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ON THE ν-SEQUENCE AND THE DISTRIBUTION OF PRIME NUMBERS

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Abstract. In this study, a different approach is proposed to analyze the distribution of prime numbers in a certain interval: for this purpose, the distribution of composite numbers is first examined and the concept of ν-sequence is defined, the number of composite numbers in the given interval is expressed by in terms of ν-sequence. Then the number of prime numbers in that interval is determined in terms of the number of composite numbers. Experimental calculations have been made with the proposed method.

Keywords: Distribution of Prime Numbers, Distribution of Composite Numbers, ν-sequence, Prime Number Theorem.

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1 Introduction

The use of prime numbers in such a crucial areas as cryptology has further increased the importance of these numbers (Crandall & Pomerance, 2005). For this reason, many studies have been done for finding prime numbers and distribution of them (Ingham, 1990; Prachar, 1957). Nowadays, intensive studies on prime numbers continue to be conducted (Tenenbaum & Mendes, 2000; Narkiewicz, 2000).

In many Number Theory Problems, it is necessary to specify the number of prime numbers in certain intervals. In such cases, we can use the number of composite numbers in this interval to calculate the number of prime numbers in the desired interval (Granville & Rudnick, 2007).

In order to calculate the number of composite numbers in a certain interval, a calculation scheme based on the Dynamic Programming Technique has been proposed in our study Nuri et al. (2019). In our study Nuri et al. (2020), the number of composite numbers in various certain intervals was analyzed and the relationship between them was determined. The results were given by making calculations with the suggested methods.

This work is a continuation of our works given in articles Nuri et al. (2019) and Nuri et al. (2020) and therefore we make use of the notations given there.

The prime-counting function $\pi(x)$ is the function, counting the number of prime numbers less than or equal to some real number x.

According to this definition, $\pi(1) = 0,\ \pi(2) = 1,\ \pi(10) = 4,\ \pi(100) = 25,\ \pi(p_n) = n$ where, p_n is the n.th prime number.

The theorem that approximately gives how many prime are less than a given real number x is known as the Prime Number Theorem (Ingham, 1990).

Theorem 1. *(Prime Number Theorem) Let $\pi(x)$ be the number of primes up to x. Then,*

$$\pi(x) \sim x/\ln x.$$

That is,

$$\lim_{n\to\infty} \frac{\pi(x)}{x/\ln x} = 1. \tag{1}$$

The following theorem is obtained as a corollary of this theorem:

Theorem 2.

$$p_n \sim n \cdot \ln n \ \text{as}\ n \to \infty. \tag{2}$$

The Prime Number Theorem was postulated by Gauss in 1792 on numerical evidence. But it was in 1896 that Hadamard and Charles Jean de la Vallée Poussin independently proved the theorem (Ingham, 1990).

It shows that $x/\ln x$ is a good approximation to $\pi(x)$ for sufficiently large numbers x. An approach better than it, is the function;

$$li\,(x) = \int_2^x \frac{dt}{\ln t}$$

In this study, a new approach based on the distribution of composite numbers is proposed to analyze the distribution of prime numbers in a certain interval. For this purpose, the concept of ν-sequence is defined and the number of composite numbers in the given interval is expressed as the ν-sequence and the number of prime numbers in the given interval is determined.

The paper consists of the following parts:

Firstly, the history of the Distribution Theory of Prime Numbers is briefly mentioned, then the definition of the ν-sequence and some of its properties are given. In the next section, an algorithm is proposed to show the distribution of composite numbers in the given interval by using the ν-sequence. At the end, experimental calculations made with the proposed method.

2 Notations

$N = \{1,\ 2, \ldots n, \ldots\}$ - Sequence of Natural Numbers.
$\overline{N} = N \setminus \{1\} = \{2,\ 3,\ 4,\ \ldots,\ n, \ldots\}$.
$P = \{p_1,\ p_2,\ p_3, \ldots\} = \{2,\ 3,\ 5, \ldots\}$ - Sequence of Prime Numbers.
$\pi(x)$ - Prime Counting Function.
$M_k = \left\{m \in \overline{N} |\ m = k \cdot n,\ n \in N\right\}, k \in \overline{N}$ - The sequence produced by by k, i.e. the sequence of multiples of k.
$\overline{M}_k = M_k \setminus \{k\}, \, k \in \overline{N}$
$M = \cup_{p\in P} \overline{M}_p$.
$\overline{\pi}(x)$ - Composite Number Counting Function.
$M = \overline{N} \setminus P = CoP$ - Sequence of Composite Numbers.
$\overline{\pi}\,(n) = (n-1) - \pi(n)$.
$P = \overline{N} \setminus M = \overline{N} \setminus (\cup_{p\in P} \overline{M}_p)$ - Sieve of Eratosthenes.
$k!! = p_1 \cdot\ p_2 \cdot\ p_3 \cdot \ldots \cdot\ p_k$ - Prime Factorial, that is, the product of the first k number of prime numbers, here p_k is the k.th prime number, for example,
$8!! = p_1 \cdot\ p_2 \cdot \ldots \cdot\ p_8 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.
$k!!^{(-1)} = (p_1 - 1) \cdot\ (p_2 - 1) \cdot\ (p_3 - 1) \cdot \ldots \cdot (p_k - 1)$.
The first k number of prime numbers that subtracted by 1 and their product is found, where p_k is the k.th prime number, for example,
$5!!^{(-1)} = (p_1 - 1) \cdot\ (p_2 - 1) \cdot \ldots \cdot (p_5 - 1) = (2-1) \cdot (3-1) \cdot (5-1) \cdot (7-1) \cdot (11-1) = 1 \cdot 2 \cdot 4 \cdot 6 \cdot 10$.

3 On the Distribution of the Prime Numbers

The positive integers other than 1 can be divided into two classes, prime numbers (such as $2, 3, 5, 7$) which cannot be factorized, and composite numbers (such as $4, 6, 8, 9$) which can. The prime numbers derive their peculiar importance from the 'fundamental theorem of arithmetic' that a composite number can be expressed in one and only one way as a product of prime factors (Crandall & Pomerance, 2005).

Theorem 3. *(Fundamental Theorem of Arithmetic) For each natural number n there is a unique factorization*

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

where exponents a_i are positive integers and $p_1 < p_2 < \ldots < p_k$ are primes.

Although the series of prime numbers exhibits great irregularities of detail, the general distribution is found to possess certain features of regularity which can be formulated in precise terms and made the subject of mathematical investigation.

We shall denote by $\pi(x)$ the number of primes not exceeding x; our problem then resolves itself into a study of the function $\pi(x)$. If we examine a table of prime numbers, we observe at once that, however extensive the table may be, the primes show no signs of coming to an end altogether, though they do appear to become on the average more widely spaced in the higher parts of the table. These observations suggest two theorems which may be taken as the starting point of our subject. Stated in terms of $\pi(x)$, these are the theorems that $\pi(x)$ tends to infinity, and $\pi(x)/x$ to zero, as x tends to infinity (Crandall & Pomerance, 2005).

Euclid may have been the first to give a proof that there are infinitely many primes.

Theorem 4. *(Euclid Theorem). There exist infinitely many primes.*

In 1737, Euler proved that the following series is divergent and showed that prime numbers are infinite:

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

This proof is based on the following identity (Ingham, 1990):

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 + p^{-s} + p^{-2s} + \ldots\right) = \prod_{p} \left(1 - p^{-s}\right)^{-1} \tag{3}$$

where the products are over all primes p.

Euler's contribution to the subject is of fundamental importance; for this identity, which may be regarded as an analytical equivalent of the fundamental theorem of arithmetic, forms the basis of nearly all subsequent work.

The question of the diminishing frequency of primes was the subject of much speculation before any definite results emerged. The problem assumed a much more precise form with the publication by Legendry in 1808 (after a less definite statement in 1798) a remarkable empirical formula for the approximate representation of $\pi(x)$. Legendry asserted that, for large values of x, $\pi(x)$ is approximately equal to

$$\frac{x}{\ln x - B} \tag{4}$$

where $\ln x$ is the natural logarithm of x and B a certain numerical constant – a theorem described by Abel (in a letter written in 1823) as the 'most remarkable in the whole of mathematics' Ingham (1990).

A similar, though not identical, formula was proposed independently by Gauss. Gauss's method, which consisted in counting the primes in blocks of a thousand consecutive integers, suggested the function $1/\ln x$ as an approximation to the average density of distribution in the neighborhood of a large number x, and thus

$$\int_2^x \frac{du}{\ln u} \tag{5}$$

as an approximation to $\pi(x)$. Gauss's observations were communicated to Encke in 1849, and first published in 1863; but they appear to have commenced as early as 1791 when Gauss was fourteen years old. In the interval the relevance of the function (Prachar, 1957) was recognized by other authors. For convenience of notation it is usual to replace this function by the 'logarithmic integral'

$$li(x) = \lim_{\eta \longrightarrow +0} \left(\int_0^{1-\eta} + \int_{1+\eta}^{\infty} \right) \frac{du}{\ln u},$$

from which it differs only by the constant $li(2) = 1,04\ldots$.

The precise degree of approximation claimed by Gauss and Legendry for their empirical formulae outside the interval of tables used in their construction is not made very explicit by either author, but we may take it that they intended to imply at any rate the 'asymptotic equivalence' of $\pi(x)$ and the approximating function $f(x)$, that is to say that $\pi(x)/f(x)$ tends to the limit 1 as x tends to infinity (Ingham, 1990).

Some calculations with these functions are given in the Table 1.

Table 1: Comparison of $\pi(x)$ with the functions $li(x)$ and $x/(ln(x)-1)$

π	$\pi(x)$	$li\ x$	$\pi(x)/li\ x$	$x/(ln\ x-1)$	$\pi(x)/(x/(ln\ x-1))$
1000	168	178	0,94382	169,269029	0,992502887
10000	1229	1245	0,987149	1217,9763	1,009050832
50000	5133	5167	0,99342	5091,76466	1,008098439
100000	9592	9630	0,996054	9512,10016	1,008399811
500000	41538	41606	0,998366	41246,0825	1,00707746
1000000	78498	78628	0,998347	78030,4456	1,005991948
2000000	148933	149055	0,999182	148053,2	1,005942461
5000000	348513	348638	0,999641	346621,689	1,005456413
10000000	664579	664918	0,99949	661458,971	1,004716889
20000000	1270607	1270905	0,999766	1264922,7	1,004493791
90000000	5216954	5217810	0,999836	5197709,24	1,003702546
100000000	5761455	5762209	0,999869	5740303,81	1,003684682
1000000000	50847478	50849235	0,999965	50701542,4	1,002878326

At first, it is seen from the table that $\pi(x) < li(x)$ for all values of x. Until recently, this inequality was thought to be true for all values of x, but in 1914 Littlewood proved that $\pi(x) > li(x)$ for some values of x and such values of x are infinitely many (Prachar, 1957).

The first theoretical results connecting $\pi(x)$ with $x/\ln x$ are due to Chebyshev. If there exists $\lim(x/\ln x)$ than it is equal to 1.

Chebyshev proved in 1850 that there are such constants a and A that for sufficiently large values of x $\pi(x)/(x/\ln x)$ is placed between the constants a and A (Prachar, 1957).

Theorem 5. *There are positive numbers a, A such that for all $x \geq 3$,*

$$\frac{a\ x}{\ln x} < \pi(x) < \frac{A\ x}{\ln x}$$

Here $a = 0,92129\ldots$, $A = 1,0555\ldots$

The constants a and A were later narrowed down by some mathematicians (especially by Silvester) Tenenbaum & Mendes (2000).

The new ideas to supply the key to the solution were introduced by Riemann in 1859, in a memoir which has become famous, not only for its bearing on the theory of primes, but also for its influence on the development of the general theory of functions (Mazur & Stein, 2016).

Euler's identity had been used by Euler himself with a fixed value of $s(s = 1)$, and by Chebyshev with s as a real variable. Riemann introduced the idea of treating s as a complex variable and studying the series on the left of (3) by the methods of the theory of analytic functions (Mazur & Stein, 2016).

This series converges only in a restricted portion of the plane of the complex variable s, but defines by continuation a single - valued analytic function regular at all finite points except for a simple pole at $s = 1$. This function is called 'zeta function of Riemann' after the notation $\zeta(s)$ adopted by its author (Tenenbaum & Mendes, 2000).

It was the brilliant leap of Riemann in the mid-19th century to ponder an entity so artfully employed by Euler,

$$\zeta\left(s\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

but to ponder with powerful generality, namely, to allow s to attain complex values.

The discoveries of Hadamard prepared the way for rapid advances in the theory of the distribution of primes. The prime number theorem was proved in 1896 by Hadamard himself and by Charles Jean de la Vallée Poussin, independently and almost simultaneously (Koukoulopoulos, 2019).

Charles Jean de la Vallée Poussin in his study published in 1899 showed that $li(x)$ is a better approximation to $\pi(x)$ than the function (4), independent of the value of B, and that the best value of B is 1 (Prachar, 1957; Koukoulopoulos, 2019).

Finally, in the 1948-1949, the Distribution Theorem of Prime Numbers is proved in an elementary way, without using the Complex Number Theory, by the P. Erdos and A. Selberg independently (Tenenbaum & Mendes, 2000).

4 On the ν-sequence

Let $A = \{a_1,\ a_2,\ a_3,\ \dots\}$ be a sequence.
Define ν-sequence created over the sequence A recursively as follows:

$$\nu_A^1 = 1/a_1,$$

$$\nu_A^k = \upsilon_A^{k-1} + 1/a_k \cdot (1 - \nu_A^{k-1}),\ k \in \overline{N};$$

In our study Nuri et al. (2020), the ν-sequence was created for the sequence of prime numbers $P = \{p_1,\ p_2,\ p_3,\ \dots\}$, direct and recursive formulas were proposed in order to calculate the k.th element of this sequence:

$$\nu_P^1 = 1/p_1 = 1/2$$

$$\nu_P^2 = \nu_P^1 + \frac{1}{p_2}\left(1 - \nu_P^1\right) = \frac{1}{2} + \frac{1}{3}\left(1 - \frac{1}{2}\right) = \frac{4}{6}$$

$$\nu_P^3 = \nu_P^2 + \frac{1}{p_2}\left(1 - \nu_P^2\right) = \frac{4}{6} + \frac{1}{5}\left(1 - \frac{4}{6}\right) = \frac{22}{30}$$

$$\nu_P^4 = \nu_P^3 + \frac{1}{p_2}\left(1 - \nu_P^3\right) = \frac{22}{30} + \frac{1}{7}\left(1 - \frac{22}{30}\right) = \frac{162}{210}$$

A recursive formula to calculate the $(k+1)$.th element of the ν-sequence for the sequence of P is as follows:

$$\nu_P^1 = 1/p_1 = 1/2 = 1/p_1 = c_1/e_1$$

$$c_1 = 1,\ e_1 = 2$$

$$\nu_P^{k+1} = c_{k+1}/e_{k+1},\ \ k \in N$$

$$c_{k+1} = c_k \cdot (p_{k+1} - 1) + e_k$$

$$e_k = p_1 \cdot p_2 \cdot \ldots \cdot p_k = k!!$$

$$c_{k+1} = c_k \cdot (p_{k+1} - 1) + p_1 \cdot p_2 \cdot \ldots \cdot p_k = c_k \cdot (p_{k+1} - 1) + k!!$$

$$\nu_P^{k+1} = (c_k \cdot (p_{k+1} - 1) + p_1 \cdot p_2 \cdot \ldots \cdot p_k)/(p_1 \cdot p_2 \cdot \ldots \cdot p_k \cdot p_{k+1}),\ k \in N$$

$$\nu_P^{k+1} = (c_k \cdot (p_{k+1} - 1) + k!!)/((k+1)!!),\ k \in N$$

A direct formula to calculate the $(k+1)$.th element of the ν-sequence, for the sequence of P is as follows:

$$\nu_P^1 = 1/p_1 = 1/2$$

Let $r_1 = (p_1 - 1)/p_1 = (2-1)/2 = 1/2$, then $\nu_P^1 = 1 - r_1 = 1 - 1/2 = 1/2$.

$$\nu_A^k = 1 - r_k, k = 1,\ 2,\ 3,\ \ldots$$

$$r_k = r_{k-1} \cdot ((p_k - 1)/p_k), k = 1,\ 2,\ 3, \ldots$$

$$r_k = ((p_1 - 1)(p_2 - 1) \ldots (p_k - 1)) / (p_1 \cdot p_2 \cdot \ldots \cdot p_k)$$

$$r_k = \left(\prod_{i=1}^{k} (p_i - 1) \right) \Bigg/ \left(\prod_{i=1}^{k} p_i \right) = (k!!^{(-1)})/(k!!)$$

If sequence A is ascending $(a_{k+1} > a_k,\ k \in N)$ and if $a_1 > 1$ then followings are true for ν_A^k:

1. $\nu_A^k < 1,\ k \in N$
2. $\nu_A^k > 1/a_1,\ k \in \overline{N}$
3. $\lim_{k \to \infty} \nu_A^k \leq 1$
4. Sup $\{\nu_A^k\} \leq 1$

5 On the Distribution of the Composite Numbers

In our study Nuri et al. (2020), the $\{\nu_A^k\}$ sequence created on the sequence P, was used to calculate the medium density of the composite numbers in the given interval:
Let define an sequence A_k as follows:

$$A_k = \left\{ a_i^k \in N |\ a_i^k = k \times i,\ i \in N \right\},\ k \in \overline{N}$$

Let $A_k(n)$ be the sequence of elements not greater than n of sequence $A_k(n)$.

$$1 \leq a_i^k \leq n,\ i \in N,\ k \in \overline{N}.$$

$d_k(n) = s\,(A_k(n))\,/n$, $k = 1,\ 2,\ 3, \ldots$.
Let $d_k(n) = s\,(A_k(n))\,/n$, $k = 1,\ 2,\ 3, \ldots$
Let $d_k(n)$ be density of a sequence A_k in interval $[1, n]$.

$$d_k = d(A_{\mathrm{k}}) \ = \lim_{n\to\infty} \{d_k(n)\} = \lim_{n\to\infty} \{s\left(A_k(n)\right)/n\}\,,\ k = 1,\, 2,\, 3, \ldots$$

Let d_k be a medium density of A_k in a sequence of natural numbers N.

$$d\left(A_k\right) = d_k = \frac{1}{k},\ k \in \overline{N}$$

Consider the concepts of densities combining several sequences:

Let $A^{(k)} = \bigcup_{i=1}^{k} A_{p_i}$.

For example,

for $k = 1$, $A^{(1)} = A_{p_i} = A_2$, for $k = 2$, $A^2 = A_{p_1} \cup A_{p_2} = A_2 \cup A_3$,

for $k = 3$, $A^3 = A_{p_1} \cup A_{p_2} \cup A_{p_3} = A_2 \cup A_3 \cup A_5$,

for $k = 4$, $A^4 = A_{p_1} \cup A_{p_2} \cup A_{p_3} \cup A_{p_4} = A_2 \cup A_3 \cup A_5 \cup A_7$,

for $k = 5$, $A^5 = A_{p_1} \cup A_{p_2} \cup A_{p_3} \cup A_{p_4} \cup A_{p_5} = A_2 \cup A_3 \cup A_5 \cup A_7 \cup A_{11}$.

Denote the sequence of elements that are not greater than n of sequence $A^{(k)}$ with $A^{(k)}(n)$.

Density in interval $[1, n]$ for $A^{(k)}$,

$$d^{(k)}(n) = d\left(A^{(k)}(n)\right) = s\left(A^{(k)}(n)\right)\Big/ n,$$

medium density can be defined as

$$d^{(k)} = d\left(A^{(k)}\right) = \lim_{n\to\infty} \left\{d^k(n)\right\} = \lim_{n\to\infty} \left\{s\left(A^{(k)}(n)\Big/ n\right)\right\}$$

The formulas given below is used to calculate the $d^{(k)}$:

$$s\left(A \cup B\right) = s\left(A\right) + s\left(B\right) - s\left(A \cap B\right)$$

$$\begin{aligned} s\left(\cup_{i=1}^{n} A_i\right) = & \sum_{i=1}^{n} s\left(A_i\right) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s\left(A_i \cap A_j\right) + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} s\left(A_i \cap A_j \cap A_k\right) - \\ & - \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=j+1}^{n} s\left(A_i \cap A_j \cap A_k \cap A_l\right) + \\ & + \sum_{i=1}^{n-4}\sum_{j=i+1}^{n-3}\sum_{k=j+1}^{n-2}\sum_{l=k+1}^{n-1}\sum_{t=l+1}^{n} s\left(A_i \cap A_j \cap A_k \cap A_l \cap A_t\right) + \\ & + \ldots + (-1)^{n+1} \cdot s\left(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{n-1} \cap A_n\right) \end{aligned}$$

To express $d^{(k)}$ in terms of the ν-sequence, the largest prime p_q less than the $\sqrt{n}$ is found. Then $\nu_P^q = d^{(q)} = d\left(A^{(q)}\right)$, will be the middle density of composite numbers in the set of natural numbers in the interval $[1, n]$. We can evaluate the number of composite numbers approximately as $n \cdot \nu_P^q$.

Let denote the number of composite numbers not exceeding x with $\overline{\pi}$. Then,

$$x \to \infty,\ \overline{\pi} \sim x \cdot v_P^q$$

Here, q is the index of the largest term p_q less than $\sqrt{n}$ of the sequence P.

It is obvius that, $\overline{\pi}\left(x\right) = (x - 1) - \pi(x)$.

The calculation results for the first 25 prime numbers are given in the Table 2.

As can be seen from Table 2, the deviation of $\widetilde{\overline{\pi}\left(x\right)}$ from $\overline{\pi}\left(x\right)$ is very small and becomes smaller as the value of n increases.

Table 2: Comparison of the results calculated with the ν-sequence with $\pi(n)$

n	n^2	$(n+1)^2$	p	$\min\{\nu \times n - \pi(n)\}$	$\min\{\nu \times n - \pi(n)\}/n^2$	$\max\{\nu \times n - \pi(n)\}$	$\max\{\nu \times n - \pi(n)\}/(n+1)^2$
2	4	8	2	0	0	0,5	0,0625
3	9	15	3	-0,3333333	-0,037037037	0,666666667	0,044444444
4	16	24		-0,3333333	-0,020833333	0,666666667	0,027777778
5	25	35	5	-0,4666667	-0,018666667	0,733333333	0,020952381
6	36	48		-0,6666667	-0,018518519	0,533333333	0,011111111
7	49	63	7	-0,2571429	-0,005247813	1,057142857	0,016780045
8	64	80		-0,0857143	-0,001339286	1,314285714	0,016428571
9	81	99		-0,9428571	-0,011640212	1,028571429	0,01038961
10	100	120		-0,8571429	-0,008571429	1,171428571	0,009761905
11	121	143	11	-0,2597403	-0,002146614	1,116883117	0,007810371
12	144	168		-0,7532468	-0,00523088	0,623376623	0,003710575
13	169	195	13	0,55644356	0,003292565	2,282717283	0,011706242
14	196	224		-0,5814186	-0,002966421	2,83016983	0,012634687
15	225	255		-0,3486513	-0,001549562	0,815184815	0,003196803
16	256	288		-0,4045954	-0,001580451	1,718281718	0,005966256
17	289	323	17	0,75924076	0,002627131	3,1060704	0,009616317
18	324	360		-0,4617735	-0,001425227	1,509784333	0,004193845
19	361	399	19	2,27448712	0,006300518	3,497799414	0,008766415
20	400	440		1,45967036	0,003649176	3,419367011	0,007771289
21	441	483		2,01304578	0,00456473	4,131781531	0,008554413
22	484	528		1,06750834	0,002205596	3,027204993	0,005733343
23	529	575	23	2,0568	0,003888091	4,4556	0,00774887
24	576	624		2,1304	0,003698611	4,7316	0,007582692
25	625	675		2,296	0,0036736	4,8604	0,007200593
26	676	728		1,2264	0,001814201	4,2612	0,005853297
27	729	783		0,9012	0,001236214	2,5372	0,003240358
28	784	840		-1,1888	-0,001516327	1,3756	0,001637619
29	841	899	29	2,7	0,003210464	4,854	0,005399333
30	900	960		1,112	0,001235556	3,062	0,003189583
31	961	1023	31	3,776	0,00392924	5,634	0,005507331
32	1024	1088		3,842	0,003751953	6,443	0,005921875
33	1089	1155		3,05	0,002800735	6,323	0,005474459
34	1156	1224		1,564	0,001352941	5,432	0,004437908
35	1225	1295		-0,228	-0,000186122	3,739	0,002887259
36	1296	1368		-0,927	-0,000715278	4,029	0,002945175
37	1369	1443	37	-1,3	-0,000949598	2,65	0,001836452
38	1444	1520		0	0	3,15	0,002072368
39	1521	1599		-0,3	-0,000197239	1,55	0,000969356
40	1600	1680		-0,4	-0,00025	2,95	0,001755952
41	1681	1763	41	5,4908	0,003266389	7,4751	0,004239989
42	1764	1848		2,1454	0,001216213	6,4161	0,003471916
43	1849	1935	43	5,94	0,003212547	9,182	0,00474522
44	1936	2024		5,58	0,002882231	7,088	0,003501976
45	2025	2115		3,64	0,001797531	6,882	0,003253901
46	2116	2208		1,316	0,000621928	6,694	0,003031703
47	2209	2303	47	3,76	0,001702128	6,42	0,002787668
48	2304	2400		3,68	0,001597222	7,14	0,002975
49	2401	2499		3,14	0,001307788	6,86	0,002745098

50	2500	2600		-0,46	-0,000184	3,16	0,001215385
51	2601	2703		-2,84	-0,001091888	1,14	0,000421754
52	2704	2808		-0,32	-0,000118343	2,66	0,000947293
53	2809	2915	53	9,0378	0,003217444	11,6951	0,004012041
54	2916	3024		6,1083	0,002094753	9,9963	0,003305655
55	3025	3135		3,3265	0,001099669	7,2975	0,002327751
56	3136	3248		-0,0528	-1,68367E-05	4,0543	0,001248245
57	3249	3363		-1,8578	-0,000571807	2,4501	0,000728546
58	3364	3480		-3,3616	-0,000999287	1,9347	0,000555948
59	3481	3599	59	3,2212	0,000925366	6,8058	0,001891025
60	3600	3720		3,954	0,001098333	6,5666	0,001765215
61	3721	3843	61	1,5752	0,000423327	5,4696	0,001423263
62	3844	3968		0,292	7,59625E-05	-0,1976	-4,97984E-05
63	3969	4095		-1,424	-0,000358781	1,9928	0,000486642
64	4096	4224		-4,456	-0,001087891	0,3736	8,8447E-05
65	4225	4355		-5,2896	-0,001251976	-0,8728	-0,000200413
66	4356	4488		-8,528	-0,001957759	-3,8968	-0,000868271
67	4489	4623	67	4,4	0,000980174	9,01	0,001948951
68	4624	4760		5,32	0,001150519	6,81	0,001430672
69	4761	4899		-0,87	-0,000182735	4,87	0,00099408
70	4900	5040		-1,64	-0,000334694	3,01	0,000597222
71	5041	5183	71	7,7852	0,001544376	11,7918	0,002275092
72	5184	5328		5,8484	0,001128164	8,8234	0,001656044
73	5329	5475	73	10,5737	0,001984181	14,15355	0,002585123
74	5476	5624		9,3469	0,001706885	14,81745	0,002634682
75	5625	5775		8,3301	0,001480907	12,33855	0,002136545
76	5776	5928		8,6831	0,001503307	13,69995	0,002311058
77	5929	6083		5,1622	0,00087067	11,64955	0,001915099
78	6084	6240		2,9942	0,000492143	7,22945	0,001158566
79	6241	6399	79	12,431416	0,001991895	14,5690315	0,002276767
80	6400	6560		8,340481	0,0013032	16,3859485	0,002497858
81	6561	6723		7,397525	0,0011275	11,9846785	0,001782638
82	6724	6888		5,991867	0,000891116	9,5702445	0,001389408
83	6889	7055	83	14,011	0,002033822	17,877	0,002533948
84	7056	7224		10,662	0,001511054	16,989	0,002351744
85	7225	7395		7,03	0,00097301	13,881	0,001877079
86	7396	7568		3,452	0,000466739	7,997	0,001056686
87	7569	7743		5,836	0,00077104	9,339	0,001206122
88	7744	7920		2,482	0,000320506	8,643	0,001091288
89	7921	8099	89	8,472	0,001069562	15,1584	0,001871638
90	8100	8280		6,6912	0,000826074	11,2432	0,001357874
91	8281	8463		4,3712	0,000527859	10,0848	0,001191634
92	8464	8648		0,5648	6,67297E-05	6,4128	0,000741536
93	8649	8835		0,7008	8,10267E-05	4,6624	0,000527719
94	8836	9024		-1,1568	-0,000130919	4,7728	0,000528901
95	9025	9215		-2,7216	-0,000301562	1,6416	0,000178144
96	9216	9408		-4,5808	-0,000497049	-0,7056	-7,5E-05
97	9409	9603	97	4,928	0,000523754	10,48672	0,001092025
98	9604	9800		3,73024	0,000388405	7,19808	0,000734498
99	9801	9999		1,92032	0,000195931	6,76512	0,00067658
100	10000	10200		-1,4192	-0,00014192	2,5968	0,000254588

Table 3: Comparison of $\widetilde{\pi(n)}$ with $\pi(n)$

n	$\pi(n)$	$\sqrt{n}$	$\widetilde{p}_k$	$\nu^k_{\widetilde{P}}$	$\overline{\pi}(n)_1$	$\widetilde{p}_{k+1}$	$\nu^{k+1}_{\widetilde{P}}$	$\overline{\pi}(n)_2$	$\widetilde{\overline{\pi}(n)}$	$\widetilde{\pi(n)}$	$\widetilde{\pi(n)}/\pi(n)$
1000	168	31,62278	29,81888	0,8382405	838,240504	33,3443416	0,8426187	842,6186772	840,4295906	159,5704094	0,949823865
10000	1229	100	97,65158	0,8771025	8771,02538	102,035921	0,878257	8782,570072	8776,797724	1223,202276	0,995282568
50000	5133	223,6068	220,4033	0,8959406	44797,0301	225,419695	0,8963921	44819,60714	44808,31862	5191,681385	1,011432181
100000	9592	316,2278	313,2035	0,9026284	90262,8366	318,500817	0,9029291	90292,90707	90277,87185	9722,128155	1,013566321
500000	41538	707,1068	703,7274	0,9154938	457746,919	709,686782	0,9156119	457805,9604	457776,4398	42223,56025	1,016504412
1000000	78498	1000	812,0278	0,9174591	917459,126	818,106103	0,9201368	920136,8458	918797,986	81202,01395	1,034446915
2000000	148933	1414,214	1161,06	0,9220268	1844053,59	1167,43773	0,9175593	1835118,548	1839586,067	160413,933	1,077087905
5000000	348513	2236,068	1859,406	0,9273849	4636924,49	1866,1817	0,9274237	4637118,337	4637021,411	362978,5886	1,041506597
10000000	664579	3162,278	2649,852	0,9309905	9309904,56	2656,93081	0,9310164	9310163,602	9310034,08	689965,9197	1,038200003
20000000	1270607	4472,136	3786,42	0,9343038	18686075,7	3793,80632	0,9343211	18686421,37	18686248,54	1313751,46	1,033955786
90000000	5216954	9486,833	8160,999	0,940517	84646533,6	8169,0527	0,9405243	84647188,32	84646860,98	5353139,02	1,026104317
100000000	5761455	10000	9993,639	0,9419769	94197689,8	10001,8703	0,9419827	94198269,47	94197979,65	5802020,354	1,007040818
1000000000	50847478	31622,78	31618,38	0,9491436	949143637	31627,6279	0,9491452	949145244,2	949144440,4	50855559,6	1,000158938

6 Calculation of $\overline{\pi}(x)$

The most important problem in calculating $\overline{\pi}(x)$ is finding the index of the largest prime number p_q smaller than $\sqrt{n}$. Finding these numbers becomes more difficult as x increases.

To overcome this difficulty, instead of the prime number p_k, we will use its approximate value, the number $\widetilde{p_k}$, which we will call the "shadow" of this prime number.

For large values of k, $p_k \sim k \cdot \ln x$ according to the Theorem 2. In order to get the proper values for small values of k we will determine $\widetilde{p}_k$ with the formula given below.

$$\widetilde{p}_k = k \cdot (1+\alpha) \cdot \ln(k \cdot (1+\beta))$$

Here, $\alpha = 1/k$ and $\beta = \gamma \cdot \alpha$.

The difference between p_k and $\widetilde{p}_k$ decreases as k increases.

To calculate the ν-sequence according to $\widetilde{p}_k$, we consider the elements of the ν-sequence over this set and calculate

$$\widetilde{P} = \{2, \widetilde{p}_2,\ \widetilde{p}_3,\ \dots\ \}$$

Here, $\widetilde{p}_1$ is taken as to 2.

The following algorithm is proposed to calculate an approximate value $\widetilde{\overline{\pi}\,(x)}$ of $\overline{\pi}(x)$:

Algorithm

Step 1. Calculate $\sqrt{x}$.

Step 2. Find the largest $\widetilde{p_k}$ less than $\sqrt{x}$.

Step 3. Calculate $\widetilde{\overline{\pi}\,(\widetilde{p}_q)} = x \cdot \left(\nu^q_{\widetilde{P}} + \nu^{q+1}_{\widetilde{P}}\right)/2$

In order to improve the results of the algorithm, in Step 3, v^q_P and v^{q+1}_P are calculated according to "$\widetilde{p}_q$" and "$\widetilde{p}_{q+1}$" and their average value is found.

7 Experimental Results

Calculation results of the algorithm are given in Table 3.

It can be seen from Table 3, that the ratio of $\widetilde{\pi(n)}$ to $\pi(n)$ is close to 1 and this ratio decreases as the value of n increases.

In Table 3, $\widetilde{\overline{\pi}(n)}$ is the average value of $\overline{\pi}(n)$ and $\widetilde{\pi(n)}$ is the approximate value of the $\pi(n)$.

$$\widetilde{\overline{\pi}(n)} = \left(\overline{\pi}(n)_1 + \overline{\pi}(n)_2\right)/2$$

$$\widetilde{\pi(n)} = (n-1) - \widetilde{\overline{\pi}(n)}$$

8 Conclusion

Table 3 shows the results of our first calculations. In future, making more comprehensive and more detailed calculations is planned. Results described in the table, we will try to improve by adjusting the parameters α and β.

References

Crandall, R., Pomerance C. (2005). *Prime Numbers. A Computational Perspective.* Springer.

Granville, A., Rudnick, Z. (Eds.), (2007). *Equidistribution in NumberTheory*, An Introduction (NatoScience Series. II:Mathematics, Physicsand Chemistry, vol. 237), Springer, Amsterdam.

Ingham, A.E. (1990). *The Distribution of Prime Numbers.* London: Cambridge University Press.

Koukoulopoulos, D. (2019). *The Distribution of Prime Numbers*. Graduate Studies in Mathematics, 203. American Mathematical Society, Providence, RI.

Mazur, B., Stein, W. (2016). *Prime Numbers and the Riemann Hypothesis*. Cambridge University Press.

Narkiewicz, W. (2000). *The Development of Prime Number Theory*. Springer monographs in Mathematics, Springer-Verlag, Berlin.

Nuri, E., Nuriyeva, F. & Nuriyev, U. (2019). On a calculation of a number of composite numbers in a given interval, *Journal of Modern Technology and Engineering*, *4*(3), 170-177.

Nuri, E., Nuriyeva, F. & Nuriyev, U. (2020). Analysis of the number of composite numbers in a given interval, *Journal of Modern Technology and Engineering*, *5*(1), 97-06.

Prachar, K., (1957). *Primzahlverteilung*. Springer.

Tenenbaum, G., Mendes France, M. (2000). The prime numbers and their distribution, *Student Mathematical Library*, *5*, American Mathematical Society.