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A STUDY OF A NEW GENERALIZED BURGERS' EQUATION: SYMMETRY SOLUTIONS AND CONSERVATION LAWS

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Abstract. In this work, we study a general form of Burgers' equation with two variable coefficients depending on space and time. Using symmetry analysis we determine certain coefficient functions for which the corresponding nonlinear partial differential equations have Lie point symmetries. For each such equation we construct its conservation laws by the use of conservation theorem owing to Ibragimov. The importance of conversations laws is also mentioned. Moreover, group invariant and power series solutions are obtained for some special cases of the equation under study.

Keywords: Generalized Burgers' equation, Lie point symmetry, self-adjointness, conservation laws, conserved vector.

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1 Introduction

It is widely believed that many problems of real-world are actualized by nonlinear partial differential equations (NLPLDLs). See for example, Adeyemo et al. (2023); Alhasanat (2023); Babajanov & Abdikarimov (2022); Simbanefayi et al. (2023); Liu et al. (2023); Ay & Yasarv (2023); Demiray & Duman (2023); Farajov (2022); Zhu (2022); Rani et al. (2022); Zhang (2022). A generalized (3+1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation of engineering and physics was studied in Adeyemo et al. (2023). This equation has wide applications, particularly in plasma physics and fluid dynamics. In Babajanov & Abdikarimov (2022), the authors studied the Zakharov-Kuznetsov equation which models the vortices in geophysical flows and has appeared in many areas of physics, applied mathematics and engineering. The sixth-order Boussinesq equation which has double dispersion governs the motion of waves on water that has a stress surface and was investigated in Farajov (2022).

Burgers' equation, sometimes called Bateman-Burgers equation is a NLPLDL that occurs in different field of applied mathematics, such as traffic flow, fluid mechanics, gas dynamics and non-linear acoustics. This equation was initially instigated by Harry Bateman during 1915 in Bateman (1915) and thereafter investigated by J.M. Burgers in 1948.

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Burgers' equation was acquired as a consequence of incorporating linear diffusion alongside nonlinear wave motion. It is the easiest model for examining amalgamated effects of nonlinear diffusion and advection (Burgers, 1948; Whitham, 1999). In this study, we investigate the generalized variable coefficients Burgers' equation (gvcB-eqn)

$$f(t)u_t + g(t,x)uu_x - u_{xx} = 0. \tag{1}$$

Here $u(t,x)$ describes density variable, functions $f(t)$ and $g(t,x)$ are non-zero functions, and $g(t,x)uu_x$ defines the time dependent and space dependent non-linear convection reactions Cao & Zhang (2020). If there is no diffusion, the term u_{xx} is dropped, and the equation (1) reduces to

$$f(t)u_t + k(t,x,u,u_x) = 0.$$

Moreover, when $f(t) = 1$ and $k = a(u)u_x$ we get

$$u_t + a(u)u_x = 0.$$

This is named as in-viscid Burgers' equation (Freire & Sampaio, 2014; Freire, 2012). This equation is a particular case of (1) when there is no diffusion, $g = 1$ and $a(u) = u$.

In nature, conservation laws often represent physical conserved quantities, including energy, mass, angular and linear momentum, together with charge and additional constants of motion Olver (1993). Theory of conservation laws for NLPLDLs plays an indispensable part in examining their uniqueness, global existence and stability of solutions. The integrability of NLPLDLs is dependent on the number of conservation laws it possesses. Another vital facet of conservation laws is that they are invaluable in numerical integration of NLPLDLs (Bluman & Kumei, 1989; Khalique & Simbanefayi, 2021; Jamal, 2019; Bruzon et al., 2021; Bruzon et al., 2022; Khalique & Simbanefayi, 2021; Chulián et al., 2020). In the literature, one can find several techniques for computing conservation laws. In 1918, Noether founded the astonishing theorem, which states that every single conservation law belonging to a system, emerging from a variational principle, originates from an associated symmetry effect Noether (1918). Nevertheless, the applicability of Noether's was limited to systems that had a Lagrangian formulation (Ibragimov, 2006, 2007). For the sake of constructing conservation laws for systems without Lagrangians, researchers developed several generalities of Noether's theorem. See for example (Olver, 1993; Steudel, 1975; Ibragimov & Kolsrud, 2004; Kara & Mahomed, 2006; Ibragimov, 2007) and the references therein. In Kara & Mahomed (2006) the authors introduced the idea of partial-Lagrangian along-with associated Noether-type symmetries with conservation laws. The multiplier approach is described in Steudel (1975); Olver (1993). In Ibragimov (2007) a straightforward method for constructing conservation laws for systems was established. It is widely known, that all local conservation laws possess the structure

$$D_tF(t,x,u_{r_1}) + D_xG(t,x,u_{r_2}) = 0 \tag{2}$$

with D_t, D_x being the total differential operators and u_{r_1} u_{r_2} denote all possible derivatives u Ibragimov (1999).

We start our study by computing point symmetries of gvcB-eqn (1) and thereafter we calculate its conservation laws. The remaining study is set out as follows. Section 2 contains symmetries of (1) for certain variable coefficients f and g.

In Section 4, we use a theorem due to Ibragimov. In Section 3, we consider the special case of the gvcB-eqn (1), present some invariant solutions and determine the conservation laws for such equations. We end the study by some discussions and conclusions which we give in Section 5.

2 Symmetries of gvcB-eqn

We contemplate the most broadest point transformations Lie groups that do not change the gvcB-eqn (1).

Firstly, we examine a 1-parameter infinitesimal transformations group

$$t \to t + \varepsilon\tau(t,x,u),\ \ x \to x + \varepsilon\xi(t,x,u),\ \ u \to u + \varepsilon\eta(t,x,u) \tag{3}$$

with vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{4}$$

By appealing to the second prolongation (see Olver (1993))

$$pr^2X = X + \rho_1\frac{\partial}{\partial u_t} + \rho_2\frac{\partial}{\partial u_x} + \rho_{11}\frac{\partial}{\partial u_{tt}} + \rho_{12}\frac{\partial}{\partial u_{tx}} + \rho_{22}\frac{\partial}{\partial u_{xx}} \tag{5}$$

to gvcB-eqn (1), we realize that the coefficient functions $\xi(t,x,u), \tau(t,x,u)$ and $\eta(t,x,u)$, for particular values of $f(t)$ and $g(t,x)$ must satisfy the symmetry condition

$$\Big[\tau f_t(t)u_t + \tau g_t u u_x + \eta g u_x + \rho_1 f(t) + \rho_2 g u - \rho_{22}\Big]_{u_{xx}=f(t)u_t+guu_x} = 0 \tag{6}$$

with

$$\begin{aligned}
\rho_1 =& \eta_t - \tau_t u_t + \eta_u u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_t u_x,\\
\rho_2 =& \eta_x - \tau_x u_t + \eta_u u_x - \xi_x u_x - \tau_u u_t u_x - \xi_u u_x^2,\\
\rho_{22} =& \eta_{xx} + 2\eta_{xu}u_x + \eta_u u_{xx} + \eta_{uu}u_x^2 - 2\xi_x u_{xx} - \xi_{xx}u_x - 2\xi_{xu}u_x^2 - 3\xi_u u_x u_{xx}\\
&- \xi_{uu}u_x^3 - 2\tau_x u_{tx} - \tau_{xx}u_t - 2\tau_{xu}u_t u_x - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx} - \tau_{uu}u_t u_x^2.
\end{aligned}$$

Invoking the values of ρ_1, ρ_2 and ρ_{22} in (6), we get the determining equations

$$\tau_u = 0,\ \ \tau_x = 0, \tag{7}$$
$$\xi_u = 0, \tag{8}$$
$$\eta_{uu} = 0, \tag{9}$$
$$\tau g_t u + \xi g_x u + \eta g - \xi_t f(t) - 2\eta_{xu} + \xi_x g u + \xi_{xx} = 0, \tag{10}$$
$$\tau f_t(t) - \tau_t f(t) - \tau_x g u + 2\xi_x f(t) = 0, \tag{11}$$
$$\eta_t f(t) + \eta_x g u - \eta_{xx} = 0. \tag{12}$$

Equations (7) give

$$\tau = \tau(t), \tag{13}$$

where as the equation (8) gives

$$\xi = \xi(t,x). \tag{14}$$

The equation (9) leads to

$$\eta = P(t,x)u + Q(t,x) \tag{15}$$

for arbitrary P, Q. Substitution of η in (10) gives

$$\tau g_t u + \xi g_x u + P g u + Q g - \xi_t f(t) - 2P_x + \xi_x g u + \xi_{xx} = 0 \tag{16}$$

and separating (16) on u, we get

$$u : \tau g_t + \xi g_x + P g + \xi_x g = 0, \tag{17}$$
$$u^0 : Q g - \xi_t f(t) - 2P_x + \xi_{xx} = 0. \tag{18}$$

We substitute the value of η in (12) and get

$$P_t f(t)u + Q_t f(t) + P_x g u^2 + Q_x g u - P_{xx} u - Q_{xx} = 0 \tag{19}$$

and separating (19) on u yields

$$u^2 : P_x(t,x) = 0, \tag{20}$$
$$u^1 : P_t(t,x)f(t) + Q_x(t,x)g(t,x) - P_{xx}(t,x) = 0, \tag{21}$$
$$u^0 : Q_t(t,x)f(t) - Q_{xx}(t,x) = 0. \tag{22}$$

From equations (17) and (20) we find that

$$\xi(t,x) = \frac{1}{g}G(t) - \frac{1}{g}\left[\tau(t)\int g_t dx + P(t)\int g dx\right]. \tag{23}$$

Considering the equations (18) and (20), we have

$$Q = \frac{\xi_t f - \xi_{xx}}{g} \tag{24}$$

and using equation (21) we find that

$$P_t = -\frac{Q_x g}{f}. \tag{25}$$

From (20) we deduce that $P = P(t)$, then we have from equation (25) that $\left(Q_x g\right)_x = 0$ as the first condition. Now equation (11) gives us $\xi_x = \frac{f(t)}{2}\left(\frac{\tau(t)}{f(t)}\right)_t$ which leads to

$$\xi_{xx} = 0 \tag{26}$$

as the second condition. From equation (22), we get $Q_t f - Q_{xx} = 0$ as the third condition. Finally, from equation (11), we obtain

$$\tau = f\left(\int \frac{2\xi_x}{f}dt + C_1\right). \tag{27}$$

Equation (21) gives

$$Q_x = -\frac{f(t)}{g(t,x)}P^{'}(t) \tag{28}$$

and

$$Q_{xx} = P^{'}(t)f(t)\frac{g_x}{g^2}, \tag{29}$$

which leads to

$$Q = -P^{'}(t)f(t)\int \frac{1}{g}dx + S(t) \tag{30}$$

with $S(t)$ an arbitrary function. Equations (29) and (22) lead to

$$Q_t = P^{'}(t)\frac{g_x}{g^2}. \tag{31}$$

Differentiate (30) with respect to t. We get

$$Q_t = -P^{''}\int \frac{f}{g}dx - P^{'}\int \frac{[f_t g - f g_t]}{g^2}dx + S^{'}(t). \tag{32}$$

From (31) and (32), we obtain

$$P''(t)\int \frac{f}{g}dx + P'(t)\left[\int \frac{(f_t g - f g_t)}{g^2}dx + \frac{g_x}{g^2}\right] - S'(t) = 0. \tag{33}$$

Finally, the equation (24) leads to

$$\xi_t = \frac{g}{f}Q(t,x). \tag{34}$$

For given functions f and g, and splitting equation (33) on x yields $S(t)$, $P(t)$. After replacing S and P in (30), we may find $Q(t,x)$. Substituting $Q(t,x)$ in (34) we can find $\xi(t,x)$ after integration. The function $\tau(t)$ can now be obtained from equation (27). Thus, the infinitesimal symmetries of the gvcB-eqn (1) can be obtained by taking account of the three given conditions.

Let us take P and Q as constants. Then equations (21) and (22) are satisfied. The equation (11) gives

$$\xi_x = \frac{1}{2}f\left(\frac{\tau}{f}\right)_t = M(t), \tag{35}$$

$$\xi = xM(t) + N(t), \tag{36}$$

$$g = \frac{1}{Q}f(t)\Big(M'(t)x + N'(t)\Big) \tag{37}$$

for some functions M and N. Similarly, from the equation (17), we get

$$-P = \tau(t)\frac{g_t}{g} + M(t) + \Big(xM(t) + N(t)\Big)\frac{g_x}{g}. \tag{38}$$

With the help of equation (37), we have

$$\frac{g_x}{g} = \frac{(g/f)_x}{g/f} = \frac{M'}{xM' + N'} \tag{39}$$

and

$$\frac{g_t}{g} = \frac{g_t/f}{g/f} = \frac{(g/f)_t + (g/f)(f'/f)}{g/f} = \frac{xM'' + N''}{xM' + N'} + \frac{f'}{f}. \tag{40}$$

For $M'(t) \neq 0$, we substitute $g(t,x)$ and its derivatives in (38) to get that

$$\begin{aligned}-P &= \tau\left(\frac{xM'' + N''}{xM' + N'} + \frac{f'}{f}\right) + M + (Mx + N)\frac{M'}{xM' + N'} \\ &= \frac{(\tau M'' + MM')}{M'}\frac{x + \dfrac{\tau N'' + NM'}{\tau M'' + MM'}}{x + \dfrac{N'}{M'}} + \tau\frac{f'}{f} + M\end{aligned}$$

or

$$-P = \frac{\tau M''}{M'} + 2M + \tau\frac{f'}{f} \tag{41}$$

with

$$\tau = \frac{M'(N'M - NM')}{N''M' - N'M''}. \tag{42}$$

From equations (35) and (41), we get

$$-P = \frac{\tau M''}{M'} + \tau', \tag{43}$$

which leads to

$$\tau = -P\left(\frac{M + C}{M'}\right) \tag{44}$$

with P and C being constants. Equating (42) and (44), we have

$$-PM'(M+N)N'' + PM''N'(M+C) - MM'^2N' + NM'^3 = 0. \tag{45}$$

Then we have

$$\tau = -P\left(\frac{M+C}{M'}\right), \ \ \xi = xM(t) + N(t), \ \ \eta = Pu + Q.$$

From equation (35) one has

$$\frac{f'}{f} = \frac{\tau'}{\tau} - \frac{2M}{\tau},$$

which gives

$$f(t) = C_2\tau e^{-2\int \frac{M}{\tau}dt}$$

with C_2 a constant of integration. By using the value of τ in the last equation, we have

$$\begin{aligned} f(t) &= C_1\left(\frac{C+M}{M'}\right) exp\left[2\int \frac{M'M}{P(C+M)}dt\right] \\ &= C_1\left(\frac{M+C}{M'}\right) exp\left[\frac{2}{P}\int \frac{M}{M+C}dM\right] \end{aligned}$$

by taking $C_1 = C_2 P$ as a constant and changing the variable. By integration, we get

$$f(t) = C_1\left(\frac{M+C}{M'}\right)\left(\frac{e^M}{(M+C)^C}\right)^{\frac{2}{P}} \tag{46}$$

and the equation (37) leads to

$$g(t,x) = \frac{C_1}{Q}\left(\frac{M+C}{M'}\right)\left(\frac{e^M}{(M+C)^C}\right)^{\dfrac{2}{P}}\left(xM' + N'\right). \tag{47}$$

Thus, according to the above calculations, we are able to state theorem 1.

Theorem 1. *For arbitrary constants P, $Q \neq 0$, C and with arbitrary functions $M(t)$, $N(t)$, (with $M'(t) \neq 0$) satisfying* (45)*, the functions*

$$\tau = -P\left(\frac{M+C}{M'}\right), \ \xi = xM(t) + N(t), \ \eta = Pu + Q$$

define the point symmetry $\mathcal{X} = \tau\partial/\partial t + \xi\partial/\partial x + \eta\partial/\partial u$ of the gvcB-eqn (1) *provided* (46) *and* (47) *are satisfied.*

For the case when $M'(t) = 0$, that is, when M is a constant, from equation (37) we have $g_x = 0$ and thus $\left(\dfrac{g_x}{g}\right)_x = 0$. By taking P as a constant and considering equation (38), we find that

$$g = C_2 exp\left[-(P+M)\int \frac{1}{\tau}dt\right]. \tag{48}$$

From equation (27), we can compute the expression

$$\int \frac{1}{\tau}dt = \frac{1}{2M}ln\left(\int \frac{1}{f}dt + \frac{C_1}{2M}\right) + C_3, \tag{49}$$

which leads to

$$g = C\left(\int \frac{1}{f}dt + C_4\right)^m, \tag{50}$$

where $C = C_2 e^{C_3}$, $C_4 = \dfrac{C_1}{2M}$ and $m = -\dfrac{P+M}{2M}$.

We see from equation (34) that

$$\xi = Mx + Q\int \frac{g}{f}dt. \tag{51}$$

From equation (27), one has

$$\tau = 2Mf\left[\int \frac{1}{f}dt + C_4\right]. \tag{52}$$

When f and g satisfy (50) for some constant m, C and C_4, then by setting $P = -M(2m+1)$, we have

$$\eta = -M(2m+1)u + Q, \tag{53}$$

and (51), (52), together with (53) define the Lie point symmetry for equation (1). We conclude the result above as follows.

Theorem 2. *If f and g satisfy equation* (50) *for some constants C, C_4 and m, then equation* (1) *admits the point symmetry $\mathcal{X} = \tau\partial/\partial t + \xi\partial/\partial x + \eta\partial/\partial u$ with τ, ξ and η defined by* (51), (52) *and* (53), *respectively, with arbitrary constants M and Q.*

We conclude that equation (1) *has Lie point symmetry if the variable coefficients $f(t)$ and $g(t,x)$ satisfy both equations* (46) *and* (47) *or equation* (50).

3 Invariant solutions

In this section we consider the special case of the gvcB-eqn (1) and present some invariant solutions.

We take $f(t) = 1$ and following theorem 2 we obtain $g(t,x) = Ct^m$. The gvcB-eqn (1) now reads

$$u_t + Ct^m u u_x - u_{xx} = 0, \tag{54}$$

where C is a constant. The Lie point symmetries of equation (54) are given by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - (2m+1)u\frac{\partial}{\partial u},\\
X_3 &= Ct^{m+1}\frac{\partial}{\partial x} + (m+1)\frac{\partial}{\partial u},
\end{aligned}$$

that form a finite Lie algebra L_3.

3.1 Symmetry reductions and solution of (54)

Case 1. For the vector $X_1 = \frac{\partial}{\partial x}$ solving the Lagrange system we have that

$$u = f(t).$$

Plugging the above equation into the gvcB-eqn (1) we get $f'(t) = 0$ which implies that $f(t) = k$, where k is a constant of integration. Thus

$$u(t,x) = k. \tag{55}$$

Case 2. Considering the vector $X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - (2m+1)u\frac{\partial}{\partial u}$ solving the associated characteristic equations we obtain the invariants

$$\begin{aligned}
J_1 &= t,\\
J_2 &= u - \frac{m+1}{Ct^{m+1}}x.
\end{aligned}$$

Hence we have the similarity solution

$$u = \frac{m+1}{Ct^{m+1}}x + \phi(t). \tag{56}$$

Substituting equation (56) into equation (1) we get the first-order ordinary differential equation

$$t\phi'(t) + (m+1)\phi(t) = 0 \tag{57}$$

whose solution is given by

$$\phi(t) = \frac{k}{t^{m+1}}, \tag{58}$$

where k is a constant. Consequently, we have the solution to the gvcB-eqn (1) as

$$u(t,x) = \frac{Ck + (m+1)x}{Ct^{m+1}}. \tag{59}$$

The wave profile of the rational solution (59), is given for parameters $C = 2, k = 2$, and $m = 1$ recorded with interval $5 < t, x < 10$, can be seen in Figure (1).

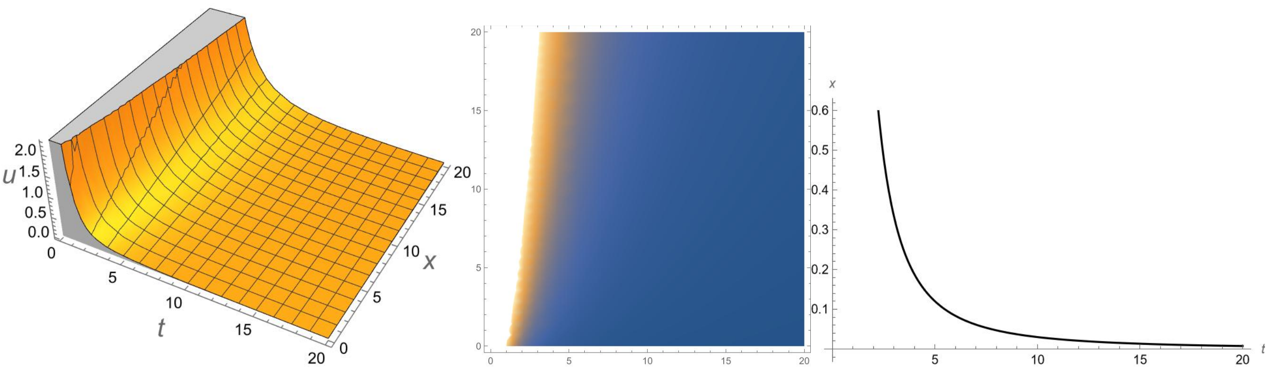

Figure 1: 3D and 2D solution profiles of (59)

Case 3. Finally from the vector $X_3 = Ct^{m+1}\frac{\partial}{\partial x} + (m+1)\frac{\partial}{\partial u}$ solving the characteristic equations leads to the invariants

$$J_1 = \frac{x}{\sqrt{t}},$$
$$J_2 = ut^{\frac{2m+1}{2}}.$$

From the above invariants we obtain the invariant solution

$$u = t^{-\frac{(2m+1)}{2}}\phi(\xi),\ \xi = \frac{x}{\sqrt{t}}. \tag{60}$$

Substituting equation (60) into equation (1) we obtain the second-order nonlinear ordinary differential equation

$$2\phi''(\xi) + \xi\phi'(\xi) - 2C\phi(\xi)\phi'(\xi) + (2m+1)\phi(\xi) = 0. \tag{61}$$

Consequently, the solution of the gvcB-eqn (1) under X_3 is $u = t^{-\frac{(2m+1)}{2}}\phi(\xi)$, where $\phi(\xi)$ satisfies

$$2\phi''(\xi) + \xi\phi'(\xi) - 2C\phi(\xi)\phi'(\xi) + (2m+1)\phi(\xi) = 0.$$

Now, we present the analytical solutions to the ordinary differential equation (61) by employing the power series method. See for example Liu & Li (2006). Our objective is to find a solution to equation (61) in the form of a power series with a prescribed structure

$$\phi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \tag{62}$$

with the first and second derivatives of $\phi(\xi)$ in (62) given as

$$\begin{aligned}\phi'(\xi) &= \sum_{n=1}^{\infty} n a_n \xi^{n-1},\\ \phi''(\xi) &= \sum_{n=2}^{\infty} n(n-1) a_n \xi^{n-2}.\end{aligned} \tag{63}$$

Substituting the different values of $\phi(\xi)$ as expressed in equation (63) into equation (61) ensures

$$\begin{aligned}&4a_2 + 12a_3\xi + 2\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}\xi^n + a_1\xi + \sum_{n=2}^{\infty} n a_n \xi^n - 2Ca_0a_1 - 4Ca_1a_2\xi\\ &- 2C\sum_{n=2}^{\infty}\left[\sum_{k=0}^{n}(n-k+1)a_k a_{n-k+1}\right]\xi^n + \nu a_0 + \nu a_1\xi + \nu\sum_{n=2}^{\infty} a_n\xi^n = 0,\end{aligned} \tag{64}$$

where $\nu = 2m+1$. Comparing coefficients from the above equation we obtain (for $n = 0$)

$$4a_2 - 2Ca_0a_1 + \nu a_0 = 0, \tag{65}$$

and (for $n = 1$)

$$12a_3 - 4Ca_1a_2 + a_1(\nu+1) = 0. \tag{66}$$

Generally, for $n \geq 2$

$$a_{n+2} = \frac{1}{2(n+1)(n+2)}\left[2C\sum_{k=0}^{n}(n-k+1)a_k a_{n-k+1} - n a_n - \nu a_n\right]. \tag{67}$$

Thus,

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \sum_{n=2}^{\infty} a_{n+2}\xi^{n+2}. \tag{68}$$

Hence, the power series solution of equation (54) is

$$\begin{aligned}u(t,x) =& t^{-\frac{(2m+1)}{2}}\left(a_0 + a_1\frac{x}{\sqrt{t}} + \frac{2Ca_0a_1 - \nu a_0}{4t}x^2 + \frac{4Ca_1a_2 - a_1(\nu+1)}{12t^{\frac{3}{2}}}x^3\right.\\ &\left.+\frac{1}{2(n+1)(n+2)}\left[2C\sum_{k=0}^{n}(n-k+1)a_k a_{n-k+1} - n a_n - \nu a_n\right]\left(\frac{x}{\sqrt{t}}\right)^{n+2}\right),\end{aligned} \tag{69}$$

where $a_i, (i = 0, 1, 2)$ are arbitrary constants. Other coefficients $a_n (n \geq 3)$ can be successively determined from the recursion relation (67).

4 Conservation laws

To investigate the conservation laws of the gvcB-eqn (1), we recall some salient features of conservation laws from Ibragimov (2007).

Consider a PDE

$$\mathcal{E}(x, \mathcal{Q}, \mathcal{Q}_{(1)}, ..., \mathcal{Q}_{(s)}) = 0 \tag{70}$$

with $\mathcal{Q}$ dependant and $x = (x^1, x^2)$ as independent variables. Here $\mathcal{Q}_i = \partial\mathcal{Q}/\partial x^i$ and $\mathcal{Q}_{ij} = \partial^2\mathcal{Q}/\partial x^i\partial x^j$, respectively, denote all partial derivatives of the first and second orders.

Definition 1. *The adjoint equation to* (70) *is*

$$\mathcal{E}^*(x, \mathcal{Q}, \mathcal{P}, \mathcal{Q}_{(1)}, \mathcal{P}_{(1)}, ..., \mathcal{Q}_{(s)}, \mathcal{P}_{(s)}) = 0 \tag{71}$$

with

$$\mathcal{E}^*(x, \mathcal{Q}, \mathcal{P}, \mathcal{Q}_{(1)}, \mathcal{P}_{(1)}, ..., \mathcal{Q}_{(s)}, \mathcal{P}_{(s)}) = \frac{\delta(\mathcal{P}\mathcal{E})}{\delta\mathcal{Q}},$$

where $\mathcal{P} = \mathcal{P}(x^1, x^2)$ *are new dependent variables,* $\mathcal{P} = \mathcal{P}(x)$ *and*

$$\frac{\delta}{\delta\mathcal{Q}} = \frac{\partial}{\partial\mathcal{Q}} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial\mathcal{Q}_{i_1\cdots i_s}}$$

is the Euler operator whereas

$$D_i = \frac{\partial}{\partial x^i} + \mathcal{Q}_i\frac{\partial}{\partial\mathcal{Q}} + \mathcal{Q}_{ij}\frac{\partial}{\partial\mathcal{Q}_j} + \cdots$$

is the total differentiation operator.

Ibragimov in his paper Ibragimov (2007) demonstrated that (71) acquires all symmetries of (70). The following Lemma will be utilized in our subsequent work.

Lemma 1. *Any symmetry*

$$\mathcal{G} = \xi^i(x, \mathcal{Q}, \mathcal{Q}_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta(x, \mathcal{Q}, \mathcal{Q}_{(1)}, \dots)\frac{\partial}{\partial\mathcal{Q}}$$

of (70) *furnishes a conservation law* $D_i(C^i) = 0$ *for* (70) *and* (71). *The components of conserved vector are specified by*

$$\begin{aligned} C^i =& \xi^i\mathcal{L} + w\left[\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ijk}}\right) - \cdots\right] \\ &+ D_j(w)\left[\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ijk}}\right) + D_kD_r\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ijkr}}\right) - \cdots\right] \\ &+ D_jD_k(w)\left[\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ijk}} - D_r\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ijkr}}\right) + \cdots\right] + \cdots \end{aligned} \tag{72}$$

with $w = \eta - \xi^j\mathcal{Q}_j$ *and Lagrangian* $\mathcal{L} = \mathcal{P}\mathcal{E}(x, \mathcal{Q}, \mathcal{Q}_{(1)}, ...\mathcal{Q}_{(s)})$. *For a second-order equation,* (72) *becomes*

$$C^i = \xi^i\mathcal{L} + w\left[\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ij}}\right)\right] + D_j(w)\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Q}_{ij}}\right).$$

We first consider the self-adjoint equations from (1).

Theorem 3. *The gvcB-eqn* (1) *is nonlinearly self-adjoint as long as* $g_x = 0$.

Proof. From (1) we have

$$\mathcal{E}^* \equiv \frac{\delta(\mathcal{P}\mathcal{E})}{\delta\mathcal{Q}} = f_t(t)\mathcal{P} + f(t)\mathcal{P}_t + g_x(t,x)\mathcal{Q}\mathcal{P} + g(t,x)\mathcal{Q}\mathcal{P}_x + \mathcal{P}_{xx}.$$

We consider $\mathcal{P} = \varphi(t, x, \mathcal{Q})$ in such a way that (1) turns into nonlinearly self-adjoint. Thereby, we suppose

$$\mathcal{E}^* |_{\mathcal{P}=\varphi(t,x,\mathcal{Q})} = \Lambda(t, x, \mathcal{Q}, \mathcal{Q}_{(1)}, ...)\mathcal{E},$$

where

$$\mathcal{P}_t = D_t[\varphi(t,x,\mathcal{Q})] = \varphi_{\mathcal{Q}}.\mathcal{Q}_t + \varphi_t,$$
$$\mathcal{P}_x = D_x[\varphi(t,x,\mathcal{Q})] = \varphi_{\mathcal{Q}}.\mathcal{Q}_x + \varphi_x,$$
$$\mathcal{P}_{xx} = D_x(\mathcal{P}_x) = \varphi_{\mathcal{Q}}.\mathcal{Q}_{xx} + \varphi_{\mathcal{Q}\mathcal{Q}}.\mathcal{Q}^2 + 2\varphi_{x\mathcal{Q}}.\mathcal{Q}_x + \varphi_{xx}.$$

Straight forward computations reveal that $\Lambda = \varphi_{\mathcal{Q}} = 0$, and so

$$f_t(t)\varphi + f(t)\varphi_t + g_x(t,x)\mathcal{Q}\varphi + g(t,x)\mathcal{Q}\varphi_x + \varphi_{xx} = 0. \tag{73}$$

Since (73) is true for every t, x and $\mathcal{Q}$, we get

$$g_x(t,x)\varphi + g(t,x)\varphi_x = 0$$

and

$$f_t(t)\varphi + f(t)\varphi_t + \varphi_{xx} = 0.$$

The calculations reveal that for $g_x = 0$, we get $\varphi = Kf(t) \neq 0$ for a constant K. This completes the proof. □

Using Lie point symmetries acquired in Theorem 1 and invoking Lemma 1, we gain conserved vectors (C^1, C^2), which solves

$$\left(D_tC^1 + D_xC^2\right)\Big|_{\mathcal{E}=0,\mathcal{E}^*=0} = 0.$$

Case 1. f, g are constants

We employ Lemma 1 to gvcB-eqn (1), by considering $f = g = 1$. This yields

$$u_t + uu_x - u_{xx} = 0 \tag{74}$$

and its adjoint equation is

$$v_t + uv_x + v_{xx} = 0. \tag{75}$$

By using Maple the Lie point symmetries of (74) are

$$\Gamma_1 = \frac{\partial}{\partial t},\ \Gamma_2 = \frac{\partial}{\partial x},\ \Gamma_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$\Gamma_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\ \Gamma_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.$$

Consider the symmetry $\Gamma_1 = \partial/\partial t$. Following the paper Ibragimov (2007), the conserved vector associated to Γ_1 is

$$C^1 = v(uu_x - u_{xx}),\quad C^2 = v(u_{tx} - uu_t) + u_tv_x.$$

Due to the presence of the arbitrary solution v of the adjoint equation (75), we gain infinitely many conservation laws of system (74) and (75).

Similarly, for $\Gamma_2 = \partial/\partial x$, associated conserved vector becomes

$$C^1 = -vu_x,\quad C^2 = vu_t + v_xu_x.$$

Likewise, one can obtain conserved vectors corresponding to the other symmetry operators Γ_3, Γ_4 and Γ_5.

Case 2. $f = 1$, $g = tx$

Now apply Lemma 1 to

$$u_t + txuu_x - u_{xx} = 0 \tag{76}$$

along with its adjoint

$$v_t - tuv - txuv_x - v_{xx} = 0. \tag{77}$$

For this case, (76) has the symmetry

$$Y_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 4u\frac{\partial}{\partial u}$$

and the associated conserved vector is

$$C^1 = (2tu + 2t^2xuu_x - 2tu_{xx} + 4u + 2tu_t)v, \quad C^2 = (xu_t - 4txu^2 + 5u_x)v + (4u + xu_x)v_x.$$

Case 3. $f(t) = t$, $g = 1$

We apply Lemma 1 to the equation

$$tu_t + uu_x - u_{xx} = 0 \tag{78}$$

in conjunction with its adjoint equation

$$v + tv_t + uv_x + v_{xx} = 0. \tag{79}$$

Equation (78) has the symmetries

$$S_1 = t\frac{\partial}{\partial t},\ S_2 = \frac{\partial}{\partial x},\ S_3 = \ln t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$S_4 = t\ln t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - \frac{u}{2}\frac{\partial}{\partial u},\ S_5 = t(\ln t)^2\frac{\partial}{\partial t} + x\ln t\frac{\partial}{\partial x} + (x - u\ln t)\frac{\partial}{\partial u}.$$

For $S_1 = t\,\partial/\partial t$, associated conserved vector becomes

$$C^1 = tv(uu_x - u_{xx}), \quad C^2 = (u_{tx} - uu_t)tv + tu_tv_x.$$

Likewise, one can obtain conserved vectors corresponding to the remaining symmetry operators S_2, S_3, S_4 and S_5.

Case 4. $f(t) = t$, $g = tx$

Let us apply Lemma 1 to the equation

$$tu_t + txuu_x - u_{xx} = 0 \tag{80}$$

whose adjoint equation is

$$v(1 + tu) + tv_t + txuv_x + v_{xx} = 0. \tag{81}$$

For this case, equation (80) has the symmetry

$$U_1 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$

and its associated conserved vector gives

$$C^1 = tv(txuu_x - u_t - u_{xx}), \quad C^2 = (u_x + tu_{tx} - txu^2 - t^2xuu_t)v + v_x.$$

5 Conclusion

We studied the gvcB-eqn (1), which had two functions $f(t)$ as well as $g(t,x)$ that are arbitrary. We performed symmetry classification and found distinct functions f and g for which equation (1) admitted Lie point symmetries. Then for each case we found conservation laws by calling on the conservation theorem owing to Ibragimov. Conservation laws are very significant in investigating differential equations as they can be employed to determine the integrability of differential equations, checking the sustainability of methods for obtaining numerical solution, determine exact solutions for differential equations, etc. Our future work will involve constructing analytic solutions to differential equations that are achieved in this work for various cases for $f(t)$ and $g(t,x)$.

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