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EXACT SOLUTIONS, WAVE DYNAMICS AND CONSERVATION LAWS OF A GENERALIZED GEOPHYSICAL KORTEWEG DE VRIES EQUATION IN OCEAN PHYSICS USING LIE SYMMETRY ANALYSIS

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Abstract. An evolution equation is a partial differential equation that describes the time evolution of a physical system starting from given initial data. Evolution equations arise from many areas of applied and engineering sciences. To this end, this article investigates the analytical studies of a generalized geophysical Korteweg-de Vries equation in ocean physics. The examination of this model is conducted via the Lie group theory of differential equations. In the first place, point symmetries, which are constituent elements of a four-dimensional Lie algebra, are systematically computed. Thereafter, one-parameter transformation groups for the algebra are calculated. Besides, going forward, a one-dimensional optimal system of subalgebras is derived in a procedural manner. Sequel to this, the subalgebras and combination of the achieved symmetries are invoked in the reduction process which enables the derivation of nonlinear ordinary differential equations associated with the generalized geophysical Korteweg-de Vries equation under study. Most of the achieved nonlinear ordinary differential equations are further solved either via direct integration or using a power series approach. Furthermore, travelling wave solutions are initially obtained. This is attained via direct integration and the use of Jacobi elliptic function approach. These techniques enable the attainment of various exact soliton solutions, including non-topological soliton solutions as well as general periodic function solutions of note, such as cosine amplitude, sine amplitude, and delta amplitude solutions of the model. Furthermore, numerical simulations of the solutions are invoked to gain a gross knowledge of the physical phenomena represented by the under-study generalized geophysical Korteweg-de Vries equation in ocean physics. In the end, the investigation further gives attention to the calculation of conserved vectors for the model using Ibragimov's theorem for conservation laws, as well as Noether's theorem.

Keywords: A generalized geophysical Korteweg de Vries equation, Lie symmetry analysis, optimal system of one-dimensional subalgebras, exact solutions, Jacobi elliptic function technique, conservation laws.
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1 Introduction

In our general surroundings, we experience a large number of perplexing actual peculiarities that show nonlinearity. These peculiarities are precisely described by nonlinear partial differential equations (NPDEs), with models going from populace environment and the study of disease transmission to science, plasma physical science, liquid mechanics, and nonlinear circuits. To acquire a profound comprehension of these peculiarities, it is vital to find answers for the differential conditions that oversee them. Thus, this necessitate the examination of solitary wave solutions of these NPDEs in exact structure. Extensive research continues to be conducted on these equations, as they play a crucial role in modelling relationships between various physical quantities found in nature and human creations. Recent advancements in computer technology have greatly improved our ability to develop algorithms for solving NPDEs. Despite this progress, it is important to acknowledge the brilliant minds that have laid the theoretical groundwork for these technologies to flourish. In recent times, numerous researchers with a strong interest in nonlinear physical phenomena have been exploring exact solutions of NPDEs due to their significance in analyzing model outcomes. It is vital that research on closed-form solutions to NPDEs plays a crucial role in understanding specific physical scenarios. The range of solutions to NPDEs holds a significant position in varieties of scientific fields. These are inclusive of electromagnetic theory, chemical physics, optical fibers, hydrodynamics, meteorology, plasma physics, biology, heat flow, chemical kinetics, and geochemistry.

Recognizing that many prominent scientists view nonlinear science as a key frontier for gaining a deeper understanding of nature, we introduce relevant models including the Boussinesq-Burgers-type system (Gao et al., 2020), which describes shallow water waves near ocean shores and lakes. Additionally, the study explored a generalized form of the KdV-ZKe model as presented in the publication by Khalique & Adeyemo (2020a). This model was used to analyze the mixing of warm isentropic fluid with cold static components and hot isothermal substances in fluid dynamics. Furthermore, an examination in another source focused on the modified and generalized ZKe model, highlighting ion-acoustic solitary waves found in a magneto-plasma environment containing electron-positron-ion particles present in a native universe (Du et al., 2020). This model was applied to study waves in dust-magneto, ion, and dust-ion acoustics within laboratory dusty plasmas. Moreover, the study delved into vector bright solitons and their interactions within the coupled Fokas-Lenells system (Zhang et al., 2020). The investigation also extended to femto-second optical pulses embedded in double-refractive optical fibers, modeled using NPDEs. The listed publications Ay & Yasarv (2023); Babajanov & Abdikarimov (2022); Alhasanat (2023); Bruzon et al. (2022); Chulián et al. (2020); Bayrakci et al. (2023); Demiray & Duman (2023); Simbanefayi et al. (2023); Zhu (2022); Zhang (2022) can be visited to peruse more of the applications of NPDEs in various ways.

After extensive research, it has been determined that there is no universal approach for achieving exact solutions to NPDEs. However, in order to address this persistent issue, mathematicians and physicists have developed several effective techniques. For instance, Sophus Lie, who lived between 1842 and 1899, made significant contributions in the field of Lie algebras, providing a unified approach to solving a wide range of differential equations (Olver, 1993; Ovsiannikov, 1982). Recent advancements in solving differential equations include Kudryashov's approach (Kudryashov, 2012), Bäcklund transformation (Gu, 1990), Hirota's bilinear approach (Hirota, 2004), simplest equation technique (Kudryashov & Loguinova, 2018), Darboux transformation method (Hyder & Barakat, 2020), sine-Gordon equation expansion approach (Chen & Yan, 2005), F-expansion approach (Zhou et al., 2018), bifurcation technique (Wen, 2020; Zhang & Khalique, 2018), and tanh-coth approach (Wazwaz, 2018).

Since the establishment of Petviashvili and Kadomtsev's hierarchy equation models over fifty years ago, numerous research papers have been published, each delving into different aspects of this complex field of equations. For instance, see Kuo & Ma (2020); Wazwaz (2012); Ma & Fan (2011); Ma (2015); Zhau & Han (2017); Khalique et al. (2020) for more of the point raised.

One of these interesting models is the popular Korteweg-de Vries equation Wazwaz (2008)

$$\psi_t + p\psi\psi_x + q\psi_{xxx} = 0, \ \ p, q \neq 0, \tag{1}$$

which is commonly known as "KdV" model has gained attraction over the years due to its applications in various physical phenomena. There are different variations of this model that incorporate the altered and summed up forms with power-regulations introduced in like manner as Wazwaz (2008); Yan (2008); Wazwaz (2006)

$$\psi_t + p\psi^2\psi_x + q\psi_{xxx} = 0 \tag{2}$$

and

$$\psi_t + p\psi^n\psi_x + r\psi^{2n}\psi_x + q\psi_{xxx} = 0, \tag{3}$$

where constant parameters r and n are non-zero real numbers. For a long time, KdV as well as KdV-related models and their single waves have been the fundamental subject of much exploration because of their job in depicting numerous actual settings. These days, there are many articles about KdV as well as KdV-related models and their lone waves, particularly the numerical hypothesis behind these sorts of model conditions is a hot subject of dynamic examination. For example, in Wazwaz (2008), the author examined (1)–(3), where in his work, he presented new plans, each consolidating two exaggerated capabilities, to concentrate on the situations. Eventually, the review uncovered that this class of conditions gives traditional solitons and occasional arrangements. It was likewise shown that the proposed plans created sets of altogether new singular wave arrangements notwithstanding the conventional arrangements. The author later thought that the examination could be applied to a wide class of nonlinear development conditions.

Besides, Yan in (Yan (2008)) explored condition (2) with the centering (+) as well as defocusing (−) branches, where many new kinds of paired voyaging wave occasional arrangements were gotten for the situation as far as Jacobi elliptic capabilities such as sine, cosine and delta abundancy arrangements and their expansions. Plus, the asymptotic properties of a portion of the found arrangements were dissected. Also, with the guide of Miura change, Yan likewise gave the comparing twofold voyaging wave occasional arrangements of the altered KdV condition (2). Besides, in (Wazwaz (2006)), Wazwaz in his work analyzed the summed up KdV model with two power law nonlinearities (3). The tanh technique and two arrangements of ansatze including exaggerated capabilities were presented for logical investigation of the situation. New sorts of single wave arrangements were officially inferred.

Later, another type of the KdV equation was proposed by Quirchmayr and Geyer in Hosseini et al. (2024); Karunakar & Chakraverty (2006), which they called the geophysical KdV equation

$$\psi_t - w_0\psi_x + \frac{3}{2}\psi\psi_x + \frac{1}{6}\psi_{xxx} = 0 \tag{4}$$

with parameter w_0 connoting the Coriolis. The determined model (4), is utilized to investigate the proliferation of maritime waves and has received a lot of consideration from scholastic researchers. For instance, in Karunakar & Chakraverty (2006), the authors explored the impact of Coriolis consistent on the arrangement of the geophysical KdV condition (4). From that point of examination, it was reasoned that the steady of Coriolis is straightforwardly corresponding to wave level and conversely relative to frequency. The presence of Coriolis expression in the situation has a noteworthy change, looking like the arrangement. In addition, in Rizvi et al. (2020), the authors recovered protuberance soliton answer for model condition (4) with the assistance of Hirota bilinear technique. They additionally got bump crimp soliton (which is a cooperation of knot). with one crimp soliton), irregularity, intermittent arrangements (which is framed by connection between occasional waves and endlessly irregularity crimps intermittent arrangements, (which is framed by the connection of occasional waves and bumps with one crimp soliton). The elements of these arrangements were additionally analyzed graphically by choosing critical boundaries. Besides, in Alharbi & Almatrafi (2002) the authors recovered a few new lone answers for (4). The acquired arrangement from carrying out the shooting strategy was effectively utilized as an underlying incentive for the versatile methodology which was used to develop the mathematical arrangement of the issue. The built-in definite arrangements harmonized with the acquired mathematical arrangements. The precision of the introduced mathematical approximations was likewise examined. Furthermore, they applied Fourier's idea to investigate the exactness and security of the mathematical plans, which they found to be genuinely steady. In Hosseini et al. (2024), the authors considered an extended version of (4) as

$$\psi_t - w_0\psi_x + \frac{3}{2}\psi\psi_x + \frac{1}{6}\psi_{xxx} = -\left(\beta_0 + \beta_1\psi + \beta_2\psi^2 + \cdots + \beta_n\psi^n\right), \tag{5}$$

where they officially presented a source, which is a polynomial of degree n in the obscure capability, in the model. Through the Painlevé investigation, it is shown that (5) with the source is not integrable. Under a few fundamental circumstances for integrability, a few crimp type singular waves to the unique instances of the overseeing model when $n = 2$ and $n = 4$ are inferred utilizing the Kudryashov's technique.

Having gone through the literature, our work delves into the exploration of the generalized form of geophysical Korteweg-de Vries equation (4), also depicted as (1+1)D-GeoKdVe, which reads

$$u_t + au_x + buu_x + cu_{xxx} = 0, \tag{6}$$

where $u = u(x,t)$ and $a, b, c \neq 0$ are real numbers. One can easily see that (4) can be recovered from (6), if $u = \psi$, $a = -w_0$, $b = 2/3$ and $c = 1/6$. Thus, the latter is a general version of the former which implies that the former is contained in the latter.

The main goal of this research is to examine the generalized (1+1)D-GeoKdVe (6) using Lie group analysis. By applying this approach, we can determine the Lie point symmetries of the understudied model, which can then be used to create a detailed set of one-dimensional subalgebras. As a result, a variety of exact general solutions for the (1+1)D-GeoKdVe model (6) could be obtained. Furthermore, besides obtaining exact solutions, we will also establish conservation principles for equation (6) by invoking the Noether theorem as well as theorem by Ibragimov for conserved currents.

Consequently, the article is structured as follows. Section 2 contains the procedural pattern through which Lie point symmetries of (1+1)D-GeoKdVe (6) is obtained, but before that the required introduction is presented. Besides, in Section 2, optimal system of one dimensional subalgebras are calculated. Meanwhile, symmetry reductions as well as various analytic travelling wave solutions are further secured. Section 3 explicates the calculated conservation laws via Ibragimov's theorem as well as Noether's theorem (Noether, 1918). These are followed by the concluding remarks in Section 4.

2 Symmetry analysis and exact solutions of (6)

We begin by deriving the Lie point symmetries of (1+1)D-GeoKdVe (6), following which we utilize them to derive exact solutions.

2.1 Lie point symmetries of (1+1)D-GeoKdVe (6)

A one-parameter transformation groups is derived for equation (6), prompting further consideration. So, we have

$$\begin{aligned}
\tilde{x} &\approx x + \epsilon\, \xi^1(x,t,u) + O(\epsilon^2),\\
\tilde{t} &\approx t + \epsilon\, \xi^2(x,t,u) + O(\epsilon^2),\\
\tilde{u} &\approx u + \epsilon\, \Psi(x,t,u) + O(\epsilon^2).
\end{aligned} \tag{7}$$

The symmetry group of (1+1)D-GeoKdVe (6) will be formed by the vector field

$$\mathfrak{S} = \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial x} + \Psi(x,t,u)\frac{\partial}{\partial u},$$

where coefficient ξ^i, $i = 1, 2$ and Ψ is a function of (x, t, u), is a Lie point symmetry of (1+1)D-GeoKdVe (6) if

$$pr^{(3)}\mathfrak{S}[u_t + au_x + buu_x + cu_{xxx}] = 0, \tag{8}$$

whenever $u_t + au_x + buu_x + cu_{xxx} = 0$. Here $pr^{(3)}\mathfrak{S}$ represents the third extension of vector $\mathfrak{S}$, which is defined by

$$pr^{(3)}\mathfrak{S} = \mathfrak{S} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} \tag{9}$$

with ζ_t, ζ_x, ζ_{tx}, ζ_{xx}, and ζ_{xxx} defined by the general formulas

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j),\\
\zeta_{ij}^\alpha &= D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k),
\end{aligned} \tag{10}$$

where D_i, are the total derivatives given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i, j = 1, ..., n. \tag{11}$$

By expanding equation (8) and separating it based on the derivatives of the function u, we are able to derive the following system of overdetermined linear partial differential equations (LPDEQs):

$$\begin{aligned}
&\xi_u^1 = 0, \quad \xi_x^1 = 0, \quad \xi_{tt}^1 = 0, \quad 3\xi_x^2 - \xi_t^1 = 0, \quad \xi_{tt}^2 = 0,\\
&\xi_u^2 = 0, \quad 3b\Psi + 2\,(a + bu)\,\xi_t^1 - 3\xi_t^2 = 0,
\end{aligned}$$

whose solutions are

$$\begin{aligned}
&\xi^1(x,t,u) = A_1 t + A_2,\ \xi^2(x,t,u) = \frac{1}{3}A_1 x + A_3 t + A_4,\\
&\Psi(x,t,u) = -\frac{2}{3b}(bu + a)A_1 + 3A_3
\end{aligned}$$

with arbitrary constants A_1, A_2 together with A_3. The above produces the following two translational symmetries and one scaling symmetry, viz.,

$$\begin{aligned}
&\mathfrak{S}_1 = \frac{\partial}{\partial t},\ \mathfrak{S}_2 = \frac{\partial}{\partial x},\ \mathfrak{S}_3 = bt\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
&\mathfrak{S}_4 = 3bt\frac{\partial}{\partial t} + bx\frac{\partial}{\partial x} - 2\,(a + bu)\,\frac{\partial}{\partial u}.
\end{aligned} \tag{12}$$

The one-parameter groups generated by the above symmetries $\mathfrak{S}_1$ and $\mathfrak{S}_2$ describes time and space-invariance of the (1+1)D-GeoKdVe (6).

Theorem 1. *If group* $\mathfrak{G}_\epsilon^k(x,t,u), k=1,2,3,4$ *defines one parameter groups of transformation computed by vectors* $\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3,\mathfrak{S}_4$ *in which each of the achieved vectors achieves a generated transformation-group, representations of these are explicated accordingly as*

$$\begin{aligned}
\mathfrak{G}_\epsilon^1 &: (\tilde{x},\tilde{t},\tilde{u}) \longrightarrow (x, t+\epsilon_1, u),\\
\mathfrak{G}_\epsilon^2 &: (\tilde{x},\tilde{t},\tilde{u}) \longrightarrow (x+\epsilon_2, t, u),\\
\mathfrak{G}_\epsilon^3 &: (\tilde{x},\tilde{t},\tilde{u}) \longrightarrow \left(x+\epsilon_3 t, t, u+\frac{\epsilon_3}{b}\right),\\
\mathfrak{G}_\epsilon^4 &: (\tilde{x},\tilde{t},\tilde{u}) \longrightarrow \left\{xe^{\frac{1}{3}\epsilon_4}, te^{\epsilon_4}, -\frac{1}{b}\Big[ae^{\frac{2}{3}\epsilon_4}-(a+bu)\Big]e^{-\frac{2}{3}\epsilon_4}\right\}.
\end{aligned}$$

Theorem 2. *If* $u(x,t)=\mathfrak{M}(x,t)$ *solves the (1+1)D-GeoKdVe (6), then so do the functions structured in the format*

$$\begin{aligned}
u^1(x,t) &= \mathfrak{M}(x, t+\epsilon_1),\\
u^2(x,t) &= \mathfrak{M}(x+\epsilon_2, t),\\
u^3(x,t) &= \mathfrak{M}(x+\epsilon_3 t, t) - \frac{\epsilon_3}{b},\\
u^4(x,t) &= \frac{1}{b}\left\{be^{\frac{2}{3}\epsilon_4}\mathfrak{M}\left(xe^{\frac{1}{3}\epsilon_4}, te^{\epsilon_4}, -\frac{1}{b}\Big[ae^{\frac{2}{3}\epsilon_4}-(a+bu)\Big]e^{-\frac{2}{3}\epsilon_4}\right)+ae^{\frac{2}{3}\epsilon_4}-a\right\}.
\end{aligned}$$

2.2 Optimal system of one-dimensional subalgebras

In this part, we investigate the balances referenced before to fabricate an ideal arrangement of 1-D subalgebras. In this manner, we utilize the subsequent ideal arrangement of one-layered subalgebra to accomplish decreases in balance and gather invariant answers for condition (6). This cycle will permit us to improve on the situation and find arrangements that are invariant under specific gatherings. We shall be following the procedure outlined in Olver (1993); Hu et al. (2015). The task involved in obtaining an optimal system of subalgebras is as well equivalent to that of achieving an optimal system of subgroups. This classification problem for one-dimensional subalgebras is fundamentally same as the problem involved in the classification of the orbits of adjoint representation. Thus, this problem is thrashed via the engagement of naive approach whereby a general element taken from the Lie algebra is subjected to different adjoint transformations so that it can be simplified as much as possible. It is important to note that the adjoint representations shall be determined through the use of Lie series.

$$\mathrm{Ad}(\exp(\varepsilon\mathfrak{S}_i))\mathfrak{S}_j = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}(\mathrm{ad}\mathfrak{S}_i)^n(\mathfrak{S}_j) = \mathfrak{S}_j - \varepsilon[\mathfrak{S}_i,\mathfrak{S}_j] + \frac{\varepsilon^2}{2!}[\mathfrak{S}_i,[\mathfrak{S}_i,\mathfrak{S}_j]] - \cdots \tag{13}$$

with a real number ε as well as the commutator $[\mathfrak{S}_i,\mathfrak{S}_j]$ is defined by

$$[\mathfrak{S}_i,\mathfrak{S}_j] = \mathfrak{S}_i\mathfrak{S}_j - \mathfrak{S}_j\mathfrak{S}_i.$$

Table 1 and Table 2 display accordingly, the commutator table of the Lie symmetries as well as the adjoint representations of the symmetry group of equation (6) on its Lie algebra.

Table 1. Commutator table of the Lie algebra of equation (6)

[,]	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{S}_1$	0	0	$\mathfrak{S}_2$	$-3\mathfrak{S}_1$
$\mathfrak{S}_2$	0	0	0	$-\mathfrak{S}_2$
$\mathfrak{S}_3$	$-\mathfrak{S}_2$	0	0	$2\mathfrak{S}_3$
$\mathfrak{S}_4$	$3\mathfrak{S}_1$	$\mathfrak{S}_2$	$-2\mathfrak{S}_3$	0

Table 2. Adjoint table of the Lie algebra of equation (6)

Ad	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$
$\mathfrak{S}_1$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3 - \epsilon_1\mathfrak{S}_2$	$\mathfrak{S}_4 + 3\epsilon_1\mathfrak{S}_1$
$\mathfrak{S}_2$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4 + \epsilon_2\mathfrak{S}_2$
$\mathfrak{S}_3$	$\mathfrak{S}_1 + \epsilon_3\mathfrak{S}_2$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4 - 2\epsilon_3\mathfrak{S}_3$
$\mathfrak{S}_4$	$e^{-3\epsilon_4}\mathfrak{S}_1$	$e^{-\epsilon_4}\mathfrak{S}_2$	$e^{2\epsilon_4}\mathfrak{S}_3$	$\mathfrak{S}_4$

Suppose, we consider $\mathfrak{S} = a_1\mathfrak{S}_1 + a_2\mathfrak{S}_2 + a_3\mathfrak{S}_3 + a_4\mathfrak{S}_4$ to be the general element belonging to the Lie algebra L_4 spanned by (12). Thus,

$$\begin{aligned} Ad\left(e^{\epsilon_1\mathfrak{S}_1}\right)\mathfrak{S} &= a_1 Ad\left(e^{\epsilon_1\mathfrak{S}_1}\right)\mathfrak{S}_1 + a_2 Ad\left(e^{\epsilon_1\mathfrak{S}_1}\right)\mathfrak{S}_2 + a_3 Ad\left(e^{\epsilon_1\mathfrak{S}_1}\right)\mathfrak{S}_3 + a_4 Ad\left(e^{\epsilon_1\mathfrak{S}_1}\right)\mathfrak{S}_4 \\ &= \left(a_1 + 3a_4\epsilon_1\right)\mathfrak{S}_1 + \left(a_2 - a_3\epsilon_1\right)\mathfrak{S}_2 + a_3\mathfrak{S}_3 + a_4\mathfrak{S}_4 \\ &= \left[\mathfrak{S}_1\,\mathfrak{S}_2\,\mathfrak{S}_3\,\mathfrak{S}_4\right]\cdot M_1^{\epsilon_1}\cdot\left[a_1\,a_2\,a_3\,a_4\right]^T, \end{aligned}$$

where matrix $M_1^{\epsilon_1}$ is calculated in this space as

$$M_1^{\epsilon_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\varepsilon_1 & 1 & 0 \\ 3\varepsilon_1 & 0 & 0 & 1 \end{pmatrix}.$$

In the same vein, other matrices $M_2^{\epsilon_2}$, $M_3^{\epsilon_3}$ and $M_4^{\epsilon_4}$ can be achieved through the application of adjoint action of $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_4$, to $\mathfrak{S}$, and these furnish

$$M_2^{\epsilon_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \varepsilon_2 & 0 & 1 \end{pmatrix}, \quad M_3^{\epsilon_3} = \begin{pmatrix} 1 & \varepsilon_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2\varepsilon_3 & 1 \end{pmatrix}, \quad M_4^{\epsilon_4} = \begin{pmatrix} e^{-3\varepsilon_4} & 0 & 0 & 0 \\ 0 & e^{-\varepsilon_4} & 0 & 0 \\ 0 & 0 & e^{2\varepsilon_4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, one has the general adjoint transformation matrix M related to the (1+1)D-GeoKdVe (6) with power-law nonlinearities as $M\left(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\right) = M_1^{\epsilon_1}\,M_2^{\epsilon_2}\,M_3^{\epsilon_3}\,M_4^{\epsilon_4}$ and this leads to

$$M = \begin{pmatrix} e^{-3\varepsilon_4} & e^{-\varepsilon_4}\varepsilon_3 & 0 & 0 \\ 0 & e^{-\varepsilon_4} & 0 & 0 \\ 0 & -e^{-\varepsilon_4}\varepsilon_1 & e^{2\varepsilon_4} & 0 \\ 3e^{-3\varepsilon_4}\varepsilon_1 & e^{-\varepsilon_4}\left(\varepsilon_2 + 3\varepsilon_1\varepsilon_3\right) & -2e^{2\varepsilon_4}\varepsilon_3 & 1 \end{pmatrix}.$$

Meanwhile, we engage the method explicated in Hu et al. (2015); Olver (1993) and on extending equation (13) to some function $\Delta_i = \Delta_i(a_1, \ldots, a_4, b_1, \ldots, b_4)$, where $b_1, \ldots, b_4$ are some arbitrary constants (see Hu et al. (2015)), we calculate $\Delta_1, \ldots, \Delta_4$, as $\Delta_1 = 3a_1b_4 - 3a_4b_1, \Delta_2 = a_3b_1 - a_4b_2 - a_1b_3 + a_2b_4, \Delta_3 = 2a_4b_3 - 2a_3b_4, \Delta_4 = 0$. We obtain the values of $b_i, i = 1, 2, 3, 4$, using the relation: $\Delta_1\partial\Omega/\partial a_1 + \Delta_2\partial\Omega/\partial a_2 + \Delta_3\partial\Omega/\partial a_3 + \Delta_4\partial\Omega/\partial a_4 = 0$, where function $\Omega = \Omega(a_1, \ldots, a_4)$. Solving the obtained equation gives $\Omega(a_1, \ldots, a_4) = G(a_4)$. We utilize the adjoint transformation equation $M\left(a_1, \ldots, a_4\right) = (q_1, \ldots, q_4)$, where $q_1, \ldots, q_4$ are accordingly equivalent to the elements of the adjoint equations (Hu et al., 2015).

Therefore, after imploring the above given information and performing some calculations, various optimal representatives were attained and combining these obtained representatives, we have the theorem given below;

Theorem 3. *An optimal system of one-dimensional subalgebras associated to the generalized (1+1)D-GeoKdVe (6) is purveyed in the following operators:* $\mathfrak{S}_1, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_3 + \mathfrak{S}_1, \mathfrak{S}_1 - \mathfrak{S}_3, \mathfrak{S}_3 + c_0\mathfrak{S}_2 + \mathfrak{S}_1$*, where* $c_0 \in \{-1, 1\}$.

2.3 Symmetry reductions and solutions of (6)

Utilizing the one-dimensional optimal system of subalgebras established in the preceding subsection, we will proceed with the symmetry reduction process to derive the exact group-invariant solutions for equation (6) by first reducing the model to ordinary differential equation (ODE).

2.3.1 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_1$

The characteristic equations related to $\mathfrak{S}_1$ are engendered as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}, \tag{14}$$

which yields two invariants $J_1 = x$ and $J_2 = u$. Thus $J_2 = f(J_1)$. Definitely

$$u = f(x). \tag{15}$$

Invoking the above value of u in equation (6) gives

$$af'(x) + bf(x)f'(x) + cf'''(x) = 0.$$

2.3.2 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_3$

The Lagrangian system associated with $\mathfrak{S}_3$ solves to give two invariants $J_1 = t$ and $J_2 = u - x/bt$. Thus $J_2 = f(J_1)$, which gives the function $u = f(t) + x/bt$. One substitutes the value of u in (6) and so it yields

$$btf'(t) + bf(t) + a = 0. \tag{16}$$

Solving the differential equation leads to

$$u(x,t) = \frac{C_0}{t}t - \frac{a}{b},$$

where C_0 is an integration constants.

2.3.3 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_4$

Now considering the third symmetry generated by the optimal system of one-dimensional subalgebras, one follows the usual process and achieve

$$\xi = \frac{x}{\sqrt[3]{t}}, \text{ and } f(\xi) = \left(u + \frac{a}{b}\right) t^{2/3},$$

which transforms (1+1)D-GeoKdVe (6) to third-order nonlinear ordinary differential equation (NODE)

$$3bf(\xi)f'(\xi) - \xi f'(\xi) - 2f(\xi) + 3cf'''(\xi) = 0. \tag{17}$$

One can observe that (17) is difficult to integrate. Therefore, in order to gain analytic solution to the nonlinear equation, we invoke the power series approach.

Analytic power series solution of (1+1)D-GeoKdVe (6)

The analytical power series solution of model (6) is attained in this subsection by invoking power series technique (Adeyemo & Khalique, 2023). In a bid to put the aforementioned to action, one seeks a series solution to solve NODE (17), in the structure

$$f(\xi) = \sum_{m=0}^{\infty} c_m \xi^m \tag{18}$$

in which constant parameters $c_m, m = 0, 1, 2, 3, 4, \ldots, \infty$, are required to be known. Meanwhile, the demanded derivatives in (17) are explicated in this context as

$$f'(\xi) = \sum_{m=0}^{\infty} mc_m\xi^{m-1}, \quad f''(\xi) = \sum_{m=0}^{\infty} m(m-1)c_m\xi^{m-2},$$
$$f'''(\xi) = \sum_{m=0}^{\infty} m(m-1)(m-2)c_m\xi^{m-3}. \tag{19}$$

Adequate replacement of the associated terms in (17) with summation expressions enunciated in (18) together with (19), establishes

$$\begin{aligned}
&360cc_6\xi^3+180cc_5\xi^2+72cc_4\xi+18cc_3+3c\sum_{m=4}^{\infty}(m+1)(m+2)(m+3)c_{m+3}\xi^m\\
&-4c_4\xi^4-3c_3\xi^3-2c_2\xi^2-c_1\xi-\sum_{m=4}^{\infty}mc_m\xi^m+12bc_3c_4\xi^6+9bc_3^2\xi^5+12bc_2c_4\xi^5\\
&+15bc_2c_3\xi^4+12bc_1c_4\xi^4+6bc_2^2\xi^3+12bc_1c_3\xi^3+12bc_0c_4\xi^3+9bc_1c_2\xi^2+9bc_0c_3\xi^2\\
&+3bc_1^2\xi+6bc_0c_2\xi+3bc_0c_1+3b\sum_{m=4}^{\infty}\sum_{k=0}^{m}c_kc_{m-k+1}\xi^m-2c_3\xi^3-2c_2\xi^2-2c_1\xi\\
&-2c_0-2\sum_{m=4}^{\infty}c_m\xi^m=0, \qquad (20)
\end{aligned}$$

from which for arbitrary c_0, c_1, and c_2, one could obtain the results; viz

$$c_3=\frac{c_0}{9c}-\frac{bc_0c_1}{6c}; \qquad (21)$$

$$c_4=\frac{c_1}{24c}-\frac{bc_1^2}{24c}-\frac{bc_0c_2}{12c}; \qquad (22)$$

$$c_5=\frac{b^2c_1c_0^2}{120c^2}-\frac{bc_0^2}{180c^2}-\frac{bc_1c_2}{20c}+\frac{c_2}{45c}; \qquad (23)$$

$$c_6=\frac{b^2c_2c_0^2}{360c^2}+\frac{b^2c_1^2c_0}{144c^2}-\frac{bc_1c_0}{135c^2}-\frac{bc_2^2}{60c}+\frac{c_0}{648c^2}. \qquad (24)$$

Besides, in general, for $m\geq 4$, one achieves the recurrence relation purveyed as

$$c_{m+3}=\frac{1}{3c(m+1)(m+2)(m+3)}\left\{2c_m-3b\sum_{k=0}^{m}c_kc_{m-k+1}+mc_m\right\}. \qquad (25)$$

Applying the recursion formula (25), successive terms $c_m, m=7,\dots,\infty$, could also be decided in a unique way. Thus, power series solution to (17) can be written as

$$\begin{aligned}
f(\xi)=&\,c_0+c_1\xi+c_2\xi^2+\left(\frac{c_0}{9c}-\frac{bc_0c_1}{6c}\right)\xi^3+\left(\frac{c_1}{24c}-\frac{bc_1^2}{24c}-\frac{bc_0c_2}{12c}\right)\xi^4\\
&+\left(\frac{b^2c_1c_0^2}{120c^2}-\frac{bc_0^2}{180c^2}-\frac{bc_1c_2}{20c}+\frac{c_2}{45c}\right)\xi^5\\
&+\left(\frac{b^2c_2c_0^2}{360c^2}+\frac{b^2c_1^2c_0}{144c^2}-\frac{bc_1c_0}{135c^2}-\frac{bc_2^2}{60c}+\frac{c_0}{648c^2}\right)\xi^6+\sum_{m=3}^{\infty}c_{m+3}\xi^{m+3}, \qquad (26)
\end{aligned}$$

Hence, the analytic power series solution to (1+1)D-GeoKdVe (6), explicates as

$$\begin{aligned}
u(x,t)=&\frac{1}{\sqrt[3]{t^2}}\Bigg\{c_0+c_1\left(\frac{x}{\sqrt[3]{t}}\right)+c_2\left(\frac{x}{\sqrt[3]{t}}\right)^2+\left(\frac{c_0}{9c}-\frac{bc_0c_1}{6c}\right)\left(\frac{x}{\sqrt[3]{t}}\right)^3+\Bigg(\frac{c_1}{24c}\\
&-\frac{bc_1^2}{24c}-\frac{bc_0c_2}{12c}\Bigg)\left(\frac{x}{\sqrt[3]{t}}\right)^4+\left(\frac{b^2c_1c_0^2}{120c^2}-\frac{bc_0^2}{180c^2}-\frac{bc_1c_2}{20c}+\frac{c_2}{45c}\right)\left(\frac{x}{\sqrt[3]{t}}\right)^5\\
&+\left(\frac{b^2c_2c_0^2}{360c^2}+\frac{b^2c_1^2c_0}{144c^2}-\frac{bc_1c_0}{135c^2}-\frac{bc_2^2}{60c}+\frac{c_0}{648c^2}\right)\left(\frac{x}{\sqrt[3]{t}}\right)^6 \qquad (27)\\
&+\frac{1}{3c}\sum_{m=3}^{\infty}\frac{m!}{(m-3)!}\left\{2c_m-3b\sum_{k=0}^{m}c_kc_{m-k+1}+mc_m\right\}\left(\frac{x}{\sqrt[3]{t}}\right)^{m+3}\Bigg\}-\frac{a}{b}.
\end{aligned}$$

which is the result attained by reverting to the original variables in t, x and u.

2.3.4 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_1 + \mathfrak{S}_3$

The next member of the optimal system of one-dimensional subalgebras $\mathfrak{S}_1 + \mathfrak{S}_3$ purveys the invariant and the associated group invariant respectively given as

$$\xi = x - \frac{1}{2}bt^2, \quad \text{and} \quad f(\xi) = u - t.$$

Utilizing the above, one successfully transforms (6) to the NODE

$$af'(\xi) + bf(\xi)f'(\xi) + cf'''(\xi) + 1 = 0. \tag{28}$$

2.3.5 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_1 - \mathfrak{S}_3$

Now we focus on $\mathfrak{S}_1 - \mathfrak{S}_3$, adopting the usual Lie theoretic approach, we gain

$$\xi = \frac{1}{2}bt^2 + x, \quad \text{with} \quad f(\xi) = u + t.$$

The function further reduces (1+1)D-GeoKdVe (6) to the ODE

$$af'(\xi) + bf(\xi)f'(\xi) + cf'''(\xi) - 1 = 0. \tag{29}$$

2.3.6 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_3 + c_0\mathfrak{S}_2 + \mathfrak{S}_1$

In seeking further to reduce (6), symmetry operator $\mathfrak{S}_3 + c_0\mathfrak{S}_2 + \mathfrak{S}_1$ with $c_0 \neq 0$ is engaged. Thus we achieve the invariant

$$\xi = x - c_0 t - \frac{1}{2}bt^2, \quad \text{where} \quad f(\xi) = u - t.$$

Eventually, application of the obtained expression of u in (6) furnishes NODE

$$af'(\xi) - c_0 f'(\xi) + bf(\xi)f'(\xi) + cf'''(\xi) + 1 = 0. \tag{30}$$

Remark 1. *We observe that combinations of $\mathfrak{S}_1$ and $\mathfrak{S}_3$ are a special case of $\mathfrak{S}_3 + c_0\mathfrak{S}_2 + \mathfrak{S}_1$, so we now seek the exact power series solution of the NODE under the latter and set aside that of the former.*

Acting on Remark 1, we seek to find the exact power series solution of NODE (30) based on the procedure earlier explicated. Therefore substituting series derivative (19) into (30) gives the equation presented in this regard as

$$\begin{aligned}
&4ac_4\xi^3 + 3ac_3\xi^2 + 2ac_2\xi + ac_1 - 4c_0c_4\xi^3 - 3c_0c_3\xi^2 - 2c_0c_2\xi - c_0c_1 + (a - c_0) \\
&\times \sum_{m=4}^{\infty}(m+1)c_{m+1}\xi^m + 4bc_3c_4\xi^6 + 3bc_3^2\xi^5 + 4bc_2c_4\xi^5 + 5bc_2c_3\xi^4 + 4bc_1c_4\xi^4 \\
&+ 2bc_2^2\xi^3 + 4bc_1c_3\xi^3 + 4bc_0c_4\xi^3 + 3bc_1c_2\xi^2 + 3bc_0c_3\xi^2 + bc_1^2\xi + 2bc_0c_2\xi \\
&+ bc_0c_1 + b\sum_{m=4}^{\infty}\sum_{j=0}^{m} c_j c_{m-j+1}\xi^m + 120cc_6\xi^3 + 60cc_5\xi^2 + 24cc_4\xi + 6cc_3 \\
&+ c\sum_{m=4}^{\infty}(m+1)(m+2)(m+3)c_{m+3}\xi^m = 0.
\end{aligned} \tag{31}$$

In general, for $m \geq 4$, one achieves the recurrence relation purveyed as

$$c_{m+3} = -\frac{1}{c(m+1)(m+2)(m+3)}\left\{(a - c_0)(m+1)c_{m+1} + b\sum_{j=0}^{m} c_j c_{m-j+1}\right\}. \tag{32}$$

In addition, for arbitrary c_0, c_1, and c_2, one could secure the values of $c_3, \ldots, c_6$ as

$$c_3 = \frac{c_0c_1}{6c} - \frac{ac_1}{6c} - \frac{bc_0c_1}{6c} - \frac{1}{6c}; \tag{33}$$

$$c_4 = \frac{c_0c_2}{12c} - \frac{ac_2}{12c} - \frac{bc_1^2}{24c} - \frac{bc_0c_2}{12c}; \tag{34}$$

$$c_5 = \frac{a^2c_1}{120c^2} + \frac{abc_0c_1}{60c^2} - \frac{ac_0c_1}{60c^2} + \frac{a}{120c^2} + \frac{b^2c_0^2c_1}{120c^2} + \frac{bc_0}{120c^2} - \frac{bc_0^2c_1}{60c^2} - \frac{bc_1c_2}{20c} - \frac{c_0}{120c^2} + \frac{c_0^2c_1}{120c^2}; \tag{35}$$

$$c_6 = \frac{a^2c_2}{360c^2} + \frac{abc_1^2}{144c^2} + \frac{abc_0c_2}{180c^2} - \frac{ac_0c_2}{180c^2} + \frac{b^2c_0c_1^2}{144c^2} + \frac{b^2c_0^2c_2}{360c^2} - \frac{bc_0c_1^2}{144c^2} + \frac{bc_1}{180c^2} - \frac{bc_0^2c_2}{180c^2} - \frac{bc_2^2}{60c} + \frac{c_0^2c_2}{360c^2}. \tag{36}$$

Utilizing (33)–(36), power series solution to (30) can be expressed as

$$\begin{aligned} f(\xi) = {} & c_0 + c_1\xi + c_2\xi^2 + \left(\frac{c_0c_1}{6c} - \frac{ac_1}{6c} - \frac{bc_0c_1}{6c} - \frac{1}{6c}\right)\xi^3 + \left(\frac{c_0c_2}{12c} - \frac{ac_2}{12c} - \frac{bc_1^2}{24c} \right. \\ & \left. - \frac{bc_0c_2}{12c}\right)\xi^4 + \left(\frac{a^2c_1}{120c^2} + \frac{abc_0c_1}{60c^2} - \frac{ac_0c_1}{60c^2} + \frac{a}{120c^2} + \frac{b^2c_0^2c_1}{120c^2} + \frac{bc_0}{120c^2} - \frac{bc_0^2c_1}{60c^2} \right. \\ & \left. - \frac{bc_1c_2}{20c} - \frac{c_0}{120c^2} + \frac{c_0^2c_1}{120c^2}\right)\xi^5 + \left(\frac{a^2c_2}{360c^2} + \frac{abc_1^2}{144c^2} + \frac{abc_0c_2}{180c^2} - \frac{ac_0c_2}{180c^2} \right. \\ & \left. + \frac{b^2c_0c_1^2}{144c^2} + \frac{b^2c_0^2c_2}{360c^2} - \frac{bc_0c_1^2}{144c^2} + \frac{bc_1}{180c^2} - \frac{bc_0^2c_2}{180c^2} - \frac{bc_2^2}{60c} + \frac{c_0^2c_2}{360c^2}\right)\xi^6 \\ & + \sum_{m=4}^{\infty} c_{m+3}\xi^{m+3}. \end{aligned} \tag{37}$$

Consequently, the analytic power series solution to (1+1)D-GeoKdVe (6), gives

$$\begin{aligned} u(x,t) = {} & \left\{ c_0 + c_1\left[x - c_0t - \frac{1}{2}bt^2\right] + c_2\left[x - c_0t - \frac{1}{2}bt^2\right]^2 + \left(\frac{c_0c_1}{6c} - \frac{ac_1}{6c} - \frac{bc_0c_1}{6c} \right.\right. \\ & \left. - \frac{1}{6c}\right)\left[x - c_0t - \frac{1}{2}bt^2\right]^3 + \left(\frac{c_0c_2}{12c} - \frac{ac_2}{12c} - \frac{bc_1^2}{24c} - \frac{bc_0c_2}{12c}\right)\left[x - c_0t - \frac{1}{2}bt^2\right]^4 \\ & + \left(\frac{a^2c_1}{120c^2} + \frac{abc_0c_1}{60c^2} - \frac{ac_0c_1}{60c^2} + \frac{a}{120c^2} + \frac{b^2c_0^2c_1}{120c^2} + \frac{bc_0}{120c^2} - \frac{bc_0^2c_1}{60c^2} - \frac{bc_1c_2}{20c} \right. \\ & \left. - \frac{c_0}{120c^2} + \frac{c_0^2c_1}{120c^2}\right)\left[x - c_0t - \frac{1}{2}bt^2\right]^5 + \left(\frac{a^2c_2}{360c^2} + \frac{abc_1^2}{144c^2} + \frac{abc_0c_2}{180c^2} - \frac{ac_0c_2}{180c^2} \right. \\ & \left. + \frac{b^2c_0c_1^2}{144c^2} + \frac{b^2c_0^2c_2}{360c^2} - \frac{bc_0c_1^2}{144c^2} + \frac{bc_1}{180c^2} - \frac{bc_0^2c_2}{180c^2} - \frac{bc_2^2}{60c} + \frac{c_0^2c_2}{360c^2}\right) \\ & \times \left[x - c_0t - \frac{1}{2}bt^2\right]^6 + \frac{1}{c}\sum_{m=4}^{\infty} \frac{m!}{(m-3)!}\left\{(a - c_0)(m+1)c_{m+1} \right. \\ & \left.\left. + b\sum_{j=0}^{m} c_jc_{m-j+1}\right\}\left[x - c_0t - \frac{1}{2}bt^2\right]^{m+3}\right\} + t. \end{aligned}$$

Next, we examine the combinations of $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4$ in reducing (6) which will give a more robust invariant solution. This will be referred to as subalgebra $\mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4$.

2.3.7 Symmetry reduction of (6) through subalgebra $\mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4$

Finally, in our quest to look for more solutions of (1+1)D-GeoKdVe (6), we consider the symmetry operator $\mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4$. Taking the usual steps, one attains the invariants

$$\xi = -\frac{bt - 2bx - 1}{2b\sqrt[3]{(3bt+1)}}, \quad \text{whereas} \quad f(\xi) = -\left(\frac{1}{2b} - \frac{a}{b} - u\right)\sqrt[3]{(3bt+1)^2}.$$

In engaging the achieved expression of u in (6), NODE furnished is

$$bf(\xi)f'(\xi) - 2bf(\xi) + cf'''(\xi) - b\xi f'(\xi) = 0. \tag{38}$$

In reference to what has been earlier invoked, one solves NODE (38) using power series method and by following the same step as earlier outlined, one achieves the analytic power series solution to (1+1)D-GeoKdVe (6) as

$$\begin{aligned} u(x,t) = {} & \frac{1}{\sqrt[3]{(3bt+1)^2}}\Bigg\{c_0 + c_1\left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right) + c_2\left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^2 \\ & + \left(\frac{bc_0}{3c} - \frac{bc_0c_1}{6c}\right)\left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^3 + \left(\frac{bc_1}{8c} - \frac{bc_1^2}{24c} - \frac{bc_0c_2}{12c}\right) \\ & \times \left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^4 + \left(\frac{b^2c_1c_0^2}{120c^2} - \frac{b^2c_0^2}{60c^2} - \frac{bc_1c_2}{20c} + \frac{bc_2}{15c}\right) \\ & \times \left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^5 + \left(\frac{b^2c_0c_1^2}{144c^2} + \frac{b^2c_0}{72c^2} - \frac{b^2c_0c_1}{45c^2} + \frac{b^2c_0^2c_2}{360c^2}\right. \\ & \left. - \frac{bc_2^2}{60c}\right)\left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^6 + \frac{1}{3}\sum_{m=3}^{\infty}\frac{m!}{(m-3)!}\Bigg\{2c_m - b\sum_{k=0}^{m}c_kc_{m-k+1} \\ & + bmc_m\Bigg\}\left(-\frac{bt-2bx-1}{2b\sqrt[3]{(3bt+1)}}\right)^{m+3}\Bigg\} + \frac{1}{2b} - \frac{a}{b}. \end{aligned} \tag{39}$$

Remark 2. *It is worthy of note that, the analytic power series approach just rendered in solving NODE (17), (30) and (38) can be utilized in entrenching exact solution. Having demonstrated the efficiency of the technique in this regard by applying it to retrieve solution to the most difficult NODE found in this work, one establishes the aforementioned. Therefore, in addendum, one could employ the approach in attaining exact series solutions to any of the other NODE here and as such to any NODE of any nature and number of terms (linear and nonlinear).*

2.4 Travelling wave solutions of (1+1)D-GeoKdVe (6)

Suppose one takes into account a linear combination of two members of the four-dimensional Lie algebra computed for (1+1)D-GeoKdVe (6) earlier as $\mathfrak{S} = \mathfrak{S}_1 + \nu\mathfrak{S}_2$. The symmetry $\mathfrak{S}$ produces two invariants entrenched as

$$\xi = x - \nu t \quad \text{and} \quad U = u,$$

which yield the group-invariant solution $U = U(\xi)$ with ξ as the new independent variable. Utilizing the above, one successfully transforms equation (6) to the third-order NODE

$$aU'(\xi) + bU(\xi)U'(\xi) + cU''' - \nu U'(\xi) = 0. \tag{40}$$

We first apply the direct integration approach to the NODE (40) to attain some solutions of (1+1)D-GeoKdVe (6). Thereafter, a standard technique will be invoked to secure more general solutions.

Solutions of (6) through direct integration approach

Using the direct integration approach, two cases are going to be looked into with a view to attaining two types of solutions to (6) and these solutions are periodic and bright soliton. Integration of (40) with reference to independent variable ξ gives

$$U''(\xi) + \frac{b}{2c}U(\xi)^2 - \alpha U(\xi) + K_1 = 0, \tag{41}$$

where $\alpha = (\nu - a)/c$, $K_1 = K_0/c$, with $K_0 \neq 0$, an integration constant.

Case 1: Weierstrass function solution

One can integrate NODE (41) (whereby $K_0 \neq 0$), easily by multiplying it by $U'(\xi)$ first. Therefore one gets

$$U'(\xi)^2 + \frac{b}{3c}U(\xi)^3 - \alpha U(\xi)^2 + 2K_1 U(\xi) + 2K_2 = 0, \tag{42}$$

where K_2 is an integration constant. One retrieves a periodic solution to NODE (40) in terms of Weierstrass function (Kudryashov (2019)) by setting a transformation as

$$U(\xi) = \frac{c}{b}\left\{\alpha - 12\wp(\xi)\right\}. \tag{43}$$

Hence, one reckons (42) as NODE with Weierstrass elliptic function (Gradshteyn & Ryzhik (2007); Akhiezer (1990))

$$\wp'(\xi)^2 - 4\wp(\xi)^3 + g_1\wp(\xi) + g_2 = 0 \tag{44}$$

with the included Weierstrass elliptic invariants g_1 as well as g_2 expressed as

$$g_1 = \frac{1}{b}\left\{24K_0 - \frac{12(\nu - a)^2}{b}\right\}, \quad g_2 = \frac{1}{b}\left\{24K_2 c + \frac{24K_0(\nu - a)}{b} - \frac{8(\nu - a)^3}{b^2}\right\}.$$

Hence, the solution to differential equation (42) produces in this regard

$$\begin{aligned} U(\xi) = \frac{1}{b}(\nu - a) + \wp\Bigg(&\frac{1}{2\sqrt{3}}\sqrt{\left|-\frac{b}{c}\right|}\xi; \frac{24K_0}{b} - \frac{12(\nu - a)^2}{b^2}, \frac{24cK_2}{b} \\ &-\frac{8(\nu - a)^3}{b^3} + \frac{24K_0(\nu - a)}{b^2}\Bigg). \end{aligned} \tag{45}$$

Bearing in mind (43) alongside (44) and reverting to previous variables, one has

$$\begin{aligned} u(x,t) = \frac{1}{b}(\nu - a) + \wp\Bigg(&\frac{1}{2\sqrt{3}}\sqrt{\left|-\frac{b}{c}\right|}(x - \nu t); \frac{24K_0}{b} - \frac{12(\nu - a)^2}{b^2}, \frac{24cK_2}{b} \\ &-\frac{8(\nu - a)^3}{b^3} + \frac{24K_0(\nu - a)}{b^2}\Bigg), \end{aligned} \tag{46}$$

where $\wp$ represents Weierstrass function (Gradshteyn & Ryzhik (2007)). The wave dynamics of the Weierstrass solution (46) is plotted in Figure 1.

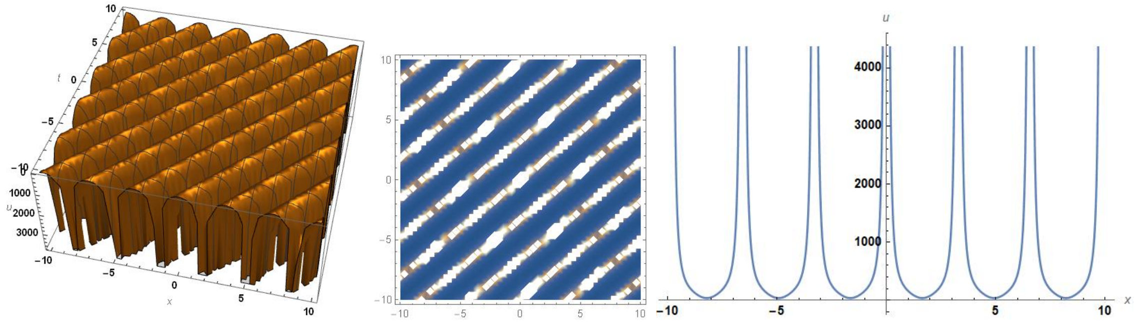

Figure 1: The singular periodic shape wave structure of Weierstrass solution (46) is plotted using the dissimilar constant values $a = 0.1$, $b = 0.1$, $c = 1$, $\nu = 10$, $K_0 = 8$, $K_2 = 10$, in the interval $-10 \leq t, x \leq 10$. The singular points are prevalent within the interval of the solution.

Case 2: Bright soliton solution

Here, one contemplates another case of NODE (41) whereby $K_0 = 0$. Thus, taking this and integrating the result as earlier demonstrated gives

$$\frac{a-\nu}{2}U(\xi)^2 + \frac{b}{6}U(\xi)^3 + \frac{c}{2}U'(\xi)^2 = C_0, \tag{47}$$

where C_0 is an integration constant. On letting $C_0 = 0$, in the first-order NODE (47), and solving the equation furnishes the soliton solution of model (6) as

$$u(x,t) = \frac{1}{b}\left\{3(a-\nu)\right\} \operatorname{sech}^2\left\{-\frac{1}{2}\left(C_1\sqrt{3\,(a-\nu)} + \sqrt{\frac{(\nu-a)}{c}}[x-\nu t]\right)\right\}, \tag{48}$$

where C_1 is an integration constant. The wave structure of the bright soliton solution (48) is the plots explicated in Figure 2.

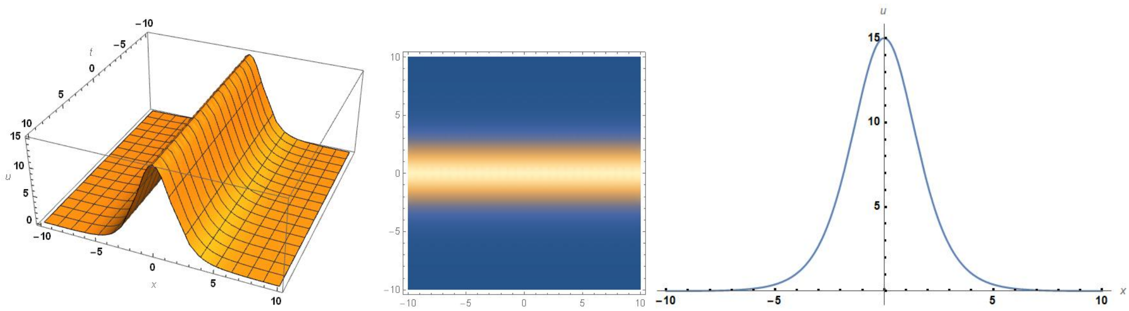

Figure 2: Bell-shaped wave structure of hyperbolic secant function solution (48) using the data values $a = 0.5$, $b = 0.1$, $c = -0.5$, $\nu = 0$, and $C_1 = 0$ in the interval $-10 \leq t, x \leq 10$.

Next, in order that one might secure various more interesting exact general solutions to (1+1)D-GeoKdVe (6) using the extended Jacobi elliptic function expansion approach.

Solutions of (6) using extended Jacobi elliptic function expansion approach

Here, we utilize the extended Jacobi elliptic function expansion technique Khalique & Adeyemo (2020a) to secure various exact general solitons and travelling wave solutions of (6). Suppose the third-order NODE (40) owns a solution of the structure

$$U(\xi) = \sum_{i=-m}^{m} A_i R(\xi)^i, \tag{49}$$

where we aim to obtain the value of positive integer m by adopting balancing procedure, see (Zhou et al. (1996)). In this regard, solving NODE (40), the expected elliptic equation to be invoked are

$$R'(\xi)+\sqrt{\{(1-R^2(\xi))(1-\omega+\omega R^2(\xi))\}}=0, \tag{50}$$
$$R'(\xi)-\sqrt{\{(1-R^2(\xi))(1-\omega R^2(\xi))\}}=0, \tag{51}$$
$$R'(\xi)+\sqrt{\{(1-R^2(\xi))(\omega-1+R^2(\xi))\}}=0, \tag{52}$$

whose solutions are expressed accordingly with regards to the Jacobi elliptic cosine, sine as well as delta amplitude functions, respectively, as

$$R(\xi)=\,\mathrm{cn}(\xi|\omega),\;\; R(\xi)=\,\mathrm{sn}(\xi|\omega),\;\text{ and }R(\xi)=\,\mathrm{dn}(\xi|\omega). \tag{53}$$

Therefore, one contemplates the subsequent solitary wave solution directions.

Cnoidal wave solutions

Here, contemplating the NODE (40), the balancing procedure produces $m=2$ and then (49) assumes structure

$$U(\xi)=A_{-2}R(\xi)^{-2}+A_{-1}R(\xi)^{-1}+A_0+A_1R(\xi)+A_2R(\xi)^2. \tag{54}$$

Substituting the value of $U(\xi)$ from (54) into (40) in conjunction with (50), we secure an algebraic equation, which splits over various powers of $R(\xi)$ and yields a system of eleven algebraic equations:

$$\begin{aligned}
&b\omega\,A_2^2-12\,c\omega^2A_2=0,\\
&b\omega\,A_1A_2-2\,c\omega^2A_1=0,\\
&24\,c\omega^2A_{-2}-2\,b\omega\,A_{-2}^2+2\,bA_{-2}^2-48\,c\omega\,A_{-2}+24\,cA_{-2}=0,\\
&6\,c\omega^2A_{-1}-3\,b\omega\,A_{-2}A_{-1}+3\,bA_{-2}A_{-1}-12\,c\omega\,A_{-1}+6\,cA_{-1}=0,\\
&2\,b\omega\,A_0A_2+b\omega\,A_1^2-4\,b\omega\,A_2^2+64\,c\omega^2A_2+2\,a\omega\,A_2+2\,bA_2^2-32\,c\omega\,A_2\\
&-2\,\nu\,\omega\,A_2=0,\\
&b\omega\,A_{-1}A_2+b\omega\,A_0A_1-6\,b\omega\,A_1A_2+14\,c\omega^2A_1+a\omega\,A_1+3\,bA_1A_2-7\,c\omega\,A_1\\
&-\nu\,\omega\,A_1=0,\\
&-2\,b\omega\,A_{-2}^2+4\,b\omega\,A_{-2}A_0+2\,b\omega\,A_{-1}^2+56\,c\omega^2A_{-2}+4\,a\omega\,A_{-2}-2\,bA_{-2}A_0\\
&-bA_{-1}^2-56\,c\omega\,A_{-2}-4\,\nu\,\omega\,A_{-2}-2\,aA_{-2}+8\,cA_{-2}+2\,\nu\,A_{-2}=0,\\
&4\,b\omega\,A_{-2}^2-2\,b\omega\,A_{-2}A_0-b\omega\,A_{-1}^2-64\,c\omega^2A_{-2}-2\,a\omega\,A_{-2}-2\,bA_{-2}^2+2\,bA_{-2}A_0\\
&+bA_{-1}^2+96\,c\omega\,A_{-2}+2\,\nu\,\omega\,A_{-2}+2\,aA_{-2}-32\,cA_{-2}-2\,\nu\,A_{-2}=0,\\
&-4\,b\omega\,A_0A_2-2\,b\omega\,A_1^2+2\,b\omega\,A_2^2-56\,c\omega^2A_2-4\,a\omega\,A_2+2\,bA_0A_2+b{A_1}^2\\
&-2\,bA_2^2+56\,c\omega\,A_2+4\,\nu\,\omega\,A_2+2\,aA_2-8\,cA_2-2\,\nu\,A_2=0,
\end{aligned}$$

$$\begin{aligned}
&6\,b\omega\,A_{-2}A_{-1} - b\omega\,A_{-2}A_1 - b\omega\,A_{-1}A_0 - 14\,c\omega^2 A_{-1} - a\omega\,A_{-1} - 3\,bA_{-2}A_{-1}\\
&+ bA_{-2}A_1 + bA_{-1}A_0 + 21\,c\omega\,A_{-1} + \nu\,\omega\,A_{-1} + aA_{-1} - 7\,cA_{-1} - \nu\,A_{-1} = 0,\\
&2\,b\omega\,A_0A_2 - 2\,b\omega\,A_{-2}A_0 - b\omega\,A_{-1}^2 + b\omega\,A_1^2 - 16\,c\omega^2 A_{-2} + 16\,c\omega^2 A_2 - 2\,a\omega\,A_{-2}\\
&+ 2\,a\omega\,A_2 - 2\,bA_0A_2 - bA_1^2 + 8\,c\omega\,A_{-2} - 24\,c\omega\,A_2 + 2\,\nu\,\omega\,A_{-2} - 2\,\nu\,\omega\,A_2\\
&- 2\,aA_2 + 8\,cA_2 + 2\,\nu\,A_2 = 0,\\
&3\,b\omega\,A_1A_2 - b\omega\,A_{-2}A_1 - b\omega\,A_{-1}A_0 - 2\,b\omega\,A_{-1}A_2 - 2\,b\omega\,A_0A_1 - 2\,c\omega^2 A_{-1}\\
&- 10\,c\omega^2 A_1 - a\omega\,A_{-1} - 2\,a\omega\,A_1 + bA_{-1}A_2 + bA_0A_1 - 3\,bA_1A_2 + c\omega\,A_{-1}\\
&+ 10\,c\omega\,A_1 + \nu\,\omega\,A_{-1} + 2\,\nu\,\omega\,A_1 + aA_1 - cA_1 - \nu\,A_1 = 0,\\
&2\,b\omega\,A_{-1}A_0 - 3\,b\omega\,A_{-2}A_{-1} + 2\,b\omega\,A_{-2}A_1 + b\omega\,A_{-1}A_2 + b\omega\,A_0A_1\\
&+ 10\,c\omega^2 A_{-1} + 2\,c\omega^2 A_1 + 2\,a\omega\,A_{-1} + a\omega\,A_1 - bA_{-2}A_1 - bA_{-1}A_0 - bA_{-1}A_2\\
&- bA_0A_1 - 10\,c\omega\,A_{-1} - 3\,c\omega\,A_1 - 2\,\nu\,\omega\,A_{-1} - \nu\,\omega\,A_1 - aA_{-1} - aA_1\\
&+ cA_{-1} + cA_1 + \nu\,A_{-1} + \nu\,A_1 = 0.
\end{aligned}$$

Employing a computer software package to solve the above system of equations gives

$$A_{-2} = \frac{12c\,(\omega-1)}{b},\;\; A_{-1}=0,\;\; A_0 = -\frac{1}{b}\left(8c\omega + a - 4c - \nu\right),\;\; A_1 = 0,\;\; A_2 = \frac{12c\omega}{b}. \tag{55}$$

Thus, we gain the solution of (6) related to (55) as

$$u(x,t) = \frac{1}{b}\left\{12c\,(\omega-1)\;\mathrm{nc}^2(\xi|\omega) - (8c\omega + a - 4c - \nu) + 12c\omega\,\mathrm{cn}^2(\xi|\omega)\right\} \tag{56}$$

with $\xi = x - \nu t$. We reveal the streaming pattern of periodic solution (56) with Figure 3.

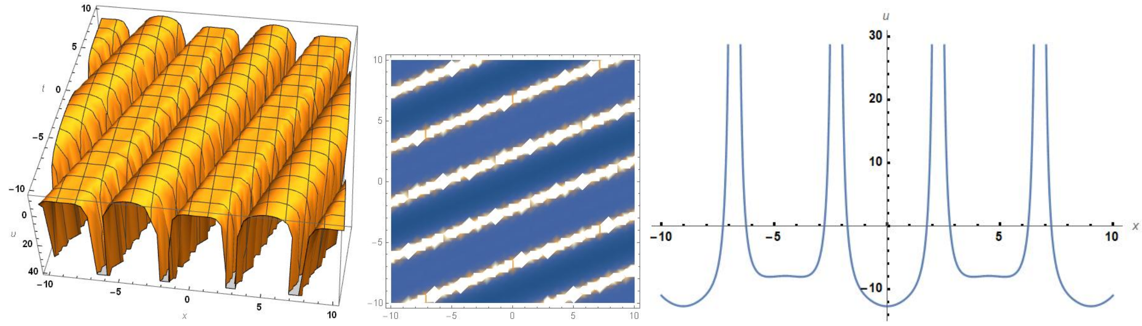

Figure 3: The periodic singular wave structure of cnoidal solution (56) at $a = 5$, $b = 0.40$, $c = -0.1$, $\nu = 0.4$, $\omega = 0.8$, in the interval $-10 \le t, x \le 10$. The singular points in the solution is embedded within $-10 \le t, x \le 10$.

Snoidal wave solutions

As shown earlier, the balancing procedure yields $m = 2$ and so the assumed solution (49) gives the same expression as (54). Following the same procedure as in the previous section but invoking (51), we obtain the values of $A_i, i = -2, \ldots, 2$, as

$$A_{-2} = -\frac{12c}{b},\;\; A_{-1} = 0,\;\; A_0 = -\frac{1}{b}\left(a - 4c\omega - 4c - \nu\right),\;\; A_1 = A_2 = 0. \tag{57}$$

Hence the solution of (6) related to (57) is

$$u(x,t) = \frac{1}{b}\left\{-12c\,\mathrm{ns}^2\left(x - \nu t\middle|\omega\right) + 4c\,(\omega+1) - a + \nu\right\}. \tag{58}$$

The dynamics of solution (58) is portrayed in Figure 4.

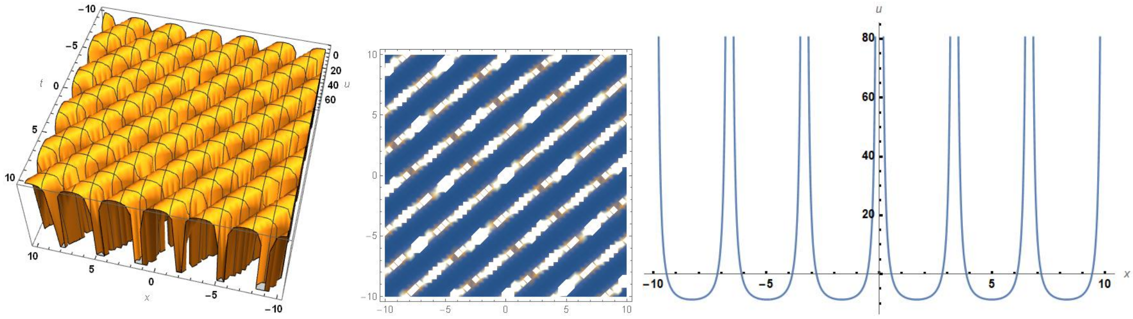

Figure 4: The periodic wave structures of snoidal wave solution 58) with singular points at $-10 \leq t, x \leq 10$ are plotted using the dissimilar values $a = 5$, $b = 0.40$, $c = -0.1$, $\nu = 0.8$, $\omega = 0.2$, in the interval $-10 \leq t, x \leq 10$.

Dnoidal wave solutions

In the same vein, the balancing procedure yields similar value of m as earlier found and so the assumed solution (49) gives the same expression as (54). Following the same steps as in the previous section but using (52), gives values of $A_{-2}, \ldots, A_2$ as

$$A_{-2} = 0, \;\; A_{-1} = 0, \;\; A_0 = -\frac{1}{b}\left(a - 4c\omega + 8c - \nu\right), \;\; A_1 = 0 \;\; A_2 = \frac{12c}{b}. \tag{59}$$

Hence, the solution of (6) corresponding to (59) is

$$u(x,t) = \frac{1}{b}\left\{\nu + 4c\omega - a - 8c + 12c\,\mathrm{dn}^2\left(x - \nu t\middle|\omega\right)\right\}. \tag{60}$$

The dynamics of solution (60) is depicted in Figure 5.

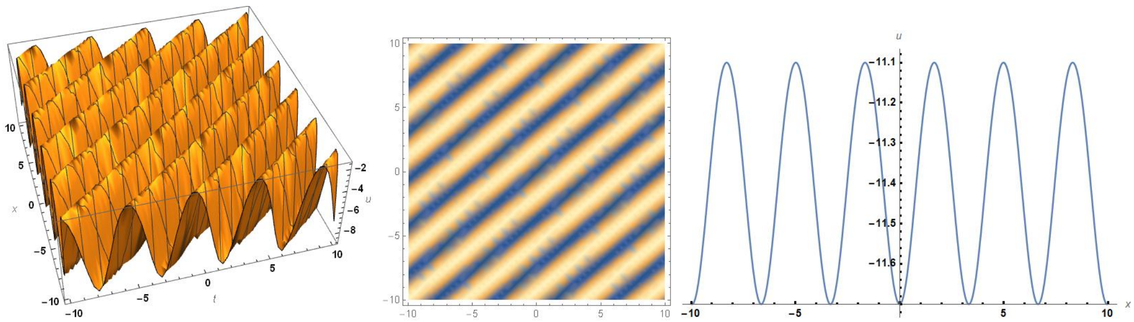

Figure 5: The smooth periodic wave structures of dnoidal wave solution 60) are plotted via the dissimilar constant values $a = 2$, $b = 0.8$, $c = -1$, $\nu = 0.8$, $\omega = 0.5$, in the interval $-10 \leq t, x \leq 10$.

3 Conservation laws of (1+1)D-GeoKdVe (6)

This segment supplies the conserved vectors of the fundamental equation by applying Ibragimov's theorem on preserved vectors, as referenced in prior works Ibragimov (2007); Khalique & Adeyemo (2020b). This is accomplished by utilizing the optimal system of Lie subalgebras.

3.1 Lagrangian and conserved vectors

Consider a system of sth-order α PDEs (Ibragimov, 2007)

$$\Xi_\sigma(x, \Theta, \Theta_{(1)}, \ldots, \Theta_{(s)}) = 0, \quad \sigma = 1, \ldots, \alpha, \tag{61}$$

with κ independent together with α dependent variables given as $x = (x^1, x^2, \ldots, x^\kappa)$ and $\Theta = (\Theta^1, \Theta^2, \ldots, \Theta^\alpha)$. The system of adjoint equations are given by

$$\Xi^*_\sigma(x, \Theta, \Omega, \ldots, \Theta_{(s)}, \Omega_{(s)}) \equiv \frac{\delta(\Omega^\beta \Xi_\beta)}{\delta\Theta^\sigma} = 0, \quad \sigma = 1, \ldots, \alpha, \tag{62}$$

where $\Omega = (\Omega^1, \ldots, \Omega^\alpha)$ are new dependent variables, $\Omega = \Omega(x)$. The operator $\delta/\delta\Theta^\sigma$, expressed for each σ, as

$$\frac{\delta}{\delta\Theta^\sigma} = \frac{\partial}{\partial\Theta^\sigma} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \ldots D_{i_s} \frac{\delta}{\delta\Theta^\sigma_{i_1,i_2,\ldots,i_s}}, \quad i = 1, \ldots, \kappa, \tag{63}$$

is the Euler-Lagrange operator and

$$D_i = \frac{\partial}{\partial x^i} + \Theta^\sigma_i \frac{\partial}{\partial\Theta^\sigma} + \Theta^\sigma_{ij} \frac{\partial}{\partial\Theta^\sigma_j} + \cdots, \ \ i = 1, \ldots, \kappa, \ \ j = 1, \ldots, \kappa \tag{64}$$

is the total differential operator.
Noether's theorem (Noether (1918)) states that suppose the variational integral with the Lagrangian $\mathcal{L}(x, \Theta, \Theta_{(1)})$ is invariant under a group G with a generator defined as

$$\mathcal{W} = \xi^i \left(x, \Theta, \Theta_{(1)}, \ldots\right) \frac{\partial}{\partial x^i} + \eta^\alpha \left(x, \Theta, \Theta_{(1)}, \ldots\right) \frac{\partial}{\partial\Theta^\alpha}, \tag{65}$$

then the vector field $T = (T^1, \ldots, T^n)$ defined by (Sarlet (2010))

$$\begin{aligned} T^k = &\mathcal{L}\tau^k + (\xi^\alpha - \Theta^\alpha_{x^j}\tau^j) \left\{ \frac{\partial\mathcal{L}}{\partial\Theta^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\Big(\frac{\partial\mathcal{L}}{\partial\Theta^\alpha_{x^l x^k}}\Big) \right\} \\ &+ \sum_{l=k}^{n}(\eta^\alpha_l - \Theta^\alpha_{x^l x^j}\tau^j) \frac{\partial\mathcal{L}}{\partial\Theta^\alpha_{x^k x^l}} - B^k, \end{aligned} \tag{66}$$

gives a conservation law for the Euler-Lagrange equations (63), that is, obeys the equation $\mathrm{div}T \equiv D_k(T^k) = 0$ for all solutions of system (61).

The derivatives of Θ with respect to x are defined as

$$\Theta^\alpha_i = D_i(\Theta^\alpha), \quad \Theta^\alpha_{ij} = D_j D_i(\Theta_i), \ldots, \tag{67}$$

where

$$D_i = \frac{\partial}{\partial x^i} + \Theta^\alpha_i \frac{\partial}{\partial\Theta^\alpha} + \Theta^\alpha_{ij} \frac{\partial}{\partial\Theta^\alpha_j} + \cdots, \quad i = 1, ..., n \tag{68}$$

is known as the operator of total differentiation. All the first derivatives Θ^α_i collected together is denoted by $\Theta_{(1)}$, i.e.,

$$\Theta_{(1)} = \{\Theta^\alpha_i\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

In the same vein

$$\Theta_{(2)} = \{\Theta^\alpha_{ij}\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $\Theta_{(3)} = \{\Theta^\alpha_{ijk}\}$ likewise $\Theta_{(4)}$ etc. Since $\Theta^\alpha_{ij} = \Theta^\alpha_{ji}$, $\Theta_{(2)}$ contains only Θ^α_{ij} for $i \leq j$.

An n-tuple $T = (T^1, T^2, \ldots, T^n)$, $1 = 1, 2, \ldots, n$, such that

$$D_i T^i = 0, \tag{69}$$

holds for all solutions of (61) is referred to as the conserved current of the equation.

The formal Lagrangian of the system (61) and its adjoint (62) is given as

$$\mathcal{L} = \Omega^\sigma \Xi_\sigma \left(x, \Theta, \Theta_{(1)}, \ldots, \Theta_{(s)}\right). \tag{70}$$

Theorem 4. *Every nonlocal symmetry, Lie-Bäcklund, as well as Lie point symmetry,*

$$\mathcal{R} = \xi^i \frac{\partial}{\partial x^i} + \varphi^\sigma \frac{\partial}{\partial \Theta^\sigma}, \;\; \xi^i = \xi^i(x, \Theta), \;\; \varphi^\sigma = \varphi^\sigma(x, \Theta), \tag{71}$$

admitted by the system (61) produces a conserved vector for equations (61) and its adjoint (62), with the conserved vectors $T = (T^1, \ldots, T^\kappa)$ *having components* T^i *given by*

$$T^i = \xi^i \mathcal{L} + \Pi^\sigma \left[\frac{\partial \mathcal{L}}{\partial \Theta_i^\sigma} - D_j \frac{\partial \mathcal{L}}{\partial \Theta_{ij}^\sigma} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial \Theta_{ijk}^\sigma} \right) + \cdots \right] + D_j(\Pi^\sigma) \left[\frac{\partial \mathcal{L}}{\partial \Theta_{ij}^\sigma} - D_k \frac{\partial \mathcal{L}}{\partial \Theta_{ijk}^\sigma} + \cdots \right] + D_j D_k(\Pi^\sigma) \frac{\partial \mathcal{L}}{\partial \Theta_{ijk}} + \cdots, \;\; i, j, k = 1, \ldots, \kappa \tag{72}$$

with Lie characteristic function Π^σ *explicated by*

$$\Pi^\sigma = \varphi^\sigma - \xi^j \Theta_j^\sigma, \quad \sigma = 1, \ldots, \alpha, \quad j = 1, \ldots, \kappa. \tag{73}$$

The multiplier Λ of system (61) has the property that

$$D_i T^i = \Lambda^\sigma \Xi_\sigma, \;\; \sigma = 1, \ldots, \alpha. \tag{74}$$

The governing equations for all multipliers involved are obtained from

$$\frac{\delta}{\delta \Theta^\sigma} \left(\Lambda^\sigma \Xi_\sigma \right) = 0, \quad \sigma = 1, \ldots, \alpha. \tag{75}$$

The moment the multipliers are generated via (75), the conserved currents can be procured using (74) as the determining equation. Now, we proceed to compute the symmetries of (6) with a view to utilizing them to calculate the conserved vectors via Theorem 4 with formula (72).

3.2 Conservation laws of (6) using Noether's theorem

In this subsection we derive the conservation laws for the modified equal-width equation (6) using the Noether theorem. This equation as it is, does not have a Lagrangian. In order to apply Noether's theorem we transform equation (6) into a fourth-order equation which has a Lagrangian. Thus using the transformation $u = w_x$, equation (6) becomes

$$w_{tx} + a w_{xx} + b w_x w_{xx} + c w_{xxxx} = 0. \tag{76}$$

It can readily be verified that the second-order Lagrangian for equation (76) is given by

$$\mathcal{L} = -\frac{1}{2} w_x w_t - \frac{1}{2} a w_x^2 - \frac{1}{6} b w_x^3 + \frac{1}{2} c w_{xx}^2 \tag{77}$$

because $\delta \mathcal{L} / \delta w = 0$ on (76). Here $\delta / \delta w$ is the Euler-Lagrange operator defined as

$$\frac{\delta \mathcal{L}}{\delta w} = \frac{\partial}{\partial w} - D_t \frac{\partial}{\partial w_t} - D_x \frac{\partial}{\partial w_x} + D_t^2 \frac{\partial}{\partial w_{tt}} + D_x^2 \frac{\partial}{\partial w_{xx}} + D_t D_x \frac{\partial}{\partial w_{tx}} - \cdots, \tag{78}$$

where the total derivatives D_t, D_x are as defined by (68) Consider the vector field

$$\mathcal{W} = \xi^1(t, x, w) \frac{\partial}{\partial t} + \xi^2(t, x, w) \frac{\partial}{\partial x} + \eta(t, x, w) \frac{\partial}{\partial w}, \tag{79}$$

where ξ^1, ξ^2 and η depend on t, x and w. To determine the Noether symmetries $\mathcal{W}$ of (76) we insert the value of $\mathcal{L}$ from (77) in

$$\mathcal{W}^{[2]}(\mathcal{L}) + \mathcal{L}[D_t(\xi^1) + D_x(\xi^2)] = D_t(B^t) + D_x(B^x), \tag{80}$$

where $B^t = B^t(t, x, w)$ and $B^x = B^x(t, x, w)$ are the gauge terms and $\mathcal{W}^{[2]}$ is the second prolongation of $\mathcal{W}$ defined as

$$\mathcal{W}^{[2]} = \mathcal{W} + \zeta_t \frac{\partial}{\partial w_t} + \zeta_x \frac{\partial}{\partial w_x} + \zeta_{tt} \frac{\partial}{\partial w_{tt}} + \zeta_{xx} \frac{\partial}{\partial w_{xx}} + \zeta_{tx} \frac{\partial}{\partial w_{tx}} \tag{81}$$

with ζ_t and ζ_x defined in this regard as (10)

Equation (80) becomes

$$-\frac{w_x}{2}\zeta_t - \frac{w_t}{2}\zeta_x - a w_x \zeta_x - \frac{1}{2} b w_x^2 \zeta_x + c w_{xx} \zeta_{xx} = B_t^t + B_x^x + w_t B_u^t + w_x B_u^x. \tag{82}$$

Expansion of the above equation gives fourteen system of differential equations:

$$\begin{aligned}
&\eta_{xx} = 0,\ \ \xi_w^1 = 0,\ \ \xi_x^1 = 0,\ \ \xi_w^2 = 0,\ \ \xi_x^1 - \eta_w = 0,\ \ \eta_{ww} - 2\xi_{xw}^2 = 0,\\
&2\eta_{xw} - \xi_{xx}^2 = 0,\ \ \eta_x + 2B_w^t = 0,\ \ B_w^t + B_x^x = 0,\ \ b\xi_x^1 + a\xi_w^1 + \xi_w^2 = 0,\\
&2\eta_w - 3\xi_x^2 + \xi_t^1 = 0,\ \ \eta_t + 2a\eta_x + 2B_w^x = 0,\ \ 2b\xi_x^2 - 3b\eta_w + 3a\xi_w^2 - b\xi_t^1 = 0,\\
&a\xi_x^2 - 2a\eta_w - b\eta_x - a\xi_t^1 + \xi_t^2 = 0.
\end{aligned} \tag{83}$$

Solving the system therefore leads to the solution presented as

$$\begin{aligned}
&\xi^1(t, x, w) = A_1,\ \ \xi^2(t, x, w) = A_2 t + A_3,\ \ \eta(t, x, w) = \frac{1}{b} A_2 x + f(t),\\
&B^t(t, x, w) = -\frac{1}{2} A_2 w + g(t, x),\ \ B^x(t, x, w) = Q(t) - \frac{1}{2} f'(t) w - a A_2 w + e(t),
\end{aligned}$$

where $Q(t) = -\int g_t(t, x)dx$, while A_1, A_2 and A_3 are arbitrary constants. Additionally, functions $f(t)$, $g(t, x)$ as well as $e(t)$ are arbitrary. One takes note that one can choose $g(t, x) = e(t) = 0$, as they contribute to the trivial part of the conserved vector, thus satisfying (83). Thus, one attains the following Noether point symmetries as well as their corresponding gauge functions:

$$\begin{aligned}
&\mathcal{W}_1 = \frac{\partial}{\partial t}, \quad B^t = 0, \quad B^x = 0,\\
&\mathcal{W}_2 = \frac{\partial}{\partial x}, \quad B^t = 0, \quad B^x = 0,\\
&\mathcal{W}_3 = bt\frac{\partial}{\partial x} + x\frac{\partial}{\partial w}, \quad B^t = -\frac{1}{2} w, \quad B^x = -aw,\\
&\mathcal{W}_f = f(t)\frac{\partial}{\partial w}, \quad B^t = 0, \quad B^x = -\frac{1}{2} f'(t) w.
\end{aligned}$$

Next, we use the above results to compute the conserved vectors of the fourth-order equation (76). Invoking the formulae for the conserved vector (T^t, T^x) expressed in (66), one could attain the four conserved vectors associated with the calculated Noether symmetries $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ and

$\mathcal{W}_f$ respectively as

$$
\begin{aligned}
T_1^t =& \frac{1}{2}cw_{xx}^2 - \frac{1}{2}aw_x^2 - \frac{1}{6}bw_x^3,\\
T_1^x =& aw_tw_x + \frac{1}{2}bw_tw_x^2 + cw_tw_{xxx} - cw_{xx}w_{tx} + \frac{w_t^2}{2};\\
T_2^t =& \frac{w_x^2}{2},\\
T_2^x =& \frac{1}{2}aw_x^2 + \frac{1}{3}bw_x^3 + cw_{xxx}w_x - \frac{1}{2}cw_{xx}^2;\\
T_3^t =& \frac{1}{2}btw_x^2 + \frac{1}{2}w - \frac{1}{2}xw_x,\\
T_3^x =& aw + \frac{1}{2}abtw_x^2 + \frac{1}{3}b^2tw_x^3 + bctw_{xxx}w_x - \frac{1}{2}bctw_{xx}^2 - \frac{1}{2}xw_t\\
&- axw_x - \frac{1}{2}bxw_x^2 - cxw_{xxx} + cw_{xx};\\
T_f^t =& -\frac{1}{2}f(t)w_x,\\
T_f^x =& -af(t)w_x - \frac{1}{2}bf(t)w_x^2 - cf(t)w_{xxx} - \frac{1}{2}f(t)w_t + \frac{1}{2}f'(t)w.
\end{aligned}
$$

Reverting to the original variables we obtain one local and three non-local conserved vectors of (6) given by

$$
\begin{aligned}
C_1^t =& \frac{1}{2}cu_x^2 - \frac{1}{2}au^2 - \frac{1}{6}bu^3,\\
C_1^x =& au\int u_tdx + \frac{1}{2}bu^2\int u_tdx + cu_{xx}\int u_tdx - cu_tu_x + \frac{1}{2}\left(\int u_tdx\right)^2;\\
C_2^t =& \frac{1}{2}u^2,\\
C_2^x =& \frac{1}{2}au^2 + \frac{1}{3}bu^3 + cu_{xx}u - \frac{1}{2}cu_x^2;\\
C_3^t =& \frac{1}{2}btu^2 + \frac{1}{2}\int udx - \frac{1}{2}xu,\\
C_3^x =& a\int udx + \frac{1}{2}abtu^2 + \frac{1}{3}b^2tu^3 + bctu_{xx}u - \frac{1}{2}bctu_x^2 - \frac{1}{2}x\int u_tdx\\
&- axu - \frac{1}{2}bxu^2 - cxu_{xx} + cu_x;\\
C_f^t =& -\frac{1}{2}f(t)u,\\
C_f^x =& -af(t)u - \frac{1}{2}bf(t)u^2 - cf(t)u_{xx} - \frac{1}{2}f(t)\int u_tdx + \frac{1}{2}f'(t)\int u_tdx.
\end{aligned}
$$

Remark: It should be noted that due to the presence of arbitrary function $f(t)$ we have infinitely many nonlocal conservation laws.

3.3 Conserved vectors of (6) via Ibragimov's theorem

Ibragimov's theorem asserts that each conserved quantity in a differential equation is uniquely related to a Lie point symmetry. Therefore, we utilize the elements of the optimal system of Lie subalgebra from section 2 to generate new conserved currents using Ibragimov's theorem (Ibragimov (2007)). Thus, we give a theorem

Theorem 5. *Given the Euler operator $\delta/\delta u$, the adjoint equation of (1+1)D-GeoKdVe (6) can be expressed through the relation (Ibragimov (2007))*

$$H^* \equiv \frac{\delta}{\delta u}\left[v\left(u_t + au_x + buu_x + cu_{xxx}\right)\right] = 0. \tag{84}$$

Further expansion of (84) secures

$$H^* \equiv v_t + (a + bu)v_x + cv_{xxx} = 0. \tag{85}$$

The formal Lagrangian of (1+1)D-GeoKdVe (6) together with its adjoint presented in (85) is expressed in the format

$$\mathcal{L} = v\left(u_t + au_x + buu_x + cu_{xxx}\right). \tag{86}$$

Therefore, the conserved vectors $(T^i, X^i), i = 1, 2, \ldots, 6$ are formulated for the Lagrangian $(\mathcal{L})$ by employing the appropriate structure of (72) applicable here, purveyed as (Ibragimov (2007))

$$\begin{aligned} T =& \xi^1\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots\right] + D_jD_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_{ijk}} + \cdots, \end{aligned} \tag{87}$$

$$\begin{aligned} X =& \xi^2\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots\right] + D_jD_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_{ijk}} + \cdots \end{aligned} \tag{88}$$

with constant $\alpha = 1, 2$ as well as $j = 1, 2, 3, 4$. $W^\alpha = \Psi^\alpha - \xi^j u_j^\alpha$ is the involved Lie characteristic function.

Now, on contemplating the Lie subalgbras previously utilized in the reduction process, one computes the conserved vectors associated to them. Thus, by using the data found in the cited-references, as demonstrated by (Khalique & Adeyemo (2020b); Ibragimov (2007)), one secures

$$\begin{aligned} T^1 &= au_xv + bu_xuv + cu_{xxx}v, \\ X^1 &= cv_xu_{tx} - au_tv - bu_tuv - cvu_{txx} - cu_tv_{xx}; \end{aligned}$$

$$\begin{aligned} T^2 &= v - btu_xv, \\ X^2 &= btu_tv + av + buv + bctu_{xx}v_x - bctu_xv_{xx} + cv_{xx}; \end{aligned}$$

$$
\begin{aligned}
T^3 =& 3abtu_xv + 3b^2tu_xuv + 3bctu_{xxx}v - bxu_xv - 2av - 2buv,\\
X^3 =& -3abtu_tv - 3b^2tu_tuv - 4bcu_{xx}v - 2bcv_{xx}u - 3bctvu_{txx} + bxu_tv\\
&- 2a^2v - 4abuv - 2b^2u^2v - 2acv_{xx} - 3bctu_tv_{xx} + 3bctv_xu_{tx} + 3bcu_xv_x\\
&+ bcxu_{xx}v_x - bcxu_xv_{xx};\\
T^4 =& au_xv - btu_xv + bu_xuv + cu_{xxx}v + v,\\
X^4 =& buv - au_tv + btu_tv - bu_tuv - cvu_{txx} + av + bctu_{xx}v_x - bctu_xv_{xx}\\
&- cu_tv_{xx} + cv_xu_{tx} + cv_{xx};
\end{aligned}
$$

$$
\begin{aligned}
T^5 =& au_xv + btu_xv + bu_xuv + cu_{xxx}v - v,\\
X^5 =& bctu_xv_{xx} - au_tv - btu_tv - bu_tuv - cvu_{txx} - av - buv - bctu_{xx}v_x\\
&- cu_tv_{xx} + cv_xu_{tx} - cv_{xx};
\end{aligned}
$$

$$
\begin{aligned}
T^6 =& au_xv - btu_xv + bu_xuv - c_0u_xv + cu_{xxx}v + v,\\
X^6 =& btu_tv - au_tv - bu_tuv + c_0u_tv - cvu_{txx} + av + buv + bctu_{xx}v_x - bctu_xv_{xx}\\
&- cu_tv_{xx} + cv_xu_{tx} + cc_0u_{xx}v_x - cc_0u_xv_{xx} + cv_{xx};
\end{aligned}
$$

$$
\begin{aligned}
T^7 =& 3abtu_xv + au_xv + 3b^2tu_xuv + 3bctu_{xxx}v - btu_xv - bxu_xv\\
&+ bu_xuv + cu_{xxx}v - u_xv - 2av - 2buv + v,\\
X^7 =& btu_tv - 3abtu_tv - au_tv - 3b^2tu_tuv - 4bcu_{xx}v - 2bcv_{xx}u - 3bctvu_{txx}\\
&+ bxu_tv - bu_tuv - cvu_{txx} + u_tv - 2a^2v - 4abuv + av - 2b^2u^2v + buv\\
&- 2acv_{xx} + bctu_{xx}v_x - bctu_xv_{xx} - 3bctu_tv_{xx} + 3bctv_xu_{tx} + 3bcu_xv_x\\
&+ bcxu_{xx}v_x - bcxu_xv_{xx} - cu_tv_{xx} + cv_xu_{tx} + cu_{xx}v_x - cu_xv_{xx} + cv_{xx}.
\end{aligned}
$$

4 Concluding remarks

In this article, copious analytical investigations carried out on generalized geophysical Korteweg de Vries equation in ocean physics is presented. In the first instance, the theory of Lie group applied to differential equations was invoked in computing the Lie point symmetries of the model which gave rise to a four-dimensional Lie algebra. Moreover, calculations are made for the algebra's one-parameter transformation groups. Additionally, moving ahead, a procedural approach is used to derive a one-dimensional optimal system of subalgebra. Following this, the subgroups and merging of the obtained symmetries are utilized during the reduction process which allows for the deduction of nonlinear ordinary differential equations related to the studied generalized geophysical Korteweg de Vries equation. The majority of these non-linear differential equations have been solved through direct integration or by utilizing the power series method. In addition, travelling wave solutions were also acquired. This is achieved through direct integration and the application of the Jacobi elliptic function method. These methods allow for the achievement of different exact soliton solutions, which include non-topological soliton solutions and general periodic function solutions like cosine, sine, and delta amplitude solutions in the model. Additionally, numerical simulations are used to develop a basic understanding of the physical phenomena described by the generalized geophysical Korteweg de Vries equation in ocean physics. Ultimately, the study also focuses on determining conserved vectors in the model through Ibragimov's theorem for conservation laws, along with Noether's theorem. In consequence, the production of conserved quantities of interest existence in physical sciences was imminent. These involve energy, and momenta. The investigation contains various rich results covering a large spectrum of applications.

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