evolve in the same way. It decreases together and becomes stationary at the same time. That is to say that when the criterion is reached, as it is clear from figure (6b) that the algorithm produces a very accurate solution, the approximate solution being very close to the exact solution.

4 Conclusion

In this paper we discussed the convergence and computational efficiency of the Nachaoui's iterative alternating method. The method is based on alternating the solutions of two different problems governed by the convection-diffusion equation. At each iteration, the method updates the boundary condition on Γ_u and solves the two problems sequentially until convergence.

The convergence of the method can be proved under some assumptions on the coefficients and the domain Ω. In particular, it is assumed that $\mu > 0$ is a constant, β is bounded and Lipschitz continuous, and Ω is a bounded and connected domain with Lipschitz boundary. Under these assumptions, Nachaoui has shown that the method converges to the unique solution of the

Cauchy problem (2) Nachaoui et al. (2021a).

The convergence rate of the method depends on the choice of the initial guess τ_0 and the tolerance ε_1. A good choice of τ_0 can lead to faster convergence, while a smaller value of ε_1 can increase the accuracy of the solution but may require more iterations. In practice, the values of τ_0 and ε_1 are chosen based on the properties of the problem and the desired accuracy.

The computational efficiency of the method is mainly determined by the cost of solving the two subproblems at each iteration.

The numerical results showed that the iterative alternating method is a robust and efficient algorithm for solving the Cauchy roblem governed by the convection-diffusion problems. Its alternating structure allows for easy implementation and parallelization, making it suitable for large-scale problems.

The computational efficiency of the method is mainly determined by the cost of solving the two subproblems at each iteration. Thus the choice of the relaxation parameter can be done in a dynamic way in order to reduce the number of direct problems to solve (as in Berdawood et al. (2023)). This will make the method even more attractive. A theoretical study of convergence as well as the search for the acceleration interval are the subject of future research. which will make the method even more attractive. A theoretical study of convergence as well as the search for the acceleration interval are the subject of future research.

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