Advanced Mathematical Models & Applications
Vol.9, No.2, 2024, pp.173-192
https://doi.org/10.62476/amma9173

A NOTE ON THE TELEGRAPH TYPE DIFFERENTIAL EQUATION WITH TIME INVOLUTION

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Abstract. In the present paper, the initial value problem for the telegraph type involutory in t second order linear partial differential equation with damping term is investigated. The equivalent initial value problem for the fourth order partial differential equations to the initial value problem for this second order linear partial differential equations with involution and damping term is obtained. Applying the operator tools, the stability estimates for the solution and its first and second order derivatives of this problem are established.

Keywords: Involutory type telegraph equation, stability, boundedness.
AMS Subject Classification: 65L20, 35L90, 35N30.
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Received: 7 November 2023; Revised: 3 February 2024; Accepted: 8 May 2024; Published: 2 August 2024.

1 Introduction

Functional differential equations, which provide mathematical models for real-world problems have been investigated by many scientists (Hernandez & Henriquez, 1998; Hale & Lunel, 1993; Hale, 1971; Kolmanovskii & Myshkis, 1992; Kolmanovskii & Nosov 1986).

Delay differential equations are universal phenomena on applied their models in engineering systems to behave like a real process (Vlasov & Rautian, 2006; Bhalekar & Patade, 2016; Srividhyaa & Gopinathan, 2006; Sriram & Gopinathan, 2004). Initial conditions in one point are not enough to get the solution of delay differential equations. In the first time Falbo (2013) in an experiment measuring the population growth of a species of water fleas, Nisbet in his study he tried to use delay differential model. He clarified the form of population equation in

$$N'(t) = aN(t-d) + bN(t).$$

The obstacle in his investigation was that he did not have enough information about reasonable history function $N(t)$ on $[-d, 0]$ to get solution of this problem. He reversed time to get the solution of functional differential equations. He used time reversal in order to seek the population

How to cite (APA): Ashyralyev, A., Ashyralyyeva, M., & Batyrova, O. (2024). A note on the telegraph type differential equation with time involution. *Advanced Mathematical Models & Applications*, *9*(2), 173-192 https://doi.org/10.62476/amma9173

before the initial time $t = 0$ (Nesbit, 1997). An involutory differential equation is a type of equation and time reversal problem is a specal case from it. It is called an involutory differential equation, if it is involving unknown function y at t and $d - t$.

Functional differential equations with involution has been studied in Przeworska-Rolewicz (1973); Wiener (1993); Cabada & Tojo, (2015); Ashyralyev & Abdalmohammed (2020); Ashyralyev et al. (2021).

In Cabada & Tojo (2015) involution differential equation studies through those discoveries related to Green's functions. They investigated the theory of Green's functions for functional differential equations with involutions in the simplest cases: order one problems with constant coefficients and reflection. Here they solve the problem with different boundary conditions, studying the specific characteristics which appear when considering periodic, anti-periodic, initial, or arbitrary boundary conditions. Computing explicitly the Green's function for a problem with nonconstant coefficients is not simple, not even in the case of ordinary differential equations. They presented a double trick. First, they reduce the case of a general involution to the case of the reflection and then they use a special change of variable that allows obtaining of the Green's function of problems with nonconstant coefficients from the Green's functions of constant-coefficient analogs.

In Ashyralyev et al. (2021), the boundedness of the solution of the initial value problem

$$y''(t) = f(t, y'(t), y(t), y(u(t)), t \in I, y(t_0) = y_0, y'(t_0) = y'_0 \tag{1}$$

for the second order ordinary differential equation with damping term and involution was investigated. Here and in future $u(t)$ is involution function, that is $u(u(t)) = t$, and t_0 is a fixed point of u and $I = (-\infty, \infty)$. Problem (1) does not seem to yield directly to any techniques that for ordinary differential equations without involution term can be used them in (1). Therefore, the second order linear differential equations with involution was considered. The equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with damping term and involution was obtained. The theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution is proved. The theorem on existence and uniqueness of bounded solution of initial value problem for the second order nonlinear ordinary differential equation with damping term and involution was established.

Presently, spectral questions of differential equations with involution were studied in Baskakov et al. (2017; 2019; 2020); Garkavenko et al. (2020); Aliev et al. (2013); Granilshchikova et al. (2022); Kritskov et al. (2019); Sarsenbi et al. (2021); Turmetov et al. (2021); Turmetov & Karachik (2021); Vladykina et al., (2019). In Sarsenbi et al. (2021) a definition of Green's function of the general boundary value problems for non-self-adjoint second order differential equation with involution was given. The sufficient conditions for the basis property of system of eigenfunctions are established in the terms of the boundary conditions. Uniform equiconvergence of spectral expansions related to the second-order differential equations with involution

$$-y''(t) + \alpha y''(-t) + q(t)y(t) = \lambda y(t), -1 < t < 1,$$

with the boundary conditions $y'(-1) + by(-1) = 0, y'(1) + dy(1) = 0$ was obtained.

Delay hyperbolic and telegraph differential equations has been investigated in several papers (Ashyralyev & Agirseven, 2019; Zhang & Zhang, 2014; Prakash & Harikrishnan, 2012; Ashyralyev & Sobolevskii, 2004; Ashyralyev & Sarsenbi, 2017). Partial differential equations with delay and involution terms have deeply different properties of solutions then without involution terms (Ashyralyev & Abdalmohammed, 2021a; 2021b). Therefore, it is important to study properties of partial differential equations with involution.

The main aim of the present paper is to study the stability of the solution of the initial value problem for telegraph type involutely partial differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = g(t,x),\ t,\ x \in I, \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in I. \end{cases} \tag{2}$$

Here $g(t,x)\,(t,x \in I)$ and $\varphi(x)$, $\psi(x)\,(x \in I)$ are given smooth and bounded functions and $|b| < a, \frac{\alpha^2}{2} < a \leq \frac{\alpha^2}{4} + \frac{b^2}{\alpha^2}, \alpha \geq 0$.

2 Stability of problem(2)

Problem (2) can be written as abstract initial value problem

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + \alpha \frac{du(t)}{dt} + aAu(t) + bAu(-t) = g(t),\ t \in I \\ u(0) = \varphi,\ u'(0) = \psi \end{cases} \tag{3}$$

in a Banach space $C(I)$ of all continuous bounded functions $f(x)$ defined on I with norm

$$\|f\|_{C(I)} = \sup_{x \in I} |f(x)|.$$

Here, positive operator A defined by the formula

$$Au = -u''(x)$$

with domain $D(A) = \{u : u(x),\ u''(x) \in C(I)\}$, $g(t) = g(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions with values in $C(I)$ and $\varphi = \varphi(x)$, $\psi = \psi(x)$ are unknown elements of $C(I)$. The normed space $C_1(I)$ is the all continuous real-valud functions $f(x)$ on I and norm defined by

$$\|f\|_{C_1(I)} = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

Theorem 1. *Assume that $|b| < a, 0 \leq \alpha,\ a \in (\frac{\alpha^2}{2}, \frac{\alpha^2}{4} + \frac{b^2}{\alpha^2}]$. Let $g(t)$ be a smooth and bounded abstract functions on I and $g(t), g_t(t), g_{tt}(t) \in C_1(I)$ and $g(t), \varphi, \psi \in D(A)$, then the problem (3) is equivalent to the following initial value problem*

$$\begin{cases} \frac{d^4 u(t)}{dt^4} + \left(2a - \alpha^2\right) A \frac{d^2 u(t)}{dt^2} + \left(a^2 - b^2\right) A^2 u(t) = F(t), \\ F(t) = aAg(t) - bAg(-t) - \alpha g_t(t) + g_{tt}(t),\ t \in I, \\ u(0) = \varphi,\ u'(0) = \psi,\ u''(0) = -(a+b) A\varphi - \alpha\psi + g(0), \\ u'''(0) = (-a+b) A\psi + \alpha (a+b) A\varphi + \alpha^2 \psi + g_t(0) - \alpha g(0) \end{cases} \tag{4}$$

for the fourth order ordinary differential equation in a Banach space $C(I)$.

Proof. Differentiating the equation (3) with respect to t, we get

$$\frac{d^3 u(t)}{dt^3} + \alpha \frac{d^2 u(t)}{dt^2} + aAu'(t) - bAu'(-t) = g_t(t), \tag{5}$$

$$\frac{d^4 u(t)}{dt^4} + \alpha \frac{d^3 u(t)}{dt^3} + aAu''(t) + bAu''(-t) = g_{tt}(t). \tag{6}$$

Using these equations and initial condition and equation in problem (3), we get

$$\begin{cases} u(0)=\varphi,\ u'(0)=\psi, \\ u''(0)=-(a+b)A\varphi-\alpha\psi+g(0), \\ u'''(0)=-(a-b)A\psi+\alpha(a+b)A\varphi+\alpha^2\psi+g_t(0)-\alpha g(0). \end{cases} \tag{7}$$

Putting $-t$ instead of t equation (3), we get

$$u_{tt}(-t)+\alpha u_t(-t)+aAu(-t)+bAu(t)=g(-t). \tag{8}$$

Applying equations (3), (6) and (8), we get

$$\frac{d^4u(t)}{dt^4}+\alpha\frac{d^3u(t)}{dt^3}+aA\frac{d^2u(t)}{dt^2}$$
$$+bA\left[-\alpha u_t(-t)-aAu(-t)-bAu(t)+g(-t)\right]=g_{tt}(t),$$
$$bAu(-t)=-\frac{d^2u(t)}{dt^2}-\alpha\frac{du(t)}{dt}-aAu(t)+g(t).$$

From these equations it follows equation

$$\frac{d^4u(t)}{dt^4}+\alpha\frac{d^3u(t)}{dt^3}+aA\frac{d^2u(t)}{dt^2}$$
$$+\alpha\left[-\frac{d^3u(t)}{dt^3}-\alpha\frac{d^2u(t)}{dt^2}-aA\frac{du(t)}{dt}+g_t(t)\right]$$
$$+aA\left[\frac{d^2u(t)}{dt^2}+\alpha\frac{du(t)}{dt}+aAu(t)-g(t)\right]-b^2A^2u(t)$$
$$=-bAg(-t)+g_{tt}(t)$$

or

$$\frac{d^4u(t)}{dt^4}+\left(2a-\alpha^2\right)A\frac{d^2u(t)}{dt^2}+\left(a^2-b^2\right)A^2u(t)$$
$$=aAg(t)-bAg(-t)-\alpha g_t(t)+g_{tt}(t).$$

So, the problem (4) is presented. Now, we will get (3) from (4). Denote that

$$L(t)=\frac{d^2u(t)}{dt^2}+\alpha\frac{du(t)}{dt}+aAu(t)+bAu(-t)-g(t),\ t\in I.$$

It is easy to see that $L(t)$ is the solution of the following problem

$$L^{''}(t)+\alpha L^{'}(t)+aAL(t)+bAL(-t)=0,\ t\in I,\ L(0)=0,\ L^{'}(0)=0.$$

From that it follows $L(t)\equiv 0$. □

Now we will obtain solution of the initial value problem (4). It is easy to see that

$$\frac{d^4u(t)}{dt^4}+\left(2a-\alpha^2\right)A\frac{d^2u(t)}{dt^2}+\left(a^2-b^2\right)A^2u(t)$$
$$=\left(\frac{d^2}{dt^2}+q^2A\right)\left(\frac{d^2}{dt^2}+p^2A\right)u(t),$$

where

$$p^2 = \left(a - \frac{\alpha^2}{2} + \sqrt{-a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right), q^2 = \left(a - \frac{\alpha^2}{2} - \sqrt{-a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right).$$

Therefore, problem (4) can be written as abstract initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} + p^2 A\right) u(t) = v(t),\ u(0) = \varphi,\ u'(0) = \psi, \\ \left(\frac{d^2}{dt^2} + q^2 A\right) v(t) = F(t), \\ F(t) = aAg(t) - bAg(-t) - \alpha g_t(t) + g_{tt}(t),\ t \in I, \\ v(0) = \left(-b - a + p^2\right) A\varphi - \alpha\psi + g(0), \\ v'(0) = \alpha(a+b) A\varphi + \left(b - a + p^2\right) A\psi + \alpha^2\psi - \alpha g(0) + g'(0) \end{cases} \tag{9}$$

for the system of second order abstract differential equations in a Banach space $C(I)$. Problem (9) can be written as initial value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - p^2 u_{xx}(t,x) = v(t,x),\ t,\ x \in I, \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\quad x \in I, \\ \frac{\partial^2 v(t,x)}{\partial t^2} - q^2 v_{xx}(t,x) = F(t,x), \\ F(t,x) = -ag_{xx}(t,x) + bg_{xx}(-t,x) + g_{tt}(t,x) - \alpha g_t(t,x),\ t,x \in I, \\ v(0,x) = \left(b + a - p^2\right)\varphi_{xx}(x) - \alpha\psi(x) + g(0,x), \\ v_t(0,x) = -\alpha(a+b)\varphi_{xx}(x) \\ -\left(b - a + p^2\right)\psi_{xx}(x) + \alpha^2\psi(x) - \alpha g(0,x) + g'(0,x),\ x \in I \end{cases} \tag{10}$$

for the system of telegraph equations. Applying the d'Alembert's formula, we get

$$u(t,x) = \frac{1}{2}\left(\varphi(x+pt) + \varphi(x-pt)\right) \tag{11}$$

$$+\frac{1}{2p}\int_{x-pt}^{x+pt} \psi(\xi)d\xi + \frac{1}{2p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} v(\tau,\xi)d\xi d\tau,$$

$$v(t,x) = \frac{1}{2}\left[\left(b + a - p^2\right)\varphi_{xx}(x+qt) - \alpha\psi(x+qt) + g(0,x+qt)\right. \tag{12}$$

$$\left. + \left(b + a - p^2\right)\varphi_{xx}(x-qt) - \alpha\psi(x-qt)\right] + g(0,x-qt)$$

$$+\frac{1}{2q}\int_{x-qt}^{x+qt} \left[-\alpha(a+b)\varphi_{\lambda\lambda}(\lambda) - \left(b - a + p^2\right)\psi_{\lambda\lambda}(\lambda) + \alpha^2\psi(\lambda) - \alpha g(0,\lambda) + g'(0,\lambda)\right] d\lambda$$

$$+\frac{1}{2q}\int_0^t \int_{x-q(t-r)}^{x+q(t-r)} F(r,\lambda)d\lambda dr.$$

Applying formulas (11) and (12), we get

$$u\left(t,x\right)=\frac{1}{2}\left(\varphi(x+pt)+\varphi(x-pt)\right)+\frac{1}{2p}\int\limits_{x-pt}^{x+pt}\psi(\xi)d\xi \tag{13}$$

$$+\frac{1}{2p}\int\limits_{0}^{t}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[\left(b+a-p^{2}\right)\varphi_{\xi\xi}(\xi+q\tau)-\alpha\psi(\xi+q\tau)\right.$$

$$\left.+\left(b+a-p^{2}\right)\varphi_{\xi\xi}\left(\xi-q\tau\right)-\alpha\psi\left(\xi-q\tau\right)\right]d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^{2}-b^{2}}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha\left(a+b\right)\varphi_{\lambda\lambda}(\lambda)\right.$$

$$\left.-\left(b-a+p^{2}\right)\psi_{\lambda\lambda}(\lambda)+\alpha^{2}\psi(\lambda)\right]d\lambda d\xi d\tau$$

$$+\frac{1}{2p}\int\limits_{0}^{t}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right]d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^{2}-b^{2}}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha g(0,\lambda)+g'(0,\lambda)\right]d\lambda d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^{2}-b^{2}}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{0}^{\tau}\int\limits_{\xi-q(\tau-r)}^{\xi+q(\tau-r)}F(r,\lambda)d\lambda dr d\xi d\tau.$$

Theorem 2. *Assume that $|b|<a, 0\leq\alpha,\ a\in(\frac{\alpha^{2}}{2},\frac{\alpha^{2}}{4}+\frac{b^{2}}{\alpha^{2}}]$. Let $g(t,x)\in C(I\times I), g(t,x)\in C_{1}(I\times I)$ and $\varphi(x),\varphi_{x}(x),\varphi_{xx}(x),\psi(x)\in C_{1}(I),\psi(x),\psi_{x}(x)\in C(I)$ and*

$$\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}\left|g(0,z)\right|dzd\tau,\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left|\psi(\lambda)\right|d\lambda d\xi d\tau,$$

$$\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left|g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)\right|d\lambda d\xi d\tau,$$

$$\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}\left|\psi(\lambda)\right|d\lambda d\tau,\int\limits_{-\infty}^{\infty}\left|g(t,x)\right|dydx<\infty$$

for any $t,x\in I$.Then, for solutions of problem (2) we have following stability estimates

$$\sup_{t,x\in I}\left|u(t,x)\right|\leq M_{1}(a,b)\left[\sup_{x\in I}\left|\varphi(x)\right|+\int\limits_{-\infty}^{\infty}\left|\psi(y)\right|dy\right.$$

$$+\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left|g(y,x)\right|dydx+\sup_{t,x\in I}\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}\left|g(0,z)\right|dzd\tau$$

$$+\alpha\int\limits_{-\infty}^{\infty}|\varphi(x)|\,dx+\alpha^{2}\sup_{t,x\in I}\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}|\psi(\lambda)|\,d\lambda d\xi d\tau$$

$$+\alpha\sup_{t,x\in I}\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left|g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)\right|d\lambda d\xi d\tau$$

$$\left.+\alpha\sup_{t,x\in I}\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(z)|\,dzd\tau\right], \tag{14}$$

$$\sup_{t,x\in I}|u_t(t,x)|+\sup_{t,x\in I}|u_x(t,x)|\leq M_2(a,b)\left[\sup_{x\in I}|\varphi_x(x)|+\sup_{x\in I}|\psi(x)|\right.$$

$$+\sup_{y\in I}\int\limits_{-\infty}^{\infty}|g(y,x)|\,dx+\alpha^{2}\sup_{t,x\in I}\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(z)|\,dzd\tau$$

$$\left.+\alpha\int\limits_{-\infty}^{\infty}|\psi(y)|\,dy+\alpha\sup_{x\in I}|\varphi(x)|+\alpha\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|g(y,x)|\,dydx\right], \tag{15}$$

$$\sup_{t,x\in I}|u_{tt}(t,x)|+\sup_{t,x\in I}|u_{xx}(t,x)|+\sup_{t,x\in I}|u_{tx}(t,x)|$$

$$\leq M_3(a,b)\left[\sup_{x\in I}|\varphi_{xx}(x)|+\sup_{x\in I}|\psi_x(x)|+\sup_{t,x\in I}|g(t,x)|\right.$$

$$\left.+\alpha\sup_{x\in I}|\psi(x)|+\alpha\sup_{x\in I}|\varphi_x(x)|+\alpha^{2}\int\limits_{-\infty}^{\infty}|\psi(y)|\,dy+\alpha\int\limits_{-\infty}^{\infty}\sup_{y\in I}|g(y,x)|\,dx\right]. \tag{16}$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha,\beta,...)$ to stress the fact that the constant depends only on $\alpha,\beta,....$

Proof. We have that

$$u(t)=J_1(t,x)+J_2(t,x)+J_3(t,x)+J_4(t,x),$$

where

$$J_1(t,x)=\frac{1}{2}\left(\varphi(x+pt)+\varphi(x-pt)\right)+\frac{1}{2p}\int\limits_{x-pt}^{x+pt}\psi(\xi)d\xi,$$

$$J_2(t,x)=\frac{1}{2p}\int\limits_{0}^{t}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[\left(b+a-p^{2}\right)\varphi_{\xi\xi}(\xi+q\tau)-\alpha\psi(\xi+q\tau)\right.$$

$$\left.+\left(b+a-p^{2}\right)\varphi_{\xi\xi}\left(\xi-q\tau\right)-\alpha\psi\left(\xi-q\tau\right)\right]d\xi d\tau,$$

$$J_3(t,x)=\int\limits_{0}^{t}\frac{1}{4\sqrt{a^{2}-b^{2}}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha\left(a+b\right)\varphi_{\lambda\lambda}(\lambda)\right.$$

$$\left.-\left(b-a+p^{2}\right)\psi_{\lambda\lambda}(\lambda)+\alpha^{2}\psi(\lambda)\right]d\lambda d\xi d\tau,$$

$$J_4(t,x)=\int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_0^{\tau} \int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)} F(r,\lambda)d\lambda dr d\xi d\tau$$

$$+\frac{1}{2p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right]d\xi d\tau$$

$$+\int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_{\xi-q\tau}^{\xi+q\tau} \left[-\alpha g(0,\lambda)+g'(0,\lambda)\right]d\lambda d\xi d\tau.$$

We prove estimate (14). We will estimate $J_k(t,x)$, $k=1,2,3,4$, separately. Applying the triangle inequality, we get

$$|J_1(t,x)|\leq M_{11}(a,b)\left[\sup_{x\in I}|\varphi(x)|+\int_{-\infty}^{\infty}|\psi(y)|\,dy\right] \tag{17}$$

for any t, $x\in I$. Now, let us estimate $J_2(t,x)$. We have that

$$\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left(\varphi_{\xi\xi}(\xi+q\tau)+\varphi_{\xi\xi}\left(\xi-q\tau\right)\right)d\xi d\tau$$

$$=\frac{1}{2}\int_0^t \Big[\big(\varphi_{x+p(t-\tau)}(x+p\left(t-\tau\right)+q\tau)+\varphi_{x+p(t-\tau)}\left(x+p\left(t-\tau\right)-q\tau\right)\big)$$

$$-\big(\varphi_{x-p(t-\tau)}(x-p\left(t-\tau\right)+q\tau)+\varphi_{x-p(t-\tau)}\left(x-p\left(t-\tau\right)-q\tau\right)\big)\Big]\,d\tau$$

$$=\frac{1}{p}\left[\varphi(x+pt)+\varphi\left(x-pt\right)-\varphi(x+qt)-\varphi\left(x-qt\right)\right].$$

Then

$$J_2(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[\varphi(x+pt)+\varphi\left(x-pt\right)-\varphi(x+qt)-\varphi\left(x-qt\right)\right] \tag{18}$$

$$-\frac{\alpha}{4p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \left(\psi(\xi+q\tau)+\psi\left(\xi-q\tau\right)\right)d\xi d\tau.$$

Applying the triangle inequality, we get

$$|J_2(t,x)|\leq M_{12}(a,b)\left[\sup_{x\in I}|\varphi(x)|+\alpha\int_0^{|t|} \int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau} |\psi(z)|\,dz d\tau\right] \tag{19}$$

for any t, $x\in I$. Third, let us estimate $J_3(t,x)$. It is easy to see that

$$J_3(t,x)=-\frac{\alpha\left(a+b\right)}{4\sqrt{a^2-b^2}}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \left[\varphi_{\xi+q\tau}(\xi+q\tau)-\varphi_{\xi-q\tau}(\xi-q\tau)\right]d\xi d\tau \tag{20}$$

$$-\frac{b-a-p^2}{4\sqrt{a^2-b^2}}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \left[\psi_{\xi+q\tau}(\xi+q\tau)-\psi_{\xi-q\tau}(\xi-q\tau)\right]d\xi d\tau$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}\psi(\lambda)d\lambda d\xi d\tau$$

$$=-\frac{\alpha\left(a+b\right)}{4\sqrt{a^2-b^2}}\int_0^t\left[\varphi(x+p\left(t-\tau\right)+q\tau)-\varphi(x-p\left(t-\tau\right)+q\tau)\right.$$

$$\left.-\varphi(x+p\left(t-\tau\right)-q\tau)+\varphi(x-p\left(t-\tau\right)-q\tau)\right]d\tau$$

$$-\frac{\left(b-a-p^2\right)}{4\sqrt{a^2-b^2}}\int_0^t\left[\psi(x+p\left(t-\tau\right)+q\tau)-\psi(x-p\left(t-\tau\right)+q\tau)\right.$$

$$\left.-\psi(x+p\left(t-\tau\right)-q\tau)+\psi(x-p\left(t-\tau\right)-q\tau)\right]d\tau$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}\psi(\lambda)d\lambda d\xi d\tau.$$

Applying the triangle inequality, we get

$$|J_3(t,x)|\leq M_{13}(a,b)\left[\int_{-\infty}^{\infty}|\psi(x)|\,dx\right.$$

$$\left.+\alpha\int_{-\infty}^{\infty}|\varphi(x)|\,dx+\alpha^2\int_0^{|t|}\int_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}|\psi(\lambda)|\,d\lambda d\xi d\tau\right] \tag{21}$$

for any t, $x\in I$. We have that

$$J_4(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_0^{\tau}\int_{\xi-p(\tau-r)}^{\xi+q(\tau-r)}\left[-ag_{\lambda\lambda}(r,\lambda)+bg_{\lambda\lambda}(-r,\lambda)\right]d\lambda drd\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_0^{\tau}\int_{\xi-p(\tau-r)}^{\xi+q(\tau-r)}\left[g_{rr}(r,\lambda)-\alpha g_r(r,\lambda)\right]d\lambda drd\xi d\tau$$

$$+\frac{1}{2p}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right]d\xi d\tau$$

$$+\int_0^t\frac{1}{4\sqrt{a^2-b^2}}\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha g(0,\lambda)+g'(0,\lambda)\right]d\lambda d\xi d\tau.$$

Applying formulas

$$\int_0^{\tau}\int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)}\left[-ag_{\lambda\lambda}(r,\lambda)+bg_{\lambda\lambda}(-r,\lambda)\right]d\lambda dr=\frac{2a}{q}g(\tau,\xi)$$

$$-\frac{a-b}{q}\left(g(0,\xi+p\tau)+g(0,\xi-p\tau)\right)-\frac{2b}{q}g(-\tau,\xi),$$

$$\int_0^{\tau}\int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)}\left[g_{rr}(r,\lambda)-\alpha g_r(r,\lambda)\right]d\lambda dr$$

$$=\int_{\xi-q\tau}^{\xi}\int_0^{\tau-\frac{1}{q}(\xi-\lambda)}\left[g_{rr}(r,\lambda)-\alpha g_r(r,\lambda)\right]d\lambda dr$$

$$+\int_{\xi}^{\xi+q\tau}\int_0^{\tau+\frac{1}{q}(\xi-\lambda)}\left[g_{rr}(r,\lambda)-\alpha g_r(r,\lambda)\right]d\lambda dr$$

$$=2qg(\tau,\xi)-qg(0,\xi-q\tau)-qg(0,\xi+q\tau)$$

$$-\int_{\xi-q\tau}^{\xi+q\tau}g'(0,\lambda)\,d\lambda+\alpha\int_{\xi-q\tau}^{\xi+q\tau}g(0,\lambda)\,d\lambda-\alpha\int_{\xi-q\tau}^{\xi+q\tau}g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)d\lambda,$$

we get

$$J_4(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\left[\frac{2a}{q}g(\tau,\xi)\right.$$

$$\left.-\frac{a-b}{q}\left(g(0,\xi+p\tau)+g(0,\xi-p\tau)\right)-\frac{2b}{q}g(-\tau,\xi)\right]d\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\left[2qg(\tau,\xi)-qg(0,\xi-q\tau)-qg(0,\xi+q\tau)\right]d\xi d\tau$$

$$+\frac{1}{2p}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right]d\xi d\tau$$

$$-\frac{\alpha}{4\sqrt{a^2-b^2}}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)d\lambda d\xi d\tau. \tag{22}$$

Applying the triangle inequality, we get

$$|J_4(t,x)|\leq M_{14}(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(y,x)|\,dydx\right.$$

$$+\int_0^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|g(0,z)|\,dzd\tau$$

$$\left.+\alpha\int_0^{|t|}\int_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}\left|g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)\right|d\lambda d\xi d\tau\right] \tag{23}$$

for any $t,\ x\in I$. Combining the estimates for $J_k(t,x), k=1,2,3,4$, we obtain estimate (14).

Now, we prove estimate (15). We will estimate $J_{k,t}(t,x)$ and $J_{k,x}(t,x)$, $k=1,2,3,4$, separately. First, we start with estimates for $J_{1,t}(t,x)$ and $J_{1,x}(t,x)$. We have that

$$J_{1,t}(t,x)$$

$$=\frac{p}{2}\left(\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)+\frac{1}{2}\left[\psi(x+pt)+\psi(x-pt)\right], \tag{24}$$

$$J_{1,x}(t,x)$$

$$=\frac{1}{2}\left(\varphi_{x+pt}(x+pt)+\varphi_{x-pt}(x-pt)\right)+\frac{1}{2p}\left[\psi(x+pt)-\psi(x-pt)\right]. \tag{25}$$

Applying the triangle inequality, we get

$$\left|J_{1,t}(t,x)\right|,\ \ \left|J_{1,x}(t,x)\right|\le M_{21}(a,b)\left[\sup_{x\in I}\left|\varphi_x(x)\right|+\sup_{x\in I}\left|\psi(x)\right|\right] \tag{26}$$

for any t, $x\in I$. Second, let us estimate $J_{2,t}(t,x)$ and $J_{2,x}(t,x)$. Applying the formula(18), we get

$$J_{2,t}(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[p\varphi_{x+pt}(x+pt)\right. \tag{27}$$
$$\left.-p\varphi_{x-pt}(x-pt)-q\varphi_{x+qt}(x+qt)+q\varphi_{x-qt}(x-qt)\right]$$
$$-\frac{\alpha}{4p}\int\limits_0^t\left[p\psi(x+p\left(t-\tau\right)+q\tau)+p(x+p\left(t-\tau\right)-q\tau)\right.$$
$$\left.+p\psi(x-p\left(t-\tau\right)+q\tau)+p\psi(x-p\left(t-\tau\right)-q\tau)\right]d\tau$$

$$J_{2,x}(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[\varphi_{x+pt}(x+pt)\right. \tag{28}$$
$$\left.+\varphi_{x-pt}(x-pt)-\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)\right]$$
$$-\frac{\alpha}{4p}\int\limits_0^t\left[\psi(x+p\left(t-\tau\right)+q\tau)+\psi\left(x+p\left(t-\tau\right)-q\tau\right)\right.$$
$$\left.-\psi(x-p\left(t-\tau\right)+q\tau)+\psi\left(x-p\left(t-\tau\right)-q\tau\right)\right]d\tau.$$

Applying the triangle inequality, we get

$$\left|J_{2,t}(t,x)\right|,\ \ \left|J_{2,x}(t,x)\right|\le M_{22}(a,b)\left[\sup_{x\in I}\left|\varphi_x(x)\right|+\alpha\int\limits_{-\infty}^{\infty}\left|\psi(y)\right|dy\right] \tag{29}$$

for any t, $x\in I$. Third, let us estimate $J_{3,t}(t,x)$ and $J_{3,x}(t,x)$.Applying the formula(20), we get

$$J_{3,t}(t,x)=-\frac{\alpha\left(a+b\right)}{4\sqrt{a^2-b^2}}\left[\frac{p}{p+q}\left(\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)-\varphi(x-pt)\right)\right.$$
$$\left.+\frac{p}{p-q}\left(\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)-\varphi(x-pt)\right)\right] \tag{30}$$
$$-\frac{\left(b-a-p^2\right)}{4\sqrt{a^2-b^2}}\left[\frac{p}{p+q}\left(\psi(x+qt)+\psi(x-qt)-\psi(x+pt)-\psi(x-pt)\right)\right.$$

$$+\frac{p}{p-q}\left(\psi(x+qt)+\psi(x-qt)-\psi(x+pt)-\psi(x-pt)\right)\Bigg]$$

$$+\frac{p\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau}\psi(\lambda)d\lambda+\int_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau}\psi(\lambda)d\lambda\right]d\tau,$$

$$J_{3,x}(t,x)=-\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}\left[\frac{1}{p+q}\left(-\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)+\varphi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(-\varphi(x+qt)+\varphi(x-qt)+\varphi(x+pt)-\varphi(x-pt)\right)\right]$$

$$-\frac{\left(b-a-p^2\right)}{4\sqrt{a^2-b^2}}\left[\frac{1}{p+q}\left(-\psi(x+qt)+\psi(x-qt)-\psi(x+pt)+\psi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(-\psi(x+qt)+\psi(x-qt)+\psi(x+pt)-\psi(x-pt)\right)\right]$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau}\psi(\lambda)d\lambda-\int_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau}\psi(\lambda)d\lambda\right]d\tau. \tag{31}$$

Applying the triangle inequality, we get

$$|J_{3,t}(t,x)|\,,\ \ |J_{3,x}(t,x)|$$

$$\le M_{23}(a,b)\left[\alpha\sup_{x\in I}|\varphi(x)|+\sup_{x\in I}|\psi(x)|+\alpha^2\int_0^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(\xi)|\,d\xi d\tau\right] \tag{32}$$

for any $t,\ x\in I$. Fourth, let us estimate $J_{4,t}(t,x)$ and $J_{4,x}(t,x)$. Applying formula (22),we get

$$J_{4,t}(t,x)=\frac{p}{4\sqrt{a^2-b^2}}$$

$$\times\int_0^t\left[\left(\frac{2a}{q}+2q\right)\left[g(\tau,x+p\,(t-\tau))+g(\tau,x-p\,(t-\tau))\right]\right.$$

$$-\frac{2b}{q}\left[g(-\tau,x+p\,(t-\tau))+g(-\tau,x-p\,(t-\tau))\right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\,(t-\tau)+q\tau)+g(0,x-p\,(t-\tau)+q\tau)\right]$$

$$\left.-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\,(t-\tau)-q\tau)+g(0,x-p\,(t-\tau)-q\tau)\right]\right]d\tau$$

$$+\frac{1}{4}\int_0^t\left[g(0,x+p\,(t-\tau)+q\tau)+g(0,x-p\,(t-\tau)+q\tau)\right.$$

$$\left.+g(0,x+p\,(t-\tau)-q\tau)+g(0,x-p\,(t-\tau)-q\tau)\right]d\tau$$

$$-\frac{\alpha p}{4\sqrt{a^2-b^2}}\int_0^t\left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau}g(\tau-\frac{1}{q}\left|x+p\,(t-\tau)-\lambda\right|,\lambda)d\lambda\right.$$

$$+\int\limits_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau} g(\tau-\frac{1}{q}\left|x-p\left(t-\tau\right)-\lambda\right|,\lambda)d\tau\Bigg] d\tau, \tag{33}$$

$$J_{4,x}(t,x)=\frac{1}{4\sqrt{a^2-b^2}}$$

$$\times\int\limits_0^t\left[\left(\frac{2a}{q}+2q\right)\left[g(\tau,x+p\left(t-\tau\right))-g(\tau,x-p\left(t-\tau\right))\right]\right.$$

$$-\frac{2b}{q}\left[g(-\tau,x+p\left(t-\tau\right))-g(-\tau,x-p\left(t-\tau\right))\right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x-p\left(t-\tau\right)+q\tau)\right]$$

$$\left.-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\left(t-\tau\right)-q\tau)-g(0,x-p\left(t-\tau\right)-q\tau)\right]\right]d\tau$$

$$+\frac{1}{4}\int\limits_0^t\left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x-p\left(t-\tau\right)+q\tau)\right.$$

$$\left.+g(0,x+p\left(t-\tau\right)-q\tau)-g(0,x-p\left(t-\tau\right)-q\tau)\right]d\tau$$

$$-\frac{\alpha}{4\sqrt{a^2-b^2}}\int\limits_0^t\left[\int\limits_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau} g(\tau-\frac{1}{q}\left|x+p\left(t-\tau\right)-\lambda\right|,\lambda)d\lambda\right.$$

$$\left.-\int\limits_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau} g(\tau-\frac{1}{q}\left|x-p\left(t-\tau\right)-\lambda\right|,\lambda)d\lambda\right]d\tau. \tag{34}$$

Applying the triangle inequality, we get

$$\left|J_{4,t}(t,x)\right|,\left|J_{4,x}(t,x)\right|$$

$$\leq M_{24}(a,b)\left[\sup_{y\in I}\int\limits_{-\infty}^{\infty}\left|g(y,x)\right|dx+\alpha\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left|g(y,x)\right|dydx\right] \tag{35}$$

for any $t,\ x\in I$. Combining the estimates for $J_{k,t}(t,x)$ and $J_{k,x}(t,x), k=1,2,3,4,$ we obtain estimate (15).

Now, we will prove estimate (16). We will estimate $J_{k,tt}(t,x)$, $J_{k,tx}(t,x)$ and $J_{k,xx}(t,x)$, $k=1,2,3,4$, separately. First, we will estimate $J_{1,tt}(t,x)$, $J_{1,tx}(t,x)$ and $J_{1,xx}(t,x)$. Using formulas (24), (25) and taking the derivative, we get

$$J_{1,tt}(t,x)=\frac{p^2}{2}\left(\varphi_{x+pt,x+pt}(x+pt)+\varphi_{x-pt,x-pt}(x-pt)\right)$$

$$+\frac{p}{2}\left[\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right],$$

$$J_{1,tx}(t,x)=\frac{p}{2}\left(\varphi_{x+pt,x+pt}(x+pt)-\varphi_{x-pt,x-pt}(x-pt)\right)$$

$$+\frac{1}{2}\left[\psi_{x+pt}(x+pt)+\psi_{x-pt}(x-pt)\right],$$

$$J_{1,xx}(t,x)=\frac{1}{2}\left(\varphi_{x+pt,x+pt}(x+pt)+\varphi_{x-pt,x+pt}(x-pt)\right)$$

$$+\frac{1}{2p}\left[\psi_{x+pt}(x+pt)-\psi_{x+pt}(x-pt)\right].$$

Applying the triangle inequality, we get

$$\left|J_{1,tt}(t,x)\right|,\left|J_{1,tx}(t,x)\right|,\ \ \left|J_{1,xx}(t,x)\right|$$

$$\leq M_{31}(a,b)\left[\sup_{x\in I}\left|\varphi_{xx}(x)\right|+\sup_{x\in I}\left|\psi_{x}(x)\right|\right] \tag{36}$$

for any $t,\ x\in I$. Second, we estimate $J_{2,tt}(t,x), J_{2,tx}(t,x)$ and $J_{2,xx}(t,x)$. Using formulas (24), (25) and taking the derivative, we get

$$J_{2,tt}(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[p^2\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.+p^2\varphi_{x-pt,x-pt}(x-pt)-q^2\varphi_{x+qt,x+qt}(x+qt)-q^2\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\left[p\psi(x+pt)+p\psi(x-qt)+p\psi(x+qt)+p\psi(x-qt)\right]$$

$$-\frac{\alpha}{4p}\int\limits_0^t\left[p^2\psi_{x+p(t-\tau)+q\tau}(x+p\left(t-\tau\right)+q\tau)+p^2\psi(x+p\left(t-\tau\right)-q\tau)\right.$$

$$\left.-p^2\psi_{x-p(t-\tau)+q\tau}(x-p\left(t-\tau\right)+q\tau)+p^2\psi_{x-p(t-\tau)-q\tau}(x-p\left(t-\tau\right)-q\tau)\right]d\tau$$

$$=\frac{\left(b+a-p^2\right)}{2p^2}\left[p^2\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.+p^2\varphi_{x-pt,x-pt}(x-pt)-q^2\varphi_{x+qt,x+qt}(x+qt)-q^2\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\left[p\psi(x+pt)+p\psi(x-qt)+p\psi(x+qt)+p\psi(x-qt)\right]$$

$$-\frac{\alpha}{4}\left[\frac{p}{-p+q}\left(\psi(x+qt)-\psi(x+pt)\right)-\frac{p}{p+q}\left(\psi(x-qt)-\psi(x+pt)\right)\right.$$

$$\left.-\frac{p}{p+q}\left(\psi(x+qt)-\psi(x-pt)\right)+\frac{p}{p-q}\left(\psi(x-qt)-\psi(x-pt)\right)\right],$$

$$J_{2,xt}(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[p\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.-p\varphi_{x-pt,x-pt}(x-pt)-q\varphi_{x+qt,x+qt}(x+qt)-q\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4}\int\limits_0^t\left[\psi_{x+p(t-\tau)+q\tau}(x+p\left(t-\tau\right)+q\tau)+\psi_{x+p(t-\tau)-q\tau}\left(x+p\left(t-\tau\right)-q\tau\right)\right.$$

$$\left.+\psi_{x-p(t-\tau)+q\tau}(x-p\left(t-\tau\right)+q\tau)-\psi_{x-p(t-\tau)-q\tau}\left(x-p\left(t-\tau\right)-q\tau\right)\right]d\tau$$

$$=\frac{\left(b+a-p^2\right)}{2p^2}\left[p\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.-p\varphi_{x-pt,x-pt}(x-pt)-q\varphi_{x+qt,x+qt}(x+qt)-q\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4}\left[\frac{1}{-p+q}\left(\psi(x+qt)-\psi(x+pt)\right)-\frac{1}{p+q}\left(\psi\left(x-qt\right)-\psi\left(x+pt\right)\right)\right.$$

$$\left.+\frac{1}{p+q}\left(\psi(x+qt)-\psi(x-pt)\right)-\frac{1}{p-q}\left(\psi\left(x-qt\right)-\psi\left(x-pt\right)\right)\right],$$

$$J_{2,xx}(t,x)=\frac{\left(b+a-p^2\right)}{2p^2}\left[\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.-\varphi_{x-pt,x-pt}(x-pt)-\varphi_{x+qt,x+qt}(x+qt)+\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\int\limits_0^t\left[\psi_{x+p(t-\tau)+q\tau}(x+p\left(t-\tau\right)+q\tau)+\psi_{x+p(t-\tau)-q\tau}\left(x+p\left(t-\tau\right)-q\tau\right)\right.$$

$$\left.-\psi_{x-p(t-\tau)+q\tau}(x-p\left(t-\tau\right)+q\tau)+\psi_{x-p(t-\tau)-q\tau}\left(x-p\left(t-\tau\right)-q\tau\right)\right]d\tau$$

$$=\frac{\left(b+a-p^2\right)}{2p^2}\left[\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.-\varphi_{x-pt,x-pt}(x-pt)-\varphi_{x+qt,x+qt}(x+qt)+\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\left[\frac{1}{-p+q}\left(\psi(x+qt)-\psi(x+pt)\right)-\frac{1}{p+q}\left(\psi\left(x-qt\right)-\psi\left(x+pt\right)\right)\right.$$

$$\left.-\frac{1}{p+q}\left(\psi(x+qt)-\psi(x-pt)\right)+\frac{1}{p-q}\left(\psi\left(x-qt\right)-\psi\left(x-pt\right)\right)\right].$$

Applying the triangle inequality, we get

$$\left|J_{2,tt}(t,x)\right|,\left|J_{2,tx}(t,x)\right|,\ \ \left|J_{2,xx}(t,x)\right|$$

$$\leq M_{32}(a,b)\left[\sup_{x\in I}\left|\varphi_{xx}(x)\right|+\alpha\sup_{x\in I}\left|\psi(x)\right|\right]$$

for any $t,\ x\in I$. Third, we estimate $J_{3,tt}(t,x)$, $J_{3,tx}(t,x)$ and $J_{3,xx}(t,x)$. Using formulas (30), (31) and taking the derivative, we get

$$J_{3,tt}(t,x)=-\frac{\alpha\left(a+b\right)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(q\varphi_{x+qt}(x+qt)-q\varphi_{x-qt}(x-qt)-p\varphi_{x+pt}(x+pt)+p\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(q\varphi_{x+qt}(x+qt)-q\varphi_{x-qt}(x-qt)-p\varphi_{x+pt}(x+pt)+p\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{\left(b-a+p^2\right)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(q\psi_{x+qt}(x+qt)-q\psi_{x-qt}(x-qt)-p\psi_{x+pt}(x+pt)+p\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(q\psi_{x+qt}(x+qt)-q\psi_{x-qt}(x-qt)-p\psi_{x+pt}(x+pt)+p\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{p\alpha^2}{2\sqrt{a^2-b^2}}\int\limits_{x-qt}^{x+qt}\psi(\lambda)d\lambda+\frac{p^2\alpha^2}{4\sqrt{a^2-b^2}}\int\limits_0^t\left[\psi(x+p\left(t-\tau\right)+q\tau)\right.$$

$$\left.-\psi(x+p\left(t-\tau\right)-q\tau)-\psi(x-p\left(t-\tau\right)+q\tau)+\psi(x-p\left(t-\tau\right)-q\tau)\right]d\tau,$$

$$J_{3,tx}(t,x) = -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{\left(b-a+p^2\right)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{p\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\psi(x+p\,(t-\tau)+q\tau)\right.$$

$$\left.-\psi(x+p\,(t-\tau)-q\tau)-\psi(x-p\,(t-\tau)+q\tau)+\psi(x-p\,(t-\tau)-q\tau)\right]d\tau,$$

$$J_{3,xx}(t,x) = -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{1}{p+q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{\left(b-a+p^2\right)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{1}{p+q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\psi(x+p\,(t-\tau)+q\tau)\right.$$

$$\left.-\psi(x+p\,(t-\tau)-q\tau)-\psi(x-p\,(t-\tau)+q\tau)+\psi(x-p\,(t-\tau)-q\tau)\right]d\tau.$$

Applying the triangle inequality, we get

$$\left|J_{3,tt}(t,x)\right|,\ \ \left|J_{3,tx}(t,x)\right|,\ \ \left|J_{3,xx}(t,x)\right| \le M_{33}(a,b)\left[\sup_{x\in I}\left|\psi_x(x)\right|\right.$$

$$\left.+\alpha\sup_{x\in I}\left|\varphi_x(x)\right|+\alpha^2\int_{-\infty}^{\infty}\left|\psi(x)\right|dx\right] \tag{37}$$

for any $t,\ x\in I$. Fourth, we estimate $J_{4,tt}(t,x), J_{4,tx}(t,x)$ and $J_{4,xx}(t,x)$. Using formulas (33), (34) and taking the derivative, we get

$$J_{4,tt}(t,x) = \frac{1}{4\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)p\left[g(0,x+pt)+g(0,x-pt)\right]\right.$$

$$-\frac{2b}{q}p\left[g(0,x+pt)+g(0,x-pt)\right]-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\left[2g(0,x+qt)\right.$$

$$\left.+\frac{p}{-p+q}\left[g(0,x+qt)-g(0,x+pt)\right]-\frac{p}{p+q}\left[g(0,x+qt)-g(0,x-pt)\right]\right]$$

$$-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\left[2g(0,x-qt)\right.$$

$$\left.+\frac{p}{p-q}\left[g(0,x-qt)-g(0,x+pt)\right]-\frac{p}{p+q}\left[g(0,x-qt)-g(0,x-pt)\right]\right]$$

$$-\frac{\alpha p}{2\sqrt{a^2-b^2}}\int\limits_{x-qt}^{x+qt}g(t-\frac{1}{q}\left|x-\lambda\right|,\lambda)d\lambda$$

$$-\frac{\alpha p^2}{4\sqrt{a^2-b^2}}\int\limits_0^t\left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right.$$

$$\left.-g(0,x-p\left(t-\tau\right)+q\tau)+g(0,x+p\left(t-\tau\right)-q\tau)\right]d\tau,$$

$$J_{4,tx}(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)\frac{1}{p}\left[g(0,x+pt)-g(0,x-pt)\right]\right.$$

$$-\frac{2b}{qp}\left[g(0,x+pt)-g(0,x-pt)\right]-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\left[\frac{1}{-p+q}\right.$$

$$\left.\times\left[g(0,x+qt)-g(0,x+pt)\right]+\frac{1}{p+q}\left[g(0,x+qt)-g(0,x-pt)\right]\right]$$

$$-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\left[-\frac{1}{p+q}\right.$$

$$\left.\times\left[g(0,x-qt)-g(0,x+pt)\right]+\frac{1}{p-q}\left[g(0,x-qt)-g(0,x-pt)\right]\right]$$

$$-\frac{\alpha p}{4\sqrt{a^2-b^2}}\int\limits_0^t\left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right.$$

$$\left.+g(0,x-p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right]d\tau,$$

$$J_{4,xx}(t,x)=\frac{1}{4p\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)\left[-2g(t,x)+g(0,x+pt)+g(0,x-pt)\right]\right.$$

$$-\frac{2b}{q}\left[-2g(-t,x)+g(0,x+pt)+g(0,x-pt)\right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[\frac{1}{-p+q}\left(g(0,x+qt)-g(0,x+pt)\right)-\frac{1}{p+q}\left(g(0,x+qt)-g(0,x-pt)\right)\right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[-\frac{1}{p+q}\left(g(0,x-qt)-g(0,x+pt)\right)-\frac{1}{p-q}\left(g(0,x-qt)-g(0,x-pt)\right)\right]$$

$$+\frac{1}{4}\left[\frac{1}{-p+q}\left(g(0,x+qt)-g(0,x+pt)\right)-\frac{1}{p+q}\left(g(0,x+qt)-g(0,x-pt)\right)\right.$$

$$\left.-\frac{1}{p+q}\left(g(0,x-qt)-g(0,x+pt)\right)-\frac{1}{p-q}\left(g(0,x-qt)-g(0,x-pt)\right)\right]$$

$$-\frac{\alpha p}{4\sqrt{a^2-b^2}}\int\limits_0^t [g(0,x+p(t-\tau)+q\tau)-g(0,x+p(t-\tau)-q\tau)$$

$$+g(0,x-p(t-\tau)+q\tau)-g(0,x+p(t-\tau)-q\tau)]\,d\tau.$$

Applying the triangle inequality, we get

$$|J_{4,tt}(t,x)|\,,\ \ |J_{4,tx}(t,x)|\,,\ \ |J_{4,xx}(t,x)| \le M_{43}(a,b)\left[\sup_{t,x\in I}|g(t,x)|\right.$$

$$\left. +\alpha\int\limits_{-\infty}^{\infty}\sup_{y\in I}|g(y,x)|\,dx\right] \tag{38}$$

for any $t,\ x\in I$. Combining the estimates for $J_{k,tt}(t,x)$, $J_{k,tx}(t,x)$ and $J_{k,xx}(t,x)$, $k=1,2,3,4$, we obtain estimate (16). □

3 Conclusion

In the present paper, the initial value problem for the telegraph type involutory linear partial differential equation is investigated. The equivalent initial value problem for the fourth order linear partial differential equations to the initial value problem for this second order linear partial differential equations with involution is established. Applying the operator tools, the stability estimates for the solution and its first and second order derivatives of this problem are proved. Note that using this method, we can get similarly statements for the solution of following problems

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+\alpha\frac{\partial^2 u(t,x)}{\partial t^2}-a\sum\limits_{r=1}^{n}a_r\frac{\partial^2 u(t,x)}{\partial x_r^2}-b\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2}=g(t,x), \\ x=(x_1,...,x_n)\in\mathbb{R}^n,\ -\infty<t<\infty, \\ u(\frac{d}{2},x)=\psi(x),\ u_t(\frac{d}{2},x)=\varphi(x),\ x\in\mathbb{R}^n \end{cases} \tag{39}$$

for a multidimensional telegraph involutory partial differential equations. Assume that $a_r > a_0 > 0$ and $g\,(t,x)\ (t\in I, x\in\mathbb{R}^n)\,,\ \psi(x),\ \varphi(x)\,(x\in\mathbb{R}^n)$ are smooth functions.

4 Acknowledgement

The publication has been prepared with the support of the "RUDN University Program 5-100" and published under target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

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