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SOLUTION OF FUZZY VOLTERRA INTEGRAL EQUATIONS USING COMPOSITE METHOD

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Abstract. With the rapid development of of fuzzy Integral equations, the problem of finding its solution becomes increasingly important for knowledge propagation and putting researchers wisdom to work. A recent development trend of fuzzy integral equations is modeling many problems in the field of science and technology. If the unknown function in the considered equation has a solution in terms of infinite series expansion, this proposed method becomes more accurate to find the exact solution. According on the parametric form of a fuzzy number, a fuzzy integral equation converts to two systems of integral equations in the crisp case, and then proposed method is achieved to obtain the exact fuzzy solutions of fuzzy Volterra integral equations. To validate the effectiveness of proposed method, some examples of considered equations are solved and the results show that it achieved the best performance.

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1 Introduction

Recent years have witnessed a growing developing the subjects that's related to fuzzy control system, it was reflected on the increasing and fast of an interest to study of fuzzy integral equations. Integral equations are important mathematical models for many problem in different fields, such as physics, chemistry, biology, engineering and etc., as especially, fuzzy integral equations.

In fact, evolution trend of the fuzzy integral equations encouraged researchers to investigate for finding accurate and efficient solving methods for it.Therefore in the literature, numerous studies have been proposed, e.g., apply of the successive approximations method Bica & Popescu (2014) and Bica & Popescu (2013), the adomian decomposition method is applied in Rouhparvar et al. (2009), Abbasbandy & Allahviranloo (2006) and Babolian et al. (2005), homotopy perturbation method is used by Narayanamoorthy & Sathiyapriya (2016), hybrid method of Laplace transform coupled with adomian decomposition method is proposed in Ullah et al. (2020), application of fuzzy differential transform method Salahshour & Allahviranloo (2013), the approximate solution is found by homotopy analysis method (Molabahrami et al., 2011), the

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fuzzy bernstein polynomials is prrsented in (Ezzati & Ziari, 2012), Laplace transform coupled with adomian decomposition method is proposed in Ullah et al. (2021), spectral method is discussed in Hooshangian, (2020).

In this paper, we introduce a composite method from a new integral transform has been propused by Jafari (2021) and adomian decomposition method (JADM) to get the analytical solution for the considered equation. Finally, to test the performance effectiveness of our method, some examples have been given and solved.

The remainder of this paper is organized as follows. Section 2 includes some elementary necessary preliminaries of the fuzzy calculus. Section 3 and Section 4 present the fuzzy Volterra integral equation and the fuzzy integral transform method respectively. Practical implementation of composite method is introduced in Section 5. In Section 6, we give three illustrative examples to implement the proposed method. Finally, Section 7 is concludes the work.

2 Preliminaries

In this section some basic definitions are presented that used through the paper.

Definition 1. *(Goetschel & Voxman, 1986) A fuzzy number is a set $y : R \longrightarrow I = [0,1]$ satisfying the following:*
i. y is upper semi-continuous.
ii. Fuzzy convex.
iii. Normal and closure $(supp\ y)$ is compact, where $supp\ y = \{t \in R : y(t) > 0\}$ represent the support of y.
Let R_F be the set of all fuzzy numbers on R. The r level set of y, is denoted by

$$[y]^r = \begin{cases} \{t \in R : y(t) \geq r\} & if\, 0 < r \leq 1, \\ cl(supp\ y) & if\, r = 0 \end{cases}$$

where $y(r) = [\underline{y}(r), \overline{y}(r)]$ is a closed bounded interval and $\underline{y}(r)$ is the left hand endpoint and $\overline{y}(r)$ is the right hand endpoint of $[y]^r$ receptively.

Definition 2. *(Friedman et al., 1996) A fuzzy number y is a pair $[\underline{y}(r), \overline{y}(r)]$, $0 \leq r \leq 1$ functions, which satisfying the following properties:*
i. $\underline{y}(r)$ is a bounded monotonic non-decreasing left continuous function,
ii. $\overline{y}(r)$ is a bounded monotonic non-increasing left continuous function,
iii. $\underline{y}(r) \leq s\overline{y}(r)$, $0 \leq r \leq s$
For arbitrary different fuzzy numbers $y(r) = [\underline{y}(r), \overline{y}(r)]$, $z(r) = [\underline{z}(r), \overline{z}(r)]$ and $k > 0$, we define various operations as follow,

Addition: $\underline{(y+z)}(r) = \underline{y}(r) + \underline{z}(r), \quad \overline{(y+z)}(r) = \overline{y}(r) + \overline{z}(r)$

Subtraction: $\underline{(y-z)}(r) = \underline{y}(r) - \underline{z}(r), \quad \overline{(y-z)}(r) = \overline{y}(r) - \overline{z}(r)$

Scaler multiplication:

$$\underline{(ky)}(r) = k\underline{y}(r), \overline{(ky)}(r) = k\overline{y}(r) \quad if\ k \geq 0$$

$$\overline{(ky)}(r) = k\overline{y}(r), \underline{(ky)}(r) = k\underline{y}(r) \quad if\ k < 0$$

Definition 3. *(Puri et al., 1993).The Hausdorff distance for arbitrary fuzzy numbers (y, z) given by $D : E \times E \longrightarrow R_+ \cup \{0\}$, and defined as:*

$$D(y, z) = \sup_{r \in [0,1]} max\{|\underline{y}(r) - \underline{z}(r)|, |\overline{y}(r) - \overline{z}(r)|\}$$

D is a metric in E and has the following properties:
i. $D(y+w, z+w) = D(y, z)$ for all $y, z, w \in E$;
ii. $D(k.y, k.z) = |k|D(y, z)$ for all $k \in R$ and $y, z \in E$;
iii. $D(y+w, z+v) \leq D(y, z) + D(w, v)$ for all $y, z, w, v \in E$;
iv. (D, E) is a complete metric space.

Definition 4. *(Friedman et al., 1999) A fuzzy function $f : [a, b] \longrightarrow R$ is denote to be continuous if for any fixed $t_0 \in R$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t - t_0| < \delta$ then $D(f(t) - f(t_0)) < \varepsilon$.*

Definition 5. *(Allahviranloo & Barkhordari, 2010) Let $y, z \in E$, if there exists $w \in E$ such that $y = z + w$, then w is said the H-difference of y and z, and denoted as $y \ominus z$.*

Definition 6. *(Park et al., 1999) Consider $y_r(t) = [y(t)]^r$ is the set valued mapping and continuous at $t = a$ with respect to the Hausdorff metric D for all $r \in [0, 1]$ then the level wise continuous mapping $y : [f_1, f_2] \subset R \to E$ is defined at $a \in [f_1, f_2]$.*

3 Fuzzy Volterra Integral Equation (FVIE)

Consider the following volterra integral equations with separable kernels

$$y(t) = f(t) + \lambda \int_a^t k(t, s) y(s) ds, \quad a \leq s, t \leq b \tag{1}$$

where $\lambda_1 \in R$, f and k are known continuous functions defined on $[a, b]$ and $[a, b] \times R$, respectively, and $y(t)$ is the unknown function. In this section, we note that we investigate analytical solution of fuzzy volterra integral equations with separable kernels. By the fuzzy concept, the given Eq.(1) becomes with the fuzzy form as follow

$$y(t, r) = f(t, r) + \lambda \int_a^t k(t, s) y(s, r) ds \tag{2}$$

Now, with respect to definition 2 we introduce the parametric form of FVIE. Let $(\underline{f}(t, r), \overline{f}(t, r))$ and $(\underline{y}(t, r), \overline{y}(t, r))$, $0 \leq r \leq 1$ and $a \leq t \leq b$, be parametric forms of $f(t)$ and $y(t)$), respectively; then the parametric form of FVIE is as follow

$$\begin{cases} \underline{y}(t, r) = \underline{f}(t, r) + \lambda \int_a^t k(t, s) \underline{y}(s, r) ds \\ \overline{y}(t, r) = \overline{f}(t, r) + \lambda \int_a^t k(t, s) \overline{y}(s, r) ds \end{cases} \tag{3}$$

we can see that Eq.(3) is a system of linear volterra integral equations in the crisp case for each $0 \leq r \leq 1$ and $t \in [a, b]$. Sufficient conditions for the existence of a unique solution to Eq. (2) are provided by the following theorem.

Theorem 1. *(Park et al., 1999) Let*
i. $y(t)$ is a level wise continuous function on $[a, a + t_0], t_0 > 0$.
ii. $k(t, s)$ is a level wise continuous function on $\triangle : a \leq s \leq t \leq a + t_0]$ and $D(z(t), y(t_0)) < t_1$, where $t_1 > 0$.
iii. For any $(t, s, y(t)), (t, s, z(t)) \in \triangle$, we have

$$D([k(t, s, y(t))]^r, [k(t, s, z(t))]^r) \leq MD([y(t)]^r, [z(t)]^r),$$

where $M > 0$ is the constant and given for any $r \in [0, 1]$. Then the level wise continuous solution $y(t)$ exist and unique for Eq.(2) and defined for $t \in (a, a + \alpha)$ where $\alpha = \{t_0, \frac{t_1}{N}\}$ and $N = D(h(t, s, y(t)), h(t, s, z(t))) \in \triangle$.

4 Fuzzy Integral Transform Method

In this section, we exhibit an integral transform which has been introduce in (Jafari, 2021) and can be expressed as

$$J\{f(t)\} = F(\rho) = p(\rho)\int_0^{\infty} f(t)e^{-q(\rho)t}dt \tag{4}$$

where $f(t)$ be a integrable function defined for $t \geq 0$, $p(\rho) \neq 0$ and $q(\rho)$ are positive real functions, and for some $q(\rho)$ provides the integral exists.
Now we suppose that f is a fuzzy valued function and ρ is a real parameter. So, we can consider Eq.(4) as definition of fuzzy integral transform of f. the notation $J(f)$ denote the fuzzy integral transform of fuzzy valued function $f(t)$, and the integral is the fuzzy Riemann improper integral.Then the lower and upper fuzzy integral transform, based on the lower and upper of fuzzy valued function f as following

$$F(\rho, r) = J\{f(t, r)\} = [J\{\underline{f}(t, r)\} \quad J\{\overline{f}(t, r)\}]$$

where

$$\begin{cases} J\{\underline{f}(t,r)\} = p(\rho)\int_0^{\infty} \underline{f}(t,r)e^{-q(\rho)t}dt, & 0 \leq \rho \leq 1 \\ J\{\overline{f}(t,r)\} = p(\rho)\int_0^{\infty} \overline{f}(t,r)e^{-q(\rho)t}dt, & 0 \leq \rho \leq 1 \end{cases}$$

Additionally, the properties of the integral transform were shown in (Jafari, 2021).

Theorem 2. *(Fuzzy Convolution Theorem) Let f_1 and f_2 are fuzzy valued function of exponential order p, which are piecewise continuous on $[0, \alpha]$, then*

$$J\{(f_1 * f_2)(t)\} = J\{f_1(t)\} \bullet J\{f_2(t)\} = \frac{1}{p(\rho)}F_1(\rho) \bullet F_2(\rho) \tag{5}$$

Proof. Let start with the product

$$J\{f1(t)\} \bullet J\{f_2(t)\} = \tfrac{1}{p(\rho)}F_1(\rho) \bullet F_2(\rho)$$

$$= \tfrac{1}{p(\rho)}\{p(\rho)\int_0^{\infty} e^{-q(\rho)\tau}f_1(\tau)d\tau\}\{p(\rho)\int_0^{\infty} e^{-q(\rho)u}f_2(u)du\}$$

$$= p(\rho)\int_0^{\infty}(\int_0^{\infty} e^{-q(\rho)(\tau+u)}f_1(\tau)f_2(u)du)d\tau$$

substituting $t = \tau + u$, and noting that τ is fixed in the interior integral, so that $du = dt$, we have

$$J\{f_1(t)\} \bullet J\{f_2(t)\} = p(\rho)\int_0^{\infty}(\int_{\tau}^{\infty} e^{-q(\rho)t}f_1(\tau)f_2(t-\tau)dt)d\tau \tag{6}$$

If we suppose $f_2(t) = \tilde{0}$ for $t < 0$, then $f_2(t-\tau) = \tilde{0}$ for $t < \tau$ then Eq.(6) becomes

$$J\{f_1(t)\} \bullet J\{f_2(t)\} = p(\rho)\int_0^{\infty}\int_0^{\infty} e^{-q(\rho)t}f_1(\tau)f_2(t-\tau)dtd\tau$$

considering the hypotheses on f_1, f_2, the fuzzy integral transform of f_1, f_2 is an absolutely converge and thus

$$\int_0^{\infty}\int_0^{\infty} |e^{-q(\rho)t}f_1(\tau)f_2(t-\tau)|dtd\tau$$

converges. In this case, the order of integration can be reversed, so that

$$\begin{aligned} J\{f_1(t)\} \bullet J\{f_2(t)\} &= p(\rho)\int_0^{\infty}\int_0^{\infty} e^{-q(\rho)t}f_1(\tau)f_2(t-\tau)d\tau dt \\ &= p(\rho)\int_0^{\infty}(\int_0^{t} e^{-q(\rho)t}f_1(\tau)f_2(t-\tau)d\tau)dt \\ &= p(\rho)\int_0^{\infty} e^{-q(\rho)t}(\int_0^{t} f_1(\tau)f_2(t-\tau)d\tau)dt \\ &= J\{(f_1 * f_2)(t)\} \end{aligned}$$

□

5 Practical Implementation of Composite Method

In previous sections, we have presented the type of equation and method of solution as a framework for our work. In this section, we will focus on the implementation issue of the proposed framework which is solving the fuzzy Volterra linear integral equations with convolution type kernel by using JADM. We assume the kernel function $k(t,s)$ is a separable kernel and has the form$k(t-s)$ i.e. $k(t,s)=\sum_{i=0}^{n} h_i(t)g_i(s)$ and satisfies the conditions of theorem 1 so that the solution of Eq.(2) exists and unique. Suppose that for any $a \le t, s \le b$ in $k(t,s)$. considered $k(t,s)=t-s$ and $\lambda=1$, Eq.(3) becomes

$$\begin{cases} \underline{y}(t,r) = \underline{f}(t,r) + \int_a^t (t-s)\underline{y}(s,r)ds \\ \overline{y}(t,r) = \overline{f}(t,r) + \int_a^t (t-s)\overline{y}(s,r)ds \end{cases} \tag{7}$$

Now we apply fuzzy integral transform to both side of Eq.(7) and using Theorem 4.1 we get

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{\underline{f}(t,r)\} + J\{\int_a^t (t-s)\underline{y}(s,r)ds\}, \\ J\{\overline{y}(t,r)\} = J\{\overline{f}(t,r)\} + J\{\int_a^t (t-s)\overline{y}(s,r)ds\}, \end{cases}$$

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{\underline{f}(t,r)\} + J\{t\}J\{\underline{y}(s,r)ds\}, \\ J\{\overline{y}(t,r)\} = J\{\overline{f}(t,r)\} + J\{t\}J\{\overline{y}(s,r)ds\}, \end{cases}$$

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{\underline{f}(t,r)\} + \frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)ds\} \\ J\{\overline{y}(t,r)\} = J\{\overline{f}(t,r)\} + \frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)ds\} \end{cases} \tag{8}$$

We then take the inverse fuzzy integral transform J^{-1} to both sides of Eq.(8), to find

$$\begin{cases} \underline{y}(t,r) = \underline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)ds\}\} \\ \overline{y}(t,r) = \overline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)ds\}\} \end{cases} \tag{9}$$

Using the Adomian decomposition method, the unknown function $y(t,r)$ of Eq.(2) decompos into a sum of an infinite number of components defined by the decomposition

$$\begin{cases} \underline{y}(t,r) = \sum_{i=0}^{\infty} \underline{y_i}(t,r) \\ \overline{y}(t,r) = \sum_{i=0}^{\infty} \overline{y_i}(t,r) \end{cases} \tag{10}$$

Substituting Eq.(10) into Eq.(9) gives the

$$\begin{cases} \sum_{i=0}^{\infty} \underline{y_i}(t,r) = \underline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \underline{y_i}(t,r)\}\} \\ \sum_{i=0}^{\infty} \overline{y_i}(t,r) = \overline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \overline{y_i}(t,r)\}\} \end{cases} \tag{11}$$

By the ADM, we can be set the zeroth component by all terms outside the integral sign and the nth components, $n \ge 1$ are given by the recurrence relation, hence we have

$$\begin{cases} \underline{y_0}(t,r) = \underline{f}(t,r) \\ \overline{y_0}(t,r) = \overline{f}(t,r) \end{cases}$$

$$\begin{cases} \underline{y_1}(t,r) = \underline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \underline{y_0}(t,r)\}\} \\ \overline{y_1}(t,r) = \overline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \overline{y_0}(t,r)\}\} \end{cases}$$

$$\begin{cases} \underline{y_2}(t,r) = \underline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\underline{y_1}(t,r)\}\} \\ \overline{y_2}(t,r) = \overline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\overline{y_1}(t,r)\}\} \end{cases}$$

and so on for other components.

$$\begin{cases} \underline{y_{n+1}}(t,r) = \underline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\underline{y_n}(t,r)\}\} \\ \overline{y_{n+1}}(t,r) = \overline{f}(t,r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\overline{y_n}(t,r)\}\} \end{cases} \tag{12}$$

Clearly seen that the decomposition method made the fuzzy integral equation into an elegant determination of computable components, then the obtained series converges very rapidly to the exact solution of Eq.(2).

6 Illustrative Examples

In this section, we will evaluate the performance of the proposed method via obtain solutions of fuzzy Volterra integral equations and show its utility. To this end, the following examples will be are provided.

Example 1. *Consider the following fuzzy linear Volterra integral equation (Ameri & Nezhad, 2017):*

$$y(t,r) = f(t,r) + \int_a^t (t-s)y(s,r)ds \tag{13}$$

where $f(t,r) = (3+r, 8-2r)$, $0 \le t \le 1$ *and the exact solution of given fuzzy integral solution is* $y(t,r) = [3+r, 8-2r]cosh(t)$. *Applying JADM to solve this fuzzy integral equation.The parametric form of Eq.(13) is as follow*

$$\begin{cases} \underline{y}(t,r) = (3+r) + \int_a^t (t-s)\underline{y}(s,r)ds \\ \overline{y}(t,r) = (8-2r) + \int_a^t (t-s)\overline{y}(s,r)ds \end{cases} \tag{14}$$

Applying the fuzzy integral transform to both side of Eq.(14) and using Theorem 4.1 *we have*

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{(3+r)\} + J\{\int_a^t (t-s)\underline{y}(s,r)ds\} \\ J\{\overline{y}(t,r)\} = J\{(8-2r)\} + J\{\int_a^t (t-s)\overline{y}(s,r)ds\} \end{cases}$$

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{(3+r)\} + J\{t\}J\{\underline{y}(s,r)\} \\ J\{\overline{y}(t,r)\} = J\{(8-2r)\} + J\{t\}J\{\overline{y}(s,r)\} \end{cases}$$

$$\begin{cases} \underline{y}(t,r) = (3+r) + \frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)\} \\ \overline{y}(t,r) = (8-2r) + \frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)\} \end{cases}$$

we then apply the inverse fuzzy integral transform J^{-1} *and simplify, to find*

$$\begin{cases} \underline{y}(t,r) = (3+r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)ds\}\} \\ \overline{y}(t,r) = (8-2r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)ds\}\} \end{cases} \tag{15}$$

Substituting the decomposition series Eq.(10) into both sides of Eq.(15) gives

$$\begin{cases} \sum_{i=0}^{\infty}\underline{y_i}(t,r) = (3+r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\underline{y_i}(t,r)\}\} \\ \sum_{i=0}^{\infty}\overline{y_i}(t,r) = (8-2r) + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\overline{y_i}(t,r)\}\} \end{cases}$$

or equivalently

$$\begin{cases} \underline{y_0}(t,r)+\underline{y_1}(t,r)+\underline{y_2}(t,r)+...=(3+r)+ \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\}+J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\}+... \\ \overline{y_0}(t,r)+\overline{y_1}(t,r)+\overline{y_2}(t,r)+...=(8-2r)+ \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\}+J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\}+... \end{cases} \tag{16}$$

This allows to set the following recurrence relation:

$$\begin{cases} \underline{y_0}(t,r)=(3+r) \\ \overline{y_0}(t,r)=(8-2r) \end{cases}$$

$$\begin{cases} \underline{y_1}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} \\ \overline{y_1}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} \end{cases}$$

$$\begin{cases} \underline{y_2}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\} \\ \overline{y_2}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\} \end{cases}$$

The general terms are given by

$$\begin{cases} \underline{y_{n+1}}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_n}(t,r)\}\} \\ \overline{y_{n+1}}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_n}(t,r)\}\} \end{cases}$$

where $n \leq 0$. Taking the lower limit solution of Eq.(14), and simplify

$$\begin{cases} \underline{y_0}(t,r)=(3+r) \\ \underline{y_1}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\}=(3+r)\frac{t^2}{2!} \\ \underline{y_2}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\}=(3+r)\frac{t^4}{4!} \\ \underline{y_3}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\}=(3+r)\frac{t^6}{6!} \\ \underline{y_4}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\}=(3+r)\frac{t^8}{8!} \end{cases} \tag{17}$$

and so on for other components may be in the same way computed. The upper limit solution of Eq.(14) is being found as

$$\begin{cases} \overline{y_0}(t,r)=(8-2r) \\ \overline{y_1}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\}=(8-2r)\frac{t^2}{2!} \\ \overline{y_2}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\}=(8-2r)\frac{t^4}{4!} \\ \overline{y_3}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\}=(8-2r)\frac{t^6}{6!} \\ \overline{y_4}(t,r)=J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\}=(8-2r)\frac{t^8}{8!} \end{cases} \tag{18}$$

In view of Eq.(17) and Eq.(18), the components were completely determined. Therefore, the solution of Eq.(14) in a series form is readily obtained by using the series assumption in Eq.(10), and it converges to the closed form solution

$$\begin{cases} \underline{y}(t,r)=(3+r)cosh(t) \\ \overline{y}(t,r)=(8-2r)cosh(t) \end{cases}$$

which is the exact solution and can be rewritten as $y(t,r)=(3+r,8-2r)cosh(t)$.

Example 2. *Consider the following fuzzy linear Volterra integral equation (Ameri & Nezhad, 2017):*

$$y(t,r) = f(t,r) + \int_a^t (t-s)y(s,r)ds \tag{19}$$

where $f(t,r) = ([r, 2-r](1 - t - \frac{t^2}{2}))$, $0 \le t \le 1$ *and the exact solution of given fuzzy integral solution is* $([r, 2-r](1 - sinh(t)))$. *Applying JADM to solve this fuzzy integral equation. The parametric form of Eq.(19) is as follow*

$$\begin{cases} \underline{y}(t,r) = r(1 - t - \frac{t^2}{2}) + \int_a^t (t-s)\underline{y}(s,r)ds \\ \overline{y}(t,r) = (2-r)(1 - t - \frac{t^2}{2}) + \int_a^t (t-s)\overline{y}(s,r)ds \end{cases} \tag{20}$$

Applying the fuzzy integral transform to both side of Eq.(20)

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{r(1 - t - \frac{t^2}{2})\} + J\{\int_a^t (t-s)\underline{y}(s,r)ds\} \\ J\{\overline{y}(t,r)\} = J\{(2-r)(1 - t - \frac{t^2}{2})\} + J\{\int_a^t (t-s)\overline{y}(s,r)ds\} \end{cases}$$

Using Theorem 4.1, *we have*

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{r(1 - t - \frac{t^2}{2})\} + J\{t\}J\{\underline{y}(s,r)\} \\ J\{\overline{y}(t,r)\} = J\{(2-r)(1 - t - \frac{t^2}{2}))\} + J\{t\}J\{\overline{y}(s,r)\} \end{cases}$$

$$\begin{cases} \underline{y}(t,r) = r(1 - t - \frac{t^2}{2}) + \frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)\} \\ \overline{y}(t,r) = (2-r)(1 - t - \frac{t^2}{2}) + \frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)\} \end{cases} \tag{21}$$

We then apply the inverse fuzzy integral transform J^{-1} *and substitute the decomposition series Eq.(10) into both sides of Eq.(21) gives*

$$\begin{cases} \sum_{i=0}^{\infty} \underline{y_i}(t,r) = & r(1 - t - \frac{t^2}{2}) + \\ & J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \underline{y_i}(t,r)\}\} \\ \sum_{i=0}^{\infty} \overline{y_i}(t,r) = & (2-r)(1 - t - \frac{t^2}{2}) + \\ & J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty} \overline{y_i}(t,r)\}\} \end{cases}$$

or equivalently

$$\begin{cases} \underline{y_0}(t,r) + \underline{y_1}(t,r) + \underline{y_2}(t,r) + ... = r(1 - t - \frac{t^2}{2}) + \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\} + ... \\ \overline{y_0}(t,r) + \overline{y_1}(t,r) + \overline{y_2}(t,r) + ... = (2-r)(1 - t - \frac{t^2}{2}) + \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\} + ... \end{cases} \tag{22}$$

This allows to set the following recurrence relation:

$$\begin{cases} \underline{y_0}(t,r) = r(1 - t - \frac{t^2}{2}) \\ \overline{y_0}(t,r) = (2-r)(1 - t - \frac{t^2}{2}) \end{cases}$$

$$\begin{cases} \underline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} \\ \overline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} \end{cases}$$

$$\begin{cases} \underline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\} \\ \overline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\} \end{cases}$$

The general terms are given by

$$\begin{cases} \underline{y_{n+1}}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_n}(t,r)\}\} \\ \overline{y_{n+1}}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_n}(t,r)\}\} \end{cases}$$

where $n \leq 0$*. Taking the lower limit solution of Eq.(20), and simplify*

$$\begin{cases} \underline{y_0}(t,r) = r(1 - t - \frac{t^2}{2!}) \\ \underline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = r(\frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!}) \\ \underline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = r(\frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^6}{6!}) \\ \underline{y_3}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = r(\frac{t^6}{6!} - \frac{t^7}{7!} - \frac{t^8}{8!}) \end{cases} \tag{23}$$

and so on for other components may be in the same way computed. The upper limit solution of Eq.(20) is being found as

$$\begin{cases} \overline{y_0}(t,r) = (2-r) \\ \overline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (2-r)(\frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!}) \\ \overline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (2-r)(\frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^6}{6!}) \\ \overline{y_3}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (2-r)(\frac{t^6}{6!} - \frac{t^7}{7!} - \frac{t^8}{8!}) \end{cases} \tag{24}$$

In view of Eq.(23) and Eq.(24), the components were completely determined. Therefore, the solution of Eq.(20) in a series form is readily obtained by using the series assumption in Eq.(10), and it converges to the closed form solution

$$\begin{cases} \underline{y}(t,r) = r(1 - sinh(t)) \\ \overline{y}(t,r) = (2-r)(1 - sinh(t)) \end{cases}$$

which is the exact solution and can be rewritten as $y(t,r) = (r, 2-r)(1 - sinh(t))$.

Example 3. *Consider the following fuzzy linear Volterra integral equation (Salahshour & Allahviranloo, 2013):*

$$y(t,r) = f(t,r) + \int_a^t (t-s)y(s,r)ds \tag{25}$$

where $f(t,r) = [r-1, 1-r]t$, $0 \leq t \leq 1$ *and the exact solution of given fuzzy integral solution is* $y(t,r) = (r-1, 1-r)(sinh(t) + cosh(t) - 1)$*. Applying JADM to solve this fuzzy integral equation. The parametric form of Eq.(25) is as follow*

$$\begin{cases} \underline{y}(t,r) = (r-1)t + \int_a^t (t-s)\underline{y}(s,r)ds \\ \overline{y}(t,r) = (1-r)t + \int_a^t (t-s)\overline{y}(s,r)ds \end{cases} \tag{26}$$

Applying the fuzzy integral transform to both side of Eq.(26) and using Theorem 4.1 *we have*

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{(r-1)t\} + J\{\int_a^t (t-s)\underline{y}(s,r)ds\} \\ J\{\overline{y}(t,r)\} = J\{(1-r)t\} + J\{\int_a^t (t-s)\overline{y}(s,r)ds\} \end{cases}$$

$$\begin{cases} J\{\underline{y}(t,r)\} = J\{(r-1)t\} + J\{t\}J\{\underline{y}(s,r)\} \\ J\{\overline{y}(t,r)\} = J\{(1-r)t)\} + J\{t\}J\{\overline{y}(s,r)\} \end{cases}$$

$$\begin{cases} \underline{y}(t,r) = (r-1)t + \frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)\} \\ \overline{y}(t,r) = (1-r)t + \frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)\} \end{cases}$$

we then apply the inverse fuzzy integral transform J^{-1} and simplify, to find

$$\begin{cases} \underline{y}(t,r) = (r-1)t + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y}(s,r)ds\}\} \\ \overline{y}(t,r) = (1-r)t + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y}(s,r)ds\}\} \end{cases} \tag{27}$$

Substituting the decomposition series Eq.(10) into both sides of Eq.(27) gives

$$\begin{cases} \sum_{i=0}^{\infty}\underline{y_i}(t,r) = (r-1)t + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\underline{y_i}(t,r)\}\} \\ \sum_{i=0}^{\infty}\overline{y_i}(t,r) = (1-r)t + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\sum_{i=0}^{\infty}\overline{y_i}(t,r)\}\} \end{cases}$$

or equivalently

$$\begin{cases} \underline{y_0}(t,r) + \underline{y_1}(t,r) + \underline{y_2}(t,r) + ... = (r-1)t + \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\} + ... \\ \overline{y_0}(t,r) + \overline{y_1}(t,r) + \overline{y_2}(t,r) + ... = (1-r)t + \\ J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} + J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\} + ... \end{cases} \tag{28}$$

This allows to set the following recurrence relation:

$$\begin{cases} \underline{y_0}(t,r) = (r-1)t \\ \overline{y_0}(t,r) = (1-r)t \end{cases}$$

$$\begin{cases} \underline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} \\ \overline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} \end{cases}$$

$$\begin{cases} \underline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_1}(t,r)\}\} \\ \overline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_1}(t,r)\}\} \end{cases}$$

The general terms are given by

$$\begin{cases} \underline{y_{n+1}}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_n}(t,r)\}\} \\ \overline{y_{n+1}}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_n}(t,r)\}\} \end{cases}$$

where $n \leq 0$. Taking the lower limit solution of Eq.(26), and simplify

$$\begin{cases} \underline{y_0}(t,r) = (r-1) \\ \underline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = (r-1)\frac{t^2}{2!} \\ \underline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = (r-1)\frac{t^4}{4!} \\ \underline{y_3}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = (r-1)\frac{t^6}{6!} \\ \underline{y_4}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\underline{y_0}(t,r)\}\} = (r-1)\frac{t^8}{8!} \end{cases} \tag{29}$$

and so on for other components may be in the same way computed. The upper limit solution of Eq.(26) is being found as

$$\begin{cases} \overline{y_0}(t,r) = (1-r) \\ \overline{y_1}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (1-r)\frac{t^2}{2!} \\ \overline{y_2}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (1-r)\frac{t^4}{4!} \\ \overline{y_3}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (1-r)\frac{t^6}{6!} \\ \overline{y_4}(t,r) = J^{-1}\{\frac{p(\rho)}{q(\rho)^2}J\{\overline{y_0}(t,r)\}\} = (1-r)\frac{t^8}{8!} \end{cases} \tag{30}$$

In view of Eq.(29) and Eq.(30), the components were completely determined. Therefore, the solution of Eq.(26) in a series form is readily obtained by using the series assumption in Eq.(10), and it converges to the closed form solution

$$\begin{cases} \underline{y}(t,r) = (r-1(sinh(t)) - cosh(t) - 1) \\ \overline{y}(t,r) = (1-r)(sinh(t)) - cosh(t) - 1) \end{cases}$$

which is the exact solution and can be rewritten as $y(t,r) = (r-1, 1-r)(sinh(t)) - cosh(t) - 1)$.

7 Conclusion

In this paper, we proposed a composite method to solve “fuzzy Volterra integral equations” with separable type kernels analytically.This propose adopted ”JADM” and hereby developed two sequences of together upper and lower limit solutions as general approach. Illustrative three different examples were presented in order to test the proposed method. As we emphasized, the solution may be found in a more easy way, remarkably, instead of using an intricate method. Finally, we showed the effectiveness of JADM and the results clearly indicate that is a powerful tool for solving fuzzy integral equations.

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