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REGULARITY MARGIN OF INTERVAL PARAMETRIC MATRICES AND APPLICATIONS

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Abstract. In this paper, a new approach to assessing regularity of interval parametric matrices is suggested. It consists in an equivalent transformation of the original problem to an interval parametric linear programming problem. Based on this approach, a new regularity measure, so-called regularity margin, is suggested. Several applications of the regularity margin are considered.

Keywords: *interval parametric matrix, regularity, singularity, regularity margin, interval parametric linear programming.*

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1. Introduction

Let $A(p)$ be a real $(n \times n)$ parametric matrix, i.e. matrix whose elements are given functions of a real m-dimensional vector $p = (p_1,.., p_m)$ belonging to a given interval vector $\boldsymbol{p} = (\boldsymbol{p}_1,.., \boldsymbol{p}_m)$. As is well known, the set of all $A(p)$ when p varies over $\boldsymbol{p}$ denoted $A(\boldsymbol{p})$, i.e. the set

$$A(\boldsymbol{p}) = \{A(p): \; p \in \boldsymbol{p}\} \qquad (1)$$

is referred to as an interval parametric (IP) matrix. The elements $a_{ij}(p)$ of $A(p)$ are, in general, nonlinear functions of p:

$$a_{ij}(p) = a_{ij}(p_1,..., p_m) \qquad (1a)$$

which are assumed continuous; in a special case, $a_{ij}(p)$ depend affine linearly on p, i.e.

$$a_{ij}(p) = \alpha_{ij} + \sum_{\mu=1}^{m} a_{ij\mu} p_\mu \, . \qquad (1b)$$

An IP matrix is called regular if each $A(p) \in A(\boldsymbol{p}), \; p \in \boldsymbol{p}$ is nonsingular; otherwise, it is said to be singular. Presently, there exist two major types of problems related to the concept of regularity.

Problem P1. This is a problem of qualitative nature: check if the matrix $A(\boldsymbol{p})$ is regular or not.

Problem P2. Now, it is necessary to determine a quantitative measure for regularity of $A(\boldsymbol{p})$.

While various methods are known for solving Problem P1 (e.g., [1, 4-6, 9, 10, 12, 14-16, 18-21]) regarding both cases (1a) and (1b), Problem 2 has seemingly been addressed only in [11] where the concept of the so-called regularity radius $r^*(A(\boldsymbol{p}))$ of $A(\boldsymbol{p})$ has been introduced. It should, however, be stressed that $r^*(A(\boldsymbol{p}))$ can be defined and computed solely in the case of IP matrices having the linear inter-parametric dependencies (1b). Indeed, on account of (1b), the parametric matrix $A(p)$ can be written in the form

$$A(p) = A^{(0)} + \sum_{\mu=1}^{m} A^{(\mu)} p_\mu \tag{2}$$

and the regularity radius of $A(\boldsymbol{p})$ is defined as [11]

$$r^*(A(\boldsymbol{p})) = \min\{r \geq 0: \quad A^{(0)} + \sum_{\mu=1}^{m} A^{(\mu)} r\, p_\mu,\ p_\mu \in \boldsymbol{p}_\mu \ \textit{ is singular}\}. \tag{3}$$

Evidently, $A(\boldsymbol{p})$ is regular if and only if $r^*(A(\boldsymbol{p})) > 1$. Knowledge of $r^*(A(\boldsymbol{p}))$, however, provides a quantitative measure: the distance from singularity.

In the present paper, a new approach to tackling the quantitative aspect of the regularity problem is suggested which is applicable to both linear and nonlinear parametric dependencies. It consists in reformulating the original problem as an equivalent interval parametric linear programming (IPLP) problem. The minimum value m^* of the associated IPLP problem provides a new measure for the distance of $A(\boldsymbol{p})$ from singularity, so-called *regularity margin* $m^*(A(\boldsymbol{p}))$.

The paper is structured as follows. The IPLP formulation of the original problem is given in Section 2. The main result of this section is Theorem 1 establishing the relation between Problem 1 and the regularity margin: it is proved that $A(\boldsymbol{p})$ is regular if and only if $m^*(A(\boldsymbol{p})) > 0$. It is shown that lower or upper bounds on $m^*(A(\boldsymbol{p}))$ provide solution to the qualitative type of the regularity problem. In Section 3, several applications of $m^*(A(\boldsymbol{p}))$ or its bounds are listed: checking if an IP matrix $A(\boldsymbol{p})$ is positive definite or if a symmetric $A(\boldsymbol{p})$ is stable and a P-matrix (Section 3.1), relationships between the real eigenvalue set *L* of the bundle $(A(\boldsymbol{p}), B(\boldsymbol{p}))$ and the regularity radius $m^*(C(\boldsymbol{p};\alpha))$ of an auxiliary IP matrix $C(\boldsymbol{p};\alpha) = A(\boldsymbol{p}) - \alpha B(\boldsymbol{p})$, $\alpha \in R$ (Section 3.2) as well as the case of complex $(A(\boldsymbol{p}), B(\boldsymbol{p}))$ (Section 3.3). Concluding remarks are given in the last section of the paper.

2. The IPLP formulation

To formulate the original regularity check problem as an equivalent IPLP, we first need an auxiliary result concerning verification of the nonsingularity of a real matrix $A = (a_{ij})$. Let $A_{i:}$ and $A_{:j}$ denote the i^{th} row and j^{th} column, respectively, of *A*. We first form a reduced $(n-1)\times n$ matrix A' by deleting from *A* its j_0 th column. Further, we remove the i_0^{th} row $A'_{i0:}$ of A' to obtain the reduced-size $(n-1)\times(n-1)$ matrix B_r. Let

$$b' = -A_{:j_0} \tag{4a}$$

and

$$c' = A_{i_0:} \; . \tag{4b}$$

Also, the reduced-length vectors b and c are formed by deleting the i_0^{th} element from b' and c', respectively. Now consider the linear system

$$B_r x = b \, . \tag{5}$$

Assuming that B_r is not a singular matrix, we find the solution $x = B_r^{-1} b$ to (5). Next, the n-dimensional column-vector

$$\xi^T = \left(x_1,.., x_{j_0-1}, 1, x_{j_0}, x_{j_0+1},.., x_{n-1} \right) \tag{6}$$

is formed and the scalar product

$$l = c'\xi \tag{7}$$

is computed. The following result is readily proved.

Lemma 1. Assume the $(n-1)\times(n-1)$ matrix B_r, associated with the matrix A, is not singular. Then A is non-singular if and only if

$$l \neq 0, \tag{8}$$

where l is computed by way of (4) to (7).

In a similar way, starting from $A(\boldsymbol{p}) = \{a_{ij}(\boldsymbol{p})\}$ and chosen indices i_0 and j_0, we form the reduced size $(n-1)\times(n-1)$ IP matrix $B_r(\boldsymbol{p})$ and the corresponding IP vectors $b(\boldsymbol{p})$ and $c'(\boldsymbol{p})$. Let $c(\boldsymbol{p})$ denote the $(n-1)$-dimensional interval vector obtained by deleting the j_0^{th} element $\boldsymbol{c}'_{j_0}$ of $\boldsymbol{c}'$. Also, let B_r^0, b^0, $(c')^0$ and $(c)^0$ denote the corresponding quantities computed for the midpoint p_0 of $\boldsymbol{p}$. Next, the interval row vector $\boldsymbol{c}' = \left(\boldsymbol{c}, \, \boldsymbol{a}_{i_0 j_0}\right)$ and its centre $\eta^0 = \left(c^0, \, a^0_{i_0 j_0}\right)$ are set up. Further, find the solution x^0 to

$$B_r^0 x = b^0 , \tag{9}$$

form the corresponding vector ξ^0 (using (6)) and compute

$$l^0 = (c')^0 \, \xi^0 . \tag{10}$$

If $l^0 = 0$, then by Lemma 1 A^0 is singular, hence $A(\boldsymbol{p})$ is singular too. Thus (without loss of generality), we assume that

$$l^0 > 0 \tag{11}$$

(if l^0 is initially negative, it suffices to choose $(b')^0 = A^0_{:j_0}$, which will lead to a new vector $\eta^0 = -\xi^0$ satisfying $(c')^0 \eta^0 > 0$).

Now consider the following IPLP problem

$$m^* = \min\left\{ c(p)\, x + c_0(p) : \; B_r(p) x = b(p), \quad p \in \boldsymbol{p} \right\}, \tag{12}$$

where

$$c_0(p) = c'_{j_0}(p) = a_{i_0 j_0}(p). \tag{13}$$

We need the following assumption.

Assumption 1. The matrix $B_r(\boldsymbol{p})$ is regular.

On account of Assumption 1, the linear system

$$B_r(p)x = b(p), \quad p \in \boldsymbol{p} \tag{14}$$

is solvable, i.e. the solution set $X(B_r(p), b(p), \boldsymbol{p}) = \{x: B_r(p)x = b(p), \ p \in \boldsymbol{p}\}$ of system (14) is bounded. Hence, the minimum m^* is also bounded. We are now ready to state the main result of this section.

Theorem 1. If A^0 is nonsingular, $l^0 > 0$ and Assumption 1 is valid, then the interval parametric matrix $A(\boldsymbol{p})$ is regular if and only if

$$m^* > 0, \tag{15}$$

where l^0 and m^* are defined by (9) to (13).

Proof. ***Sufficiency***. If (15) holds, then

$$l(p) = c(p)x + a_{i_0 j_0}(p) > 0$$

for any p, $p \in \boldsymbol{p}$ and corresponding $x \in X(B_r(p), b(p), \boldsymbol{p})$. By Lemma 1 the related $A(p)$ is non-singular and, hence, $A(\boldsymbol{p})$ is regular.

Necessity. Assume that (15) is valid but $A(\boldsymbol{p})$ is singular. Since A^0 is nonsingular and $l^0 > 0$, the singularity of $A(\boldsymbol{p})$ entails that for some $p^1 \in \boldsymbol{p}$ there exists a matrix $A^1 \in A(\boldsymbol{p})$ different from A^0 such that the corresponding product

$$l^1 = c^1 x^1 + a^1_{i_0 j_0} = 0, \tag{16}$$

where x^1 is the solution of the associated system $B^1_r x \in b^1$. Since m^* is the minimum value of $l(p) = c(p)x + c_0(p)$ in (12) (attained at the solution pair p^* and $x^* = x(p^*)$)

$$c^1 x^1 + c^1_0 > m^*. \tag{17}$$

It follows from (17) and the expression $l^1 = c^1 x^1 + a^1_{i_0 j_0}$ that

$$l^1 > m^* > 0 \tag{18}$$

which is a contradiction with $l^1 = 0$ in (16).

The value of m^* provides a quantitative measure for regularity of the IP matrix $A(\boldsymbol{p})$ considered. The number m^* will be called *the regularity margin* of $A(\boldsymbol{p})$ and will be denoted $m^*(A(\boldsymbol{p}))$.

Corollary 1. Under the assumptions of Theorem 1, the interval parametric matrix $A(\boldsymbol{p})$ is singular if and only if

$$m^*(A(\boldsymbol{p})) \le 0. \tag{19}$$

The regularity margin $m^*(A(\boldsymbol{p}))$ is an alternative regularity measure for $A(\boldsymbol{p})$ with respect to the regularity radius $r^*(A(\boldsymbol{p}))$ introduced in [11].

Let $\underline{m}(A(\boldsymbol{p}))$ denote a lower bound on $m^*(A(\boldsymbol{p}))$, i.e. $\underline{m}(A(\boldsymbol{p})) < m^*(A(\boldsymbol{p}))$ and $\overline{m}(A(\boldsymbol{p}))$ denote an upper bound on $m^*(A(\boldsymbol{p}))$, i.e. $\overline{m}(A(\boldsymbol{p})) > m^*(A(\boldsymbol{p}))$. On account of Theorem 1 and Corollary 1, the following sufficient conditions are valid.

Corollary 2. Under the assumptions of Theorem 1, the interval parametric matrix $A(\boldsymbol{p})$ is regular if

$$\underline{m}(A(\boldsymbol{p})) > 0\,; \tag{20}$$

it is singular if

$$\overline{m}(A(\boldsymbol{p})) \le 0\,. \tag{21}$$

Remark 1. As shown above, computing $m^*(A(\boldsymbol{p}))$ reduces to solving the IPLP problem (12). Nowadays, the latter problem does not seem to have been considered in the literature for general nonlinear parametric dependencies in $c(\boldsymbol{p})$, $B_r(\boldsymbol{p})$ and $b(\boldsymbol{p})$. The case of an IPLP problem having linear parametric dependencies has been recently addressed in [10, Section 3.3]; the interval (nonparametric) case has been handled in a number of publications (cf., e.g. [2] and the references therein cited).

Determining the bounds $\underline{m}(A(\boldsymbol{p}))$ or $\overline{m}(A(\boldsymbol{p}))$ is a much easier task since these are found as a two-sided approximate solution of the IPLP problem (12). The linear parametric dependencies case can be solved using the approach of [10] (Theorem 1 and Corollary 1 extended to problem (12)).

Remark 2. The validity of Assumption 1 can be checked using some of the available methods for outer solution of (14)**,** accounting for the structural specificity of the IP matrix $B_r(\boldsymbol{p})$ [12].

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3. Applications

3.1. Checking properties of $A(\boldsymbol{p})$

An IP matrix $A(\boldsymbol{p})$ is called positive definite (p.d.), stable or P-matrix, if each real $A(p) \in A(\boldsymbol{p})$ is p.d., stable or P-matrix. It will be now shown that these properties of $A(\boldsymbol{p})$ can be checked using the regularity margin $m^*(A(\boldsymbol{p}))$ of $A(\boldsymbol{p})$.

An interval parametric matrix $A(\boldsymbol{p})$ is called symmetric if each $A(p) \in A(\boldsymbol{p})$ is symmetric.

Theorem 2. An interval parametric matrix $A(\boldsymbol{p})$ is positive definite if and only if

$$m^*((S(\boldsymbol{p})) > 0\,, \tag{22}$$

$$S(\boldsymbol{p}) = (1/2)(A(\boldsymbol{p}) + A(\boldsymbol{p})^T) \tag{22a}$$

and the symmetric IP matrix $S(\boldsymbol{p})$ contains at least one real positive definite matrix for a $p \in \boldsymbol{p}$.

Proof. Sufficiency is obvious. To prove necessity, assume to the contrary that, on account of (22), $A(\boldsymbol{p})$ is regular and contains a positive definite matrix $A(p^0)$ but is not positive definite, hence $x^T A(p^1)\, x \le 0$ for some $p^1 \in \boldsymbol{p}$ and $x \ne 0$. Denote

$$f(p) = (x^T A(p)\, x)/x^T x, \quad x \ne 0, \quad p \in \boldsymbol{p}\,, \quad A_0 = A(p^0)\,, \quad A_1 = A(p^1)$$

and let

$$A_0 = A(p^0) = (1/2)(A_0 + A_0^T), \quad A_1 = A(p^1) = (1/2)(A_1 + A_1^T).$$

As is easily seen

$$f(p^0) = A(p^0) = A(p^0) > 0, \quad f(p^1) = A(p^1) = A(p^1) \le 0.$$

Now define a real function of one variable by

$$\varphi(t) = f(A[tp^0 + (1-t)p^1]), \quad t \in [0, 1].$$

Since $a_{ij}(p)$ are assumed continuous, $\varphi(t)$ is also continuous and because $\varphi(0)\varphi(1) \le 0$ there exists a $t_0 \in [0, 1]$ with $\varphi(t_0) = 0$. So

$$A' = A(t_0) = A[t_0 p^0 + (1-t_0) p^1])$$

is symmetric, $A' \in A(\boldsymbol{p})$ and $f(A') = 0$, hence A' is singular. Thus, $A(\boldsymbol{p})$ is singular so $m^*((1/2)(A(\boldsymbol{p}) + A(\boldsymbol{p})^T)) \le 0$, which is contradiction with (22).

Theorem 3. A symmetric IP matrix $A(\boldsymbol{p})$ is stable if and only if

$$m^*(-A(\boldsymbol{p})) > 0 \qquad (23)$$

and it contains at least one stable matrix.

Proof. Since $A(\boldsymbol{p})$ is symmetric, each $A(p)$ has only real eigenvalues. If each eigenvalue λ of $A(p)$ is negative to ensure stability, then each $-A(p)$ is p.d., hence $-A(\boldsymbol{p})$ is p.d., which by Theorem 2 entails the validity of the present theorem.

Theorem 4. A symmetric interval parametric matrix $A(\boldsymbol{p})$ is a P-matrix if and only if

$$m^*(A(\boldsymbol{p})) > 0 \qquad (24)$$

and it contains at least one real P-matrix matrix.

The proof is similar to that of Theorem 3.

On account of Theorems 2 to 4, the following sufficient conditions are valid.

Corollary 3. An interval parametric matrix $A(\boldsymbol{p})$ is positive definite if

$$\underline{m}((S(\boldsymbol{p})) > 0 \qquad (25)$$

and the IP matrix $S(\boldsymbol{p})$ (defined in (22a)) contains at least one real positive definite matrix for a $p \in \boldsymbol{p}$.

Corollary 4. A symmetric IP matrix $A(\boldsymbol{p})$ is stable if

$$\underline{m}(-A(\boldsymbol{p})) > 0 \qquad (26)$$

and it contains at least one stable matrix.

Corollary 5. A symmetric interval parametric matrix $A(\boldsymbol{p})$ is a P-matrix if

$$\underline{m}(A(\boldsymbol{p})) > 0 \qquad (27)$$

and it contains at least one real P-matrix matrix.

3.2. Real eigenvalue problems

We revisit several problems [11, Section 3] related to bounding or determining the set L of all real eigenvalues of the bundle ($A(\boldsymbol{p}), B(\boldsymbol{p})$), i.e. the set of all real eigenvalues of the following parametric generalized eigenvalue problem

$$A(p)x = \lambda B(p)x, \quad p \in \boldsymbol{p}. \qquad (28)$$

The first problem is: how to establish whether a real number α belongs to L or not. In [11], it has been established that the answer to this question is given by the numerical value of the regularity radius $r^*(C(\boldsymbol{p};\alpha))$ of the following auxiliary IP matrix

$$C(\boldsymbol{p};\alpha) = A(\boldsymbol{p}) - \alpha B(\boldsymbol{p}). \tag{29}$$

We now show that a similar result can be obtained if the regularity margin $m^*(C(\boldsymbol{p};\alpha))$ is used.

From (28), the set L of all real eigenvalues of the bundle $(A(\boldsymbol{p}), B(\boldsymbol{p}))$ is defined as follows:

$$L = \{\lambda \in R: \quad A(p)x = \lambda B(p),\ p \in \boldsymbol{p},\ x \neq 0\}. \tag{30}$$

Lemma 2. [11] A real number α is an eigenvalue of the bundle $(A(\boldsymbol{p}), B(\boldsymbol{p}))$ if and only if the IP matrix (29) is singular.

On account of Theorem 1, Corollary 1 and Lemma 2, the following theorem is valid.

Theorem 5. $\alpha \in L$ if and only if

$$m^*(C(\boldsymbol{p};\alpha)) \leq 0. \tag{31}$$

Conversely $\alpha \notin L$ if and only if

$$m^*(C(\boldsymbol{p};\alpha)) > 0. \tag{32}$$

Let ∂L denote the boundary of L. To prove a result concerning the case $\alpha \in \partial L$, we need the following additional facts. The relation (28) defines each eigenvalue λ_k as an implicit function of p, i.e. $\lambda_k = \lambda_k(p)$ for $p \in \boldsymbol{p}$. In case of real eigenvalues, the range $\boldsymbol{\lambda}_k^* = [\underline{\lambda}_k^*, \overline{\lambda}_k^*]$ is given by the real set $\boldsymbol{\lambda}_k^* = \{\lambda_k : A(p)x = \lambda B(p)x,\ p \in \boldsymbol{p}\}$. Each endpoint of the interval $\boldsymbol{\lambda}_k^* = [\underline{\lambda}_k^*, \overline{\lambda}_k^*]$ is found as the global solution of a respective optimization problem:

$$\underline{\lambda}_k^* = \min\{\lambda_k(p) : A(p)x = \lambda B(p)x,\ p \in \boldsymbol{p}\}, \tag{33a}$$

$$\overline{\lambda}_k^* = \max\{\lambda_k(p) : A(p)x = \lambda B(p)x,\ p \in \boldsymbol{p}\}. \tag{33b}$$

As in [11], the intervals $\boldsymbol{\lambda}_k^*$ are assumed disjoint. Thus, ∂L is made up of the union of the points $\underline{\lambda}_k^*$ and $\overline{\lambda}_k^*$, $k \in \{1,...,K\}$ where K is the total number of intervals $\boldsymbol{\lambda}_k^*$.

An interval parametric matrix $A(\boldsymbol{p})$ is referred to as minimally singular if [11] $r^*(A(\boldsymbol{p})) = 1$. Obviously, an alternative condition is

$$m^*(A(\boldsymbol{p})) = 0. \tag{34}$$

Theorem 6. $\alpha \in \partial L$ if and only if

$$m^*(C(\boldsymbol{p};\alpha)) = 0. \tag{35}$$

The proof of the theorem is similar to that of Theorem 4 in [11] and is therefore omitted.

Remark 3. Let the upper bound $\underline{\lambda}_k$ on $\underline{\lambda}_k^*$ have been found using some local optimization technique for locating $\underline{\lambda}_k^*$. Theorem 6 provides a simple global optimality check for $\underline{\lambda}_k$: if condition $m^*(A(\boldsymbol{p}) - \underline{\lambda}_k B(\boldsymbol{p})) = 0$ is fulfilled, then

$\underline{\lambda}_k$ is, in fact, equal to $\underline{\lambda}_k^*$. Similarly, the lower bound $\overline{\lambda}_k$ on the right end-point $\overline{\lambda}_k^*$ of the real eigenvalue range $\boldsymbol{\lambda}_k^*$ determines $\overline{\lambda}_k^*$ itself, i.e. $\overline{\lambda}_k = \overline{\lambda}_k^*$, if $m^*(A(\boldsymbol{p}) - \overline{\lambda}_k B(\boldsymbol{p})) = 0$.

Next, sufficient conditions for a given α to belong to L or not are suggested. They are based on the use of a lower $\underline{m}(C(\boldsymbol{p};\alpha))$ or upper $\overline{m}(C(\boldsymbol{p};\alpha))$ bound on $m^*(C(\boldsymbol{p};\alpha))$. On account of Theorem 5, the following result is valid.

Corollary 6. If

$$\overline{m}(C(\boldsymbol{p};\alpha)) \le 0, \tag{36}$$

then $\alpha \in L$. Conversely, if

$$\underline{m}(C(\boldsymbol{p};\alpha)) > 0, \tag{36a}$$

then $\alpha \notin L$.

The second assertions of Theorem 5 and Corollary 6 can be extended in the following way. Let $\alpha \in \boldsymbol{a}$ where $\boldsymbol{a}$ is a real interval. On account of (29), define an augmented parameter vector $\boldsymbol{p}'$ and a corresponding matrix

$$C(\boldsymbol{p}') = A(\boldsymbol{p}) - \boldsymbol{a}B(\boldsymbol{p}) \ , \quad \boldsymbol{p}' = (\boldsymbol{p}, \boldsymbol{a}) \ . \tag{38}$$

We provide a criterion certifying that the whole interval $\boldsymbol{a}$ does not belong to L.

Theorem 7. Let $\boldsymbol{a}$ be a real interval. Then $\boldsymbol{a} \notin L$ if and only if

$$m^*(C(\boldsymbol{p}')) > 0. \tag{39}$$

where $C(\boldsymbol{p}')$ is defined in (38).

Corollary 6 is modified in the same manner.

Theorem 7 (Corollary 6) can be useful in developing algorithms for finding two-sided bounds on each real eigenvalue of (41) deleting “superfluous parts” of the real axis.

3.3. Complex eigenvalue problems

The results of the previous subsection can be extended to the case of complex matrices

$$A(p) = A_1(p) + iA_2(p) \tag{40}$$

of real parameters (in some cases, matrices containing complex parameters can be transformed to the form (40) where the vector p regroups the real and imaginary parts of the corresponding complex components of the initial p). Now the parametric generalized eigenvalue problem (28) becomes

$$[A_1(p) + iA_2(p)][x_1 + ix_2] = [\lambda_1 + i\lambda_2]B(p)[x_1 + ix_2], \quad p \in \boldsymbol{p}. \tag{41}$$

where, for simplicity of presentation, $B(p)$ is assumed to be real. The set of all eigenvalues of (41) will be denoted L_c. We are interested in verifying whether a complex number $\alpha = \alpha_1 + i\alpha_2$ belongs to L_c or not.

To answer this question using the approach of Section 3.2, we first transform the complex $(n \times n)$ system (41) into an equivalent real $(2n \times 2n)$ system by separating real and imaginary parts in (41). As is readily seen, the augmented-size real system is

$$C(p;\lambda_1,\lambda_2)\,x=0\,,\quad x=\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad p\in\boldsymbol{p} \tag{42}$$

where

$$C(p;\lambda_1,\lambda_2)=\begin{bmatrix}A_1-\lambda_1 B & -A_2+\lambda_2 B\\ A_2-\lambda_2 B & A_1-\lambda_1 B\end{bmatrix}. \tag{42a}$$

By analogy with Lemma 2, the following result is valid.

Lemma 3. A complex number $\alpha=\alpha_1+i\alpha_2$ is an eigenvalue of the bundle $(A(\boldsymbol{p}),\ B(\boldsymbol{p}))$ where $A(\boldsymbol{p})$ is a complex and $B(\boldsymbol{p})$ is a real IP matrix if and only if the real IP matrix $C(\boldsymbol{p};\alpha_1,\alpha_2)$ defined as in (42a) is singular.

On account of Theorem 5 and Lemma 3, the following theorem holds.

Theorem 8. The complex number $\alpha\in L_c$ if and only if

$$m^*(C(\boldsymbol{p};\alpha_1,\alpha_2))\le 0\,. \tag{43a}$$

Conversely, $\alpha\notin L_c$ if and only if

$$m^*(C(\boldsymbol{p};\alpha_1,\alpha_2))>0\,. \tag{43b}$$

In a similar manner, extension of Theorems 5 and 7 as well as Corollary 6 can be obtained. More specifically, we refer to the extended version of Theorem 7. Let $\alpha_1\in\boldsymbol{a}_1$ and $\alpha_2\in\boldsymbol{a}_2$ where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are real intervals. Using (42a), define the matrix

$$C(\boldsymbol{p};\boldsymbol{a}_1,\boldsymbol{a}_2)=\begin{bmatrix}A_1(\boldsymbol{p})-\boldsymbol{a}_1 B(\boldsymbol{p}) & -A_2(\boldsymbol{p})+\boldsymbol{a}_2 B(\boldsymbol{p})\\ A_2(\boldsymbol{p})-\boldsymbol{a}_2 B(\boldsymbol{p}) & A_1(\boldsymbol{p})-\boldsymbol{a}_1 B(\boldsymbol{p})\end{bmatrix}. \tag{44}$$

Theorem 7 is modified as follows.

Theorem 9. Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be real intervals. Then $\boldsymbol{a}=\boldsymbol{a}_1+i\boldsymbol{a}_2\notin L_c$ if and only if

$$m^*(C(\boldsymbol{p};\boldsymbol{a}_1,\boldsymbol{a}_2))>0 \tag{45}$$

where $C(\boldsymbol{p};\boldsymbol{a}_1,\boldsymbol{a}_2)$ is defined in (44).

Theorem 9 can be useful in various ways. One example is developing algorithms for: (i) locating a complex interval enclosing all possible values for a given complex eigenvalue $\lambda=\lambda_1+i\lambda_2$ of (41) over $\boldsymbol{p}$ or (ii) finding bounds $\boldsymbol{b}_1,\boldsymbol{b}_2$ on the whole set L_c.

Another possibility is to help solving the following robustness problem related to (41). Let for a fixed $p_0\in\boldsymbol{p}$ the eigenvalues of (41) be denoted $\lambda^{(k)}(p_0)=\lambda_1^{(k)}(p_0)+i\lambda_2^{(k)}(p_0)$. Assume that they have the following Property P: for a given $r\in R$ the real parts $\lambda_1^{(k)}(p_0)$ of n_1 eigenvalues are smaller than r whereas the real parts of the remaining eigenvalues are larger than *r*. The problem is to check whether the eigenvalues of (41) are robust, i.e. if Property P remains valid for all $p\in\boldsymbol{p}$. Let

$$C(\boldsymbol{p};r,\boldsymbol{a}_2)=\begin{bmatrix}A_1(\boldsymbol{p})-rB(\boldsymbol{p}) & -A_2(\boldsymbol{p})+\boldsymbol{a}_2 B(\boldsymbol{p})\\ A_2(\boldsymbol{p})-\boldsymbol{a}_2 B(\boldsymbol{p}) & A_1(\boldsymbol{p})-rB(\boldsymbol{p})\end{bmatrix}. \tag{46}$$

The following theorem offers a solution to the problem stated.

Theorem 10. The eigenvalues of (41) are robust with respect to Property P if

$$m^*(C(\boldsymbol{p}; r, \boldsymbol{b}_2)) > 0 \tag{47}$$

where $\boldsymbol{b}_2 = [0, b_2^u]$ is an upper bound on the imaginary parts of the n_1 eigenvalues of (41) and (41) contains at least one instance having the property P.

An illustration of the above theorem will now be given for a particular case where $A(\boldsymbol{p})$ is real, $B(\boldsymbol{p}) = I$, $n_1 = n$ and $r \le r_0 \le 0$. Obviously, these conditions define the problem of checking if a real (non-symmetric) IP matrix is (critically) stable ($r_0 = 0$) or stable with a stability margin r_0 ($r_0 < 0$). In this case

$$C(\boldsymbol{p}; r_0, \boldsymbol{b}_2) = \begin{bmatrix} A(\boldsymbol{p}) - r_0 I & \boldsymbol{b}_2 I \\ -\boldsymbol{b}_2 I & A(\boldsymbol{p}) - r_0 I \end{bmatrix}. \tag{48}$$

Theorem 11. A real IP matrix $A(\boldsymbol{p})$ is stable ($r_0 = 0$) or stable with a stability margin r_0 ($r_0 < 0$) if

$$m^*(C(\boldsymbol{p}; r_0, \boldsymbol{b}_2)) > 0 \tag{49}$$

where $C(\boldsymbol{p}; r_0, \boldsymbol{b}_2)$ is defined by (48) while $\boldsymbol{b}_2 = [0, b_2^u]$ is an upper bound on the imaginary parts of the complex eigenvalues of $A(\boldsymbol{p})$, and $A(\boldsymbol{p})$ contains at least one real matrix that is stable or stable with a stability margin r_0.

This theorem is an alternative of another result [7] where the stability (stability measure) of $A(\boldsymbol{p})$ is established by determining a so-called stability radius, which is a more expensive approach. It should be also mentioned that Theorem 11 can be extended to the case of the real bundle

$$A(p)x = \lambda Bx, \quad p \in \boldsymbol{p}. \tag{50}$$

Such an approach seems to be a better alternative than the solution suggested in [8].

Finally, the quasi-aperiodic property of a real IP matrix (or bundle (50)) will be considered. A real $A(\boldsymbol{p})$ is called aperiodic if all eigenvalues of any $A(p)$ (bundle), $p \in \boldsymbol{p}$ are real and negative. This property has been considered for the special case of interval (nonparametric) matrices in [13] where so-called robust linear algebra is used. Here the general approach of (44) and Theorem 9 is modified in the following manner

$$C(\boldsymbol{p}; \boldsymbol{b}_1, \varepsilon) = \begin{bmatrix} A(\boldsymbol{p}) - \boldsymbol{b}_1 I & \varepsilon I \\ -\varepsilon I & A(\boldsymbol{p}) - \boldsymbol{b}_1 I \end{bmatrix} \tag{51}$$

where $\boldsymbol{b}_1$ is a two-sided bound along the real axis on all eigenvalues of $A(\boldsymbol{p})$ and $\varepsilon > 0$ is a small constant. A real $A(\boldsymbol{p})$ will be called *quasi-aperiodic* if all eigenvalues of any $A(p)$ (bundle), $p \in \boldsymbol{p}$ have negative real parts and any imaginary part remains smaller in magnitude than ε. We have the following result.

Theorem 12. A real IP matrix $A(\boldsymbol{p})$ is quasi-aperiodic if

$$m^*(C(\boldsymbol{p}; \boldsymbol{b}_1, \varepsilon)) > 0, \tag{52}$$

where $C(\boldsymbol{p};\boldsymbol{b}_1,\varepsilon)$ is defined by (51), $\boldsymbol{b}_1$ is a two-sided bound along the real axis on all eigenvalues of $A(\boldsymbol{p})$, and $A(\boldsymbol{p})$ contains at least one real matrix that is aperiodic.

The proof of the theorem is based on the fact that if $A(\boldsymbol{p})$ is quasi-aperiodic, then the equivalent complex representation (51), cannot have solutions on the line parallel to the real axis within $\boldsymbol{b}_1$ and distant at ε.

It is to be noted that Theorems 8 to 12 can be modified to give only sufficient conditions if the corresponding m^* is replaced with a bound $\underline{m}$ or $\overline{m}$.

4. Conclusion

The concept of regularity margin $m^*(A(\boldsymbol{p}))$ of an interval parametric (IP) matrix $A(\boldsymbol{p})$ has been suggested, which is an alternative to another quantitative regularity measure, the regularity radius $r^*(A(\boldsymbol{p}))$ [11] of $A(\boldsymbol{p})$. While $r^*(A(\boldsymbol{p}))$ can be defined only for IP matrices having linear parametric dependencies (1b), $m^*(A(\boldsymbol{p}))$ can be introduced for matrices of the general non linear parametric dependencies type (1a).

It has been shown (Section 2) that $m^*(A(\boldsymbol{p}))$ can be determined as the minimum of an associated interval parametric linear programming problem (12). The use of $m^*(A(\boldsymbol{p}))$ leads to a new necessary and sufficient condition (Theorem 1) for ascertaining regularity of $A(\boldsymbol{p})$. Sufficient conditions for $A(\boldsymbol{p})$ to be regular or singular, based on upper $\overline{m}(A(\boldsymbol{p}$) on lower $\underline{m}(A(\boldsymbol{p}))$ bounds on $m^*(A(\boldsymbol{p}))$, are given in Corollary 2.

Several applications of the regularity margin or its bounds related to various robustness problems are illustrated in Theorems 2 to 12 and Corollaries 3 to 6 in Section 3.

It should be mentioned that the above concepts and results suggested for interval parametric matrices are applicable, after obvious modifications, to the case of interval (nonparametric) matrices.

It is expected that development of more efficient methods for computing $m^*(A(\boldsymbol{p}))$ or the bounds $\overline{m}(A(\boldsymbol{p}))$ and $\underline{m}(A(\boldsymbol{p}))$ would lead to a broader applicability of the present approach to assessing regularity quantitatively or qualitatively.

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