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TOTAL DOMINATION INTEGRITY OF GRAPHS

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Abstract. Domination concept has been widely used in fields such as Science, Technology, Engineering, Communication Networks, etc. Total domination is one of well-known domination concepts. Integrity is also another important parameter in network design. A subset S of $V(G)$ is called a total dominating set (TD-set) if every vertex of G is adjacent to some vertex in S. In this paper, the concept of total domination integrity is introduced as a new parameter of vulnerability and some properties, bounds and total domination integrity of some graph classes are determined. Total domination integrity of a graph G with no isolated vertices is denoted by $TDI\,(G)$ and defined as $TDI\,(G) = min\,\{|S| + m\,(G-S) : S \subseteq V\,(G)\}$ where S is a total dominating set of G and $m\,(G-S)$ is an order of maximum component of $G-S$. In this study, all the graphs are considered simple, finite, without isolated vertices and undirected.

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1 Introduction

Networks and network design have become more important with the increasing demand for information transfer. A communication network consists of nodes and links that connect these nodes. The efficiency of the network decreases when the nodes or the links of the communication network are damaged. In order to ensure the continuity of the data flow, the stability of network is important after the damages that may occur in the network. Here, the concept of vulnerability comes to mind. Vulnerability is the resistance of any communication network after any failures on its nodes or links.

A communication network can be modeled with a graph where nodes are represented by vertices and links by edges. Therefore, these parameters on graph models are studied in vulnerability analysis of networks. Many parameters have been introduced for the measurement of vulnerability. Some of them are connectivity, tenacity, toughness, integrity, domination integrity.

In analysis of the vulnerability of a communication network, some of the fundamental questions are (i) what is the number of elements that must be removed to disconnect a network (ii) what is the number of elements that are not functioning and (iii) what is the size of the largest remaining component, in which mutual communication still exists.

The connectivity is the parameter that gives the answer to the first question. The connectivity of G, written $\kappa(G)$, is the minimum order of a vertex set S such that $G-S$ is disconnected or has only one vertex (West, 2001).

Integrity is one of the well-known vulnerability parameters that tries to find answers to second and third question. Integrity was introduced by Barefoot et al. (1987) and it is defined as $I(G) = min\{|S| + m(G-S) : S \subset V(G)\}$ where $m(G-S)$ denotes the maximum order of a component of $G-S$ (Goddard & Swart, 1990).

Failure on vertices or edges which have special properties plays a great role in the vulnerability analysis. For example, domination is a well-known concept in network design and it has a wide range of applications to many areas like Science, Technology, Engineering.

Definition 1. *The open neighborhood of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and closed neighborhood of v is $N[v] = \{v\} \cup N(v)$ (Henning, 2009).*

Definition 2. *For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = S \cup N(S)$ (Henning, 2009).*

Definition 3. *A subset S of $V(G)$ is called dominating set if for every $v \in V - S$, there exist a $u \in S$ such that v is adjacent to u (Haynes et al., 1998).*

Definition 4. *The minimum cardinality of a minimal dominating set in G is called the domination number of G denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a γ-set of G (Haynes et al., 1998).*

There are various types of domination defined by researchers in recent studies. Some of them are connected domination, independent domination, efficient domination, total domination (Hedetniemi & Laskar, 1990).

Later on, Sundareswaran and Swaminathan (2010a) have introduced the concept of domination integrity as a new vulnerability parameter and it is defined as follows.

Definition 5. *The domination integrity of a connected graph G is denoted by $DI(G)$ and defined as $DI(G) = \min\{|S| + m(G-S) \ : S \text{ is a dominating set}\}$ where $m(G-S)$ is the order of a maximum component of $G-S$ (Sundareswaran, 2010).*

Many new results domination integrity were found by Sundareswaran and Swaminathan (2010b, 2012). Vaidya and Kothari have discussed domination integrity in the context of some graph operations (Vaidya & Kothari, 2012) and splitting graph of path P_n and cycle C_n (Vaidya & Kothari, 2013). Vaidya and Shah determined the domination integrity of total graphs of path P_n, cycle C_n and star $K_{1,n}$ (Vaidya & Shah, 2014a) and also determined the domination integrity of square graph of path (Vaidya & Shah, 2014b). Computational complexity of domination integrity in graphs is studied by Sundareswaran and Swaminathan (2015). Beşirik and Kılıç (2018) determined domination integrity of wheel $W_{1,n}$, double star $S_{m,n}$, friendship F_n, ladder L_n, thorn graphs of P_n and C_n.

2 Total Domination

Total domination is one of well-known domination concepts and it was introduced by Cockayne, Dawes and Hedetniemi (1980) as follows.

Definition 6. *A total dominating set, abbreviated TD-set, of a graph $G = (V(G), E(G))$ with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Thus a set $S \subseteq V$ is a TD-set in G if $N(S) = V(G)$ (Henning & Yeo, 2013).*

Definition 7. *The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set (Henning & Yeo, 2013).*

The following results are used in proofs of main results.

Proposition 1. *For $n \geq 3$,*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \ (mod\ 4) \\ \frac{n+2}{2}, & n \equiv 2 \ (mod\ 4) \\ \frac{n+1}{2}, & otherwise \end{cases}$$

(Henning & Yeo, 2013).

Proposition 2. *Let G be a graph with no isolated vertices. Then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$ (Henning & Yeo, 2013).*

Proposition 3. *Let G be a graph of order n with no isolated vertices.Then $\gamma_t \geq \frac{n}{\Delta}$ (Henning & Yeo, 2013).*

Proposition 4. *If G is a connected graph order at least two, then $\gamma_t(G) \geq rad(G)$ (Henning & Yeo, 2013).*

Proposition 5. *If G is a connected graph order at least two, then $\gamma_t(G) \geq \frac{diam(G)+1}{2}$ (Henning & Yeo, 2013).*

Proposition 6. *If G is a graph of girth g, then $\gamma_t(G) \geq g/2$ (Henning & Yeo, 2013).*

3 Total Domination Integrity

Domination has been used to many problem models such as location, monitoring communication or networks, routing, etc. Total domination plays a role in the problem of placing monitoring devices in a system. Every site in the system, including the monitors, is adjacent to a monitor site. If a monitor is damaged, then an adjacent monitor can still protect the system (Cockayne & Hedetniemi, 1977). But if the adjacent monitor is damaged, then the system becomes vulnerable. Thus, when total dominating sets are removed in a network, the damage is vital. This motivated introducing total domination integrity as a new parameter of stability when total dominating sets are damaged.

Definition 8. *The total domination integrity of a graph G with no isolated vertices is $TDI(G) = min\{|S| + m(G-S) : S \subseteq V(G)\}$ where $m(G-S)$ denotes the order of a maximum component of $G-S$ and S is a total dominating set of G.*

Definition 9. *A set $S \subseteq V(G)$ is a TDI -set if $TDI(G) = |S| + m(G-S)$ and S is a total dominating set of G.*

If two graphs have same connectivity, integrity and domination integrity values, then these parameters are not enough to distinguish them. So, a new parameter that distinguishes these graphs is needed. Then, the following questions arise: How can a network designer determine which network is more stable than the other? Is the total domination integrity a vulnerability parameter that compare these graphs in resistance?

Let's see this with a simple comparison between two graphs. Assume that G_1 and G_2 are graphs with same order as follows in Fig. 1.

For G_1 and G_2, $\kappa(G_1) = \kappa(G_2) = 1$, $I(G_1) = I(G_2) = 3$ and $DI(G_1) = DI(G_2) = 3$. So, connectivity, integrity and domination integrity do not distinguish between G_1 and G_2. Total domination integrity values of these graphs are computed as follows.

Consider $S_1 = \{u_2, u_3, u_5\}$ as a total dominating set of G_1 then $m(G_1 - S_1) = 1$. There does not exist a total dominating set of G_1 such that $|X| + m(G_1 - X) < |S_1| + m(G_1 - S_1)$. So, $TDI(G_1) = 3 + 1 = 4$.

Consider $S_2 = \{v_1, v_3\}$ as a total dominating set of G_2 then $m(G_2 - S_2) = 1$. There does not exist a total dominating set of G_2 such that $|X| + m(G_2 - X) < |S_2| + m(G_2 - S_2)$. So, $TDI(G_2) = 2 + 1 = 3$.

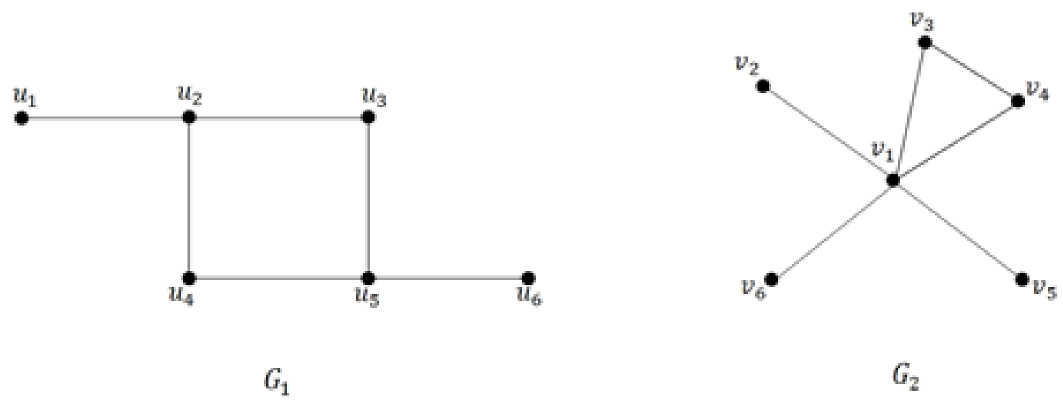

Figure 1: Graph G_1 and graph G_2

For G_1 and G_2, $TDI(G_1) = 4$ and $TDI(G_2) = 3$. So, it can be said that G_1 is more stable than G_2. Then, total domination integrity is a suitable measure of vulnerability which distinguishes between these graphs.

Observation 1. *$2 \leq TDI(G) \leq n$. $TDI(G) = 2$, if and only if $G = K_2$. For upper bound, equality holds for $G = K_n$. If $G \neq K_2$, then $3 \leq TDI(G) \leq n$.*

Observation 2. *Let G be a graph of order n such that neither G nor $\overline{G}$ contains isolated vertices. Then, $3 \leq TDI\left(\overline{G}\right) \leq n$. For upper bound, equality holds if G or $\overline{G}$ consists of disjoint copies of K_2.*

Observation 3. *$I(G) \leq DI(G) \leq TDI(G)$.*

Proof. From Proposition 2, we know that $\gamma(G) \leq \gamma_t(G)$, then $DI(G) \leq TDI(G)$. Since $I(G) \leq DI(G)$ (Sundareswaran, 2010), then $I(G) \leq DI(G) \leq TDI(G)$.
For example; for $n \geq 2$, consider $G = K_{1,n}$. Then $I(G) = DI(G) = 2$ and $TDI(G) = 3$. □

Observation 4. *For all graph G order of $n \geq 2$ with no isolated vertices, $TDI(G) \geq \chi(G)$ where $\chi(G)$ is the chromatic number of G.*

Proof. Let G be a graph order of $n \geq 2$ with no isolated vertices. Since $I(G) \geq \chi(G)$ (Bagga et al., 1992), then $TDI(G) \geq \chi(G)$ where $\chi(G)$ is the chromatic number of G. □

Observation 5. *For every graph G order of $n \geq 2$ with no isolated vertices, $\delta(G) + 1 \leq TDI(G) \leq \alpha(G) + 1$ where $\delta(G)$ is the minimum vertex degree of G and $\alpha(G)$ is the covering number of G.*

Proof. Let G be a graph order of $n \geq 2$ with no isolated vertices. Since $\delta(G) + 1 \leq I(G) \leq DI(G) \leq \alpha(G) + 1$ (Sundareswaran, 2010) and from *Observation 3*, we obtain $\delta(G) + 1 \leq I(G) \leq DI(G) \leq TDI(G) \leq \alpha(G) + 1$. Hence, $\delta(G) + 1 \leq TDI(G) \leq \alpha(G) + 1$. □

Theorem 1. *Let G be a graph of order $n > 2$ with no isolated vertices. Then $TDI(G) \geq \gamma_t + 1$.*

Proof. Let G be a graph of order $n > 2$ with no isolated vertices and S is a γ_t-set of G. So, $|S| = \gamma_t$ and $m(G - S) \geq 1$. Then,

$$\begin{aligned} TDI(G) &= \min_{S \subseteq V(G)} \{|S| + m(G - S)\} \\ &\geq \min\{\gamma_t + 1\} \\ &\geq \gamma_t + 1 \end{aligned}$$

□

Theorem 2. *For any connected graph G with no isolated vertices, $TDI(G) = \gamma_t(G)$ if and only if $G = K_2$.*

Proof. Let G be a connected graph with no isolated vertices. If $TDI(G) = \gamma_t(G)$, then $|S| + m(G - S) = \gamma_t(G)$.
Case 1: If $\gamma_t(G) = 2$, then $|S| + m(G - S) = 2$. Since S is a total dominating set of G and $\gamma_t(G) \geq 2$, then $|S| \geq 2$. Hence, $|S| = 2$ and $m(G - S) = 0$. So $G \cong K_2$.
Case 2: Let $\gamma_t(G) \geq 3$ and $|S| + m(G - S) = \gamma_t(G)$. Then, $|S| = \gamma_t(G)$ and $m(G - S) = 0$. So, $E(G) = \emptyset$ (if $E(G) \neq \emptyset$, then $TDI(G) \geq \gamma_t + 1$ from *Theorem 1*). Since G is a connected graph with no isolated vertices, it is impossible that $E(G) = \emptyset$. Hence, if $TDI(G) = \gamma_t(G)$, then $G = K_2$. The converse is obvious. □

Theorem 3. *Let G be a graph of order $n > 2$ with no isolated vertices. Then, $TDI(G) \geq \frac{n}{\Delta} + 1$.*

Proof. Let G be a connected graph of order $n > 2$ with no isolated vertices and S is a γ_t-set of G. So, $|S| = \gamma_t$ and $m(G - S) \geq 1$. From *Theorem 1*, $TDI(G) \geq \gamma_t + 1$.
It has been proved that $\gamma_t \geq \frac{n}{\Delta}$ in *Proposition 3*. Since $\gamma_t \geq \frac{n}{\Delta}$, then we have $TDI(G) \geq \gamma_t + 1 \geq \frac{n}{\Delta} + 1$. □

Theorem 4. *Let G be a connected graph of order $n > 2$ with no isolated vertices. Then $TDI(G) \geq rad(G) + 1$.*

Proof. Let G be a connected graph of order $n > 2$ with no isolated vertices and S is a γ_t-set of G. So, $|S| = \gamma_t$ and $m(G - S) \geq 1$. From *Theorem 1*, $TDI(G) \geq \gamma_t + 1$.
It has been proved that $\gamma_t \geq rad(G)$ in *Proposition 4*. Since $\gamma_t \geq rad(G)$, then we have $TDI(G) \geq \gamma_t + 1 \geq rad(G) + 1$. Hence, $TDI(G) \geq rad(G) + 1$. □

Theorem 5. *Let G be a connected graph of order $n > 2$ with no isolated vertices. Then $TDI(G) \geq \frac{diam(G)+3}{2}$.*

Proof. Let G be a connected graph of order $n > 2$ with no isolated vertices and S is a γ_t-set of G. So, $|S| = \gamma_t$ and $m(G - S) \geq 1$. From *Theorem 1*, $TDI(G) \geq \gamma_t + 1$.
It has been proved that $\gamma_t \geq \frac{diam(G)+1}{2}$ in *Proposition 5*. Since $\gamma_t \geq \frac{diam(G)+1}{2}$, then we have $TDI(G) \geq \gamma_t + 1 \geq \frac{diam(G)+1}{2} + 1 = \frac{diam(G)+3}{2}$. Hence, $TDI(G) \geq \frac{diam(G)+3}{2}$. □

Theorem 6. *Let G be a connected graph of girth g and order $n > 2$ with no isolated vertices. Then $TDI(G) \geq \gamma_t + 1 \geq \frac{g+2}{2}$.*

Proof. Let G be a connected graph of girth g and order $n > 2$ with no isolated vertices and S is a γ_t-set of G. So, $|S| = \gamma_t$ and $m(G - S) \geq 1$. From *Theorem 1*, $TDI(G) \geq \gamma_t + 1$.
It has been proved that $\gamma_t \geq \frac{g}{2}$ in *Proposition 6*. Since $\gamma_t \geq \frac{g}{2}$, then we have $TDI(G) \geq \gamma_t + 1 \geq \frac{g}{2} + 1 = \frac{g+2}{2}$. Hence, $TDI(G) \geq \frac{g+2}{2}$. □

Theorem 7. *For any graph G of order n such that contains no isolated vertices, if $TDI(G) = n$, then $diam(G) \leq 2$.*

Proof. Let G be a graph of order n such that contains no isolated vertices. Assume that $diam(G) \geq 3$, then G contains a path P_4. Thus, $TDI(G) \leq n - 1$ and it is a contradiction. Hence, $diam(G) \leq 2$. □

Theorem 8. *Let G be a graph of order n such that neither G nor $\overline{G}$ contains isolated vertices.*

(i) $6 \leq TDI(G) + TDI\left(\overline{G}\right) \leq 2n$.
(ii) $9 \leq TDI(G).TDI\left(\overline{G}\right) \leq n^2$.
In both cases, for upper bound, equality holds if and only if G or $\overline{G}$ consists of disjoint copies of K_2. For lower bound, equality holds for $G = P_4$.

Proof. From *Observation 1* and *Observation 2*, by applying algebraic operations on inequalities we obtain $6 \leq TDI\,(G) + TDI\,(\overline{G}) \leq 2n$ and $9 \leq TDI\,(G)\,.TDI\,(\overline{G}) \leq n^2$. □

Theorem 9. *For $n \geq 2$, $TDI\,(K_{1,n}) = 3$.*

Proof. Let $S \subseteq V\,(K_{1,n})$ is a total dominating set of star graph $K_{1,n}$ and $n \geq 2$. Since $|S| \geq 2$ and $m(K_{1,n} - S) \geq 1$, then $TDI\,(K_{1,n}) \geq |S| + m\,(K_{1,n} - S) = 2 + 1 = 3$. If $S \subseteq V\,(K_{1,n})$ is constructed by adding central vertex to any other vertex of $K_{1,n}$, then $TDI\,(K_{1,n}) = 3$. □

Theorem 10. *For $n \geq 2$, $TDI\,(K_n) = n$.*

Proof. Let $S \subseteq V\,(K_n)$ is a total dominating set of complete graph K_n and $n \geq 2$. Since K_n is a complete graph, S can be selected from any vertex set of K_n at least two vertices and $m\,(K_n - S) = n - |S|$. Hence,

$$TDI\,(K_n) = \min_{S \subseteq V(K_n)} \{|S| + m\,(K_n - S)\}$$

$$= |S| + n - |S| = n$$

Hence the result is obtained. □

Theorem 11. *For $3 \leq n \leq 7$,*

$$TDI\,(P_n) = \begin{cases} \frac{n}{2} + 1, & n \equiv 0 \ (mod\ 4) \\ \frac{n+2}{2} + 1, & n \equiv 2 \ (mod\ 4) \\ \frac{n+1}{2} + 1, & otherwise \end{cases}$$

Proof. Let P_n be a path graph and $3 \leq n \leq 7$. It has been proved that $\gamma_t\,(P_n) = \frac{n}{2}$ when $n \equiv 0\ (mod\ 4)$, $\gamma_t\,(P_n) = \frac{n+2}{2}$ when $n \equiv 2\ (mod\ 4)$, $\gamma_t\,(P_n) = \frac{n+1}{2}$ when $n \equiv 1\ (mod\ 4)$ or $n \equiv 3\ (mod\ 4)$ in *Proposition 1.* For any minimum total dominating set S of P_n, $m\,(P_{\mathrm{n}} - S) = 1$. Therefore, $TDI\,(P_n) \leq \gamma_t\,(P_n) + 1$. If X is any total dominating set of P_{n}, then $|X| + m\,(P_n - X) \geq \gamma_t\,(P_n) + 1$. Hence the result is obtained. □

Theorem 12. *For $n \geq 8$,*

$$TDI\,(P_n) = \begin{cases} \frac{n}{2} + 2, & n \equiv 0 \ (mod\ 4) \\ \frac{n+2}{2} + 2, & n \equiv 2 \ (mod\ 4) \\ \frac{n+1}{2} + 2, & otherwise \end{cases}.$$

Proof. Let P_n be a path graph and $n \geq 8$. It has been proved that $\gamma_t\,(P_n) = \frac{n}{2}$ when $n \equiv 0\ (mod\ 4)$, $\gamma_t\,(P_n) = \frac{n+2}{2}$ when $n \equiv 2\ (mod\ 4)$, $\gamma_t\,(P_n) = \frac{n+1}{2}$ when $n \equiv 1\ (mod\ 4)$ or $n \equiv 3\ (mod\ 4)$ in *Proposition 1.* For any minimum total dominating set S of P_n, $m\,(P_{\mathrm{n}} - S) = 2$. Therefore, $TDI\,(P_n) \leq \gamma_t\,(P_n) + 2$. If X is any total dominating set of P_{n}, then $|X| + m\,(P_n - X) \geq \gamma_t\,(P_n) + 2$. Hence the result is obtained. □

Theorem 13. *For $3 \leq n \leq 6$,* $TDI\,(C_n) = \begin{cases} 3, & n = 3 \\ 4, & n = 4 \\ 5, & n = 5, 6 \end{cases}$

Proof. Let $V\,(C_n) = \{u_1, u_2, \ldots, u_n\}$ where $3 \leq n \leq 6$.

Case 1: For $n = 3$, consider $S = \{u_1, u_2\}$ as a total dominating set of C_3 then $m\,(C_3 - S) = 1$. There does not exist a total dominating set of C_3 such that $|X| + m\,(C_3 - X) < |S| + m(C_3 - S)$. So, $TDI(C_3) = 3$.

Case 2: For $n = 4$, consider $S = \{u_1, u_2\}$ as a total dominating set of C_4 then $m\,(C_4 - S) = 2$. There does not exist a total dominating set of C_4 such that $|X| + m\,(C_4 - X) < |S| + m(C_4 - S)$. So, $TDI(C_4) = 4$.

Case 3: For $n = 5$, consider $S = \{u_1, u_2, u_3\}$ as a total dominating set of C_5 then $m(C_5 - S) = 2$. There does not exist a total dominating set of C_5 such that $|X| + m(C_5 - X) < |S| + m(C_5 - S)$. So, $TDI(C_5) = 5$.

For $n = 6$, consider $S = \{u_1, u_2, u_4, u_5\}$ as a total dominating set of C_6 then $m(C_6 - S) = 1$. There does not exist a total dominating set of C_6 such that $|X| + m(C_6 - X) < |S| + m(C_6 - S)$. So, $TDI(C_6) = 5$. □

Theorem 14. *For $n \geq 7$,*

$$TDI(C_n) = \begin{cases} \frac{n}{2} + 2, & n \equiv 0 \ (mod\ 4) \\ \frac{n+2}{2} + 2, & n \equiv 2 \ (mod\ 4) \\ \frac{n+1}{2} + 2, & otherwise \end{cases}.$$

Proof. Let C_n be a cycle graph and $n \geq 7$. It has been proved that $\gamma_t(C_n) = \frac{n}{2}$ when $n \equiv 0 \ (mod\ 4)$ and $\gamma_t(C_n) = \frac{n+2}{2}$ when $n \equiv 2 \ (mod\ 4)$and $\gamma_t(C_n) = \frac{n+1}{2}$ when $n \equiv 1 \ (mod\ 4)$ or $n \equiv 3 \ (mod\ 4)$ in *Proposition 1*. For any minimum total dominating set S of C_n, $m(C_n - S) = 2$. Therefore, $TDI(C_n) \leq \gamma_t(C_n) + 2$. If X is any total dominating set of C_n, then $|X| + m(C_n - X) \geq \gamma_t(C_n) + 2$. Hence the result is obtained. □

Theorem 15. *For $m, n \geq 2$,*

$$TDI(K_{m,n}) = \begin{cases} 4, & m = n = 2 \\ min\{m, n\} + 2, & otherwise \end{cases}.$$

Proof. Let $V(K_{m,n}) = V_1(K_{m,n}) \bigcup V_2(K_{m,n})$ and $V_1(K_{m,n}) = \{u_1, \ldots, u_m\}$, $V_2(K_{m,n}) = \{v_1, \ldots, v_n\}$ where $m, n \geq 2$. It is easy to verify that $TDI(K_{2,2}) = 4$. Let consider other situations except $m = n = 2$. Let $m \geq n$. $S = \{v_1, \ldots, v_n, u_1\} \subseteq V(K_{m,n})$ is a total dominating set of $K_{m,n}$. So, $|S| = n + 1 = min\{m, n\} + 1$ and $m(K_{m,n} - S) = 1$. If X is any total dominating set of $K_\mathrm{m,n}$, then $|X| + m(K_{m,n} - X) \geq |S| + m(K_{m,n} - S) = \min\{m, n\} + 1 + 1 = \min\{m, n\} + 2$. Hence the result is obtained. □

Proposition 7. $I(K_{a_1, a_2, \ldots, a_r}) = \sum_{i=1}^{r} a_i + 1 - \max_i a_i$ *(Goddard & Swart, 1990).*

Theorem 16. *For $r \geq 3$, $TDI(K_{a_1, a_2, \ldots, a_r}) = \sum_{i=1}^{r} a_i - \max_i a_i + 1$.*

Proof. Let $K_{a_1, a_2, \ldots, a_r}$ be a complete multipartite graph and $r \geq 3$. In *Observation 3*, $TDI(G) \geq I(G)$ and in *Proposition 7*, $I(K_{a_1, a_2, \ldots, a_r}) = \sum_{i=1}^{r} a_i + 1 - \max_i a_i$. Therefore, $TDI(K_{a_1, a_2, \ldots, a_r}) \geq \sum_{i=1}^{r} a_i - \max_i a_i + 1$. Let $S = V(K_{a_1, a_2, \ldots, a_r}) - X$ where X is the largest partite set of $K_{a_1, a_2, \ldots, a_r}$. Since $N(S) = V(K_{a_1, a_2, \ldots, a_r})$, then S is a total dominating set of $K_{a_1, a_2, \ldots, a_r}$ and $m(K_{a_1, a_2, \ldots, a_r} - S) = 1$. Hence,

$$TDI(K_{a_1, a_2, \ldots, a_r}) \geq |S| + m(K_{a_1, a_2, \ldots, a_r} - S) = \sum_{i=1}^{r} a_i - \max_i a_i + 1.$$

Then, the result is obtained. □

Theorem 17. *Let T be a tree. Then $TDI(T) = n$ if and only if either $T \cong P_2$ or $T \cong P_3$.*

Proof. Let T be a tree. If $TDI(T) = n$, from *Theorem 7*, then $diam(G) \leq 2$. If $diam(G) = 2$, then $T = K_{1,n}$. Since $TDI(K_{1,n-1}) = 3$, then $n - 1 = 2$. So, $T = K_{1,2} = P_3$. If $diam(G) = 1$, then $T = P_2$. Conversely, let T be P_2 or P_3. If $T = P_2$, $TDI(P_2) = 2$ and if $T = P_3$, $TDI(P_3) = 3$. □

Proposition 8. *For any graphs G and H,*
$I(G + H) = \min\{I(G) + |V(H)|, \ I(H) + |V(G)|\}$ *(Bagga et al., 1992).*

Proposition 9. *For any two graphs* G*and* H*,* $DI\left(G+H\right)=I\left(G+H\right)$ *(Sundareswaran, 2010).*

Theorem 18. *For any graphs* G*and* H*,* $TDI\left(G+H\right)=I\left(G+H\right)=DI(G+H)$*.*

Proof. Let G and H be any two graphs. Consider $S=V\left(H\right)\cup X$ where X is an I-set of G. Since $N\left(S\right)=V(G+H)$, then S is a total dominating set of $G+H$.

$$TDI\left(G+H\right)\leq|S|+m((G+H)-S)$$

$$=|V(H)|+|X|+m\left(G-X\right)=|V(H)|+I(G)$$

In similar way, $TDI\left(G+H\right)\leq|V(G)|+I(H)$ can be obtained.
As a result, $TDI\left(G+H\right)\leq\min\left\{|V\left(H\right)|+I\left(G\right),|V\left(G\right)|+I\left(H\right)\right\}$. So from *Proposition 8*, $TDI\left(G+H\right)\leq\min\left\{|V\left(H\right)|+I\left(G\right),|V\left(G\right)|+I\left(H\right)\right\}=I(G+H)$. Since $I(G+H)\leq TDI(G+H)$, we obtain $TDI\left(G+H\right)\leq I\left(G+H\right)\leq TDI(G+H)$. Hence, $TDI\left(G+H\right)=I\left(G+H\right)$.

From *Proposition 9*, we know that $DI\left(G+H\right)=I\left(G+H\right)$. Therefore, $TDI\left(G+H\right)=I\left(G+H\right)=DI(G+H)$. □

Theorem 19. *For* $m\geq n$*,* $TDI\left(\overline{K_{m,n}}\right)=m+2$*.*

Proof. Let $V\left(\overline{K_{m,n}}\right)=V_1(\overline{K_{m,n}})\cup V_2(\overline{K_{m,n}})$ and $V_1\left(\overline{K_{m,n}}\right)=\{u_1,\ldots,u_m\}$, $V_2\left(\overline{K_{m,n}}\right)=\{v_1,\ldots,v_n\}$. Let $m\geq n$. Since $\overline{K_{m,n}}\cong K_m\cup K_n$, then $S=\{v_1,v_n,u_1,u_m\}\subseteq V(\overline{K_{m,n}})$ is a total dominating set of $\overline{K_{m,n}}$ and $m\left(\overline{K_{m,n}}-S\right)=m-2$. Then, we have

$$TDI\left(\overline{K_{m,n}}\right)\leq|S|+m\left(\overline{K_{m,n}}-S\right)=4+m-2=m+2.$$

There does not exist a total dominating set of $\overline{K_{m,n}}$ such that $|X|+m\left(\overline{K_{m,n}}-X\right)<|S|+m(\overline{K_{m,n}}-S)$. So, $TDI\left(\overline{K_{m,n}}\right)=m+2$. □

The consequence of *Theorem 15* and *Theorem 19* is given as follows.

Corollary 1. $TDI\left(K_{m,n}\right)=TDI\left(\overline{K_{m,n}}\right)$ *if and only if* $m=n$*.*

4 Conclusion

In this paper, total domination integrity is introduced as a new vulnerability parameter and total domination integrity values of P_n, C_n, K_n, $K_{1,n}$, $K_{m,n}$, $\overline{K_{m,n}}$ and $K_{a_1,a_2,\ldots,a_r}$ are obtained. The bounds and some properties for total domination integrity value of a graph are determined and total domination integrity of the join of two graphs is found.

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