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COMPUTATION OF THE REGULARIZED GREEN'S FUNCTION FOR VIBRATION TRANSPORT IN TWO-LAYERED RODS

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Abstract. The main topic of the article is the partial differential equation of the transverse oscillation in composite materials. An approach for deriving the nonstationary Green's functions of the considered differential equation is described. This approach is concerned with finding the regularized Green's function in the form of a finite Fourier series. Here, the eigenfunctions are a complete set of orthonormal eigenfunctions for the Sturm-Liouville operator, which also satisfies the boundary conditions. Computational experiments support the reliability of the claimed approach.

Keywords: Singular boundary value problem, vibration transport, two-step rod, eigenvalues, eigenfunctions, generalized solution.
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1 Introduction

The Green's functions are a powerful method for solving many problems of electromagnetic, acoustics, elasticity and other applied area. The derivation of the Green's function for linear anisotropic elastodynamic materials has been made by many authors (see, for example, Ba et al., 2020; Ghadi et al., 2009; Pan, 2019; Yakhno, 2018a, 2018b; Yakhno & Altunkaynak, 2018; Yakhno, 2020; Zhan at al., 2019, and references of these papers). The Green's functions have been applied to solve of the wave propagation problems in composite elastic materials. For instance, an exact solution of equations for two-clamped-free rods has been found in (Inceoglu & Gurgoze 2000). The Green's functions are constructed for a second-order linear partial differential equation with constant coefficients in (Faydaoglu & Guseinov, 2003, 2010, Yakhno, 2018a, 2018b; Yakhno & Altunkaynak, 2018; Yakhno, 2020).

The construction of the Green function for higher order equations can be found in (Polyanin, 2002) for the case when the coefficients of the equations are constants which means that materials are not composite. We note that the transverse vibration of the composite rod is modeled by partial differential equations of the fourth order with piecewise constant coefficients and the techniques of the derivation of the Green's function for these equations have not been developed so far.

The aim of our research is to derive the regularized Green's function of the time-dependent equation of the transverse oscillation of a composite material. We propose a new analytical approach for deriving the regularized Green's function. This approach is based on the generalization of the Fourier series expansion. First, the ordinary differential equation with boundary and matching conditions corresponding to the given partial differential equation for vibration transmission is obtained. It is found eigenvalues and eigenfunctions of this equation. Similar problems for ordinary differential equations have been studied in (Allahverdiev & Tuna, 2019a, 2019b; Kulaev, 2016; Iraniparast at al., 2017; Faydaoglu & Yakhno, 2016; Mukhtarov & Yucel, 2020; Mukhtarov et al. 2018, 2020; Faydaoglu, 2018, 2019; Vladimirov, 1971). These eigenfunctions used in the Fourier series expansion form the set of orthogonal basis functions. The Green's function is derived in the form of a formal Fourier series, including this set of eigenfunctions. An regularization of this occuring Green's function is in the form of a finite Fourier series.

2 The Green's Function for Vibration Transport in Two-Layered Rods

Let us consider piecewise constant functions which are defined by the following equalities

$$\eta(x) = \begin{cases} \eta_1, & 0 \le x < p \\ \eta_2, & p < x \le q \end{cases}, \alpha(x) = \begin{cases} \alpha_1, & 0 \le x < p \\ \alpha_2, & p < x \le q \end{cases},$$

$$\beta(x) = \begin{cases} \beta_1, & 0 \le x < p \\ \beta_2, & p < x \le q \end{cases}, \gamma(x) = \begin{cases} \gamma_1, & 0 \le x < p \\ \gamma_2, & p < x \le q \end{cases},$$

where $\eta_i, \alpha_i, \beta_i, \gamma_i$ are constants and $\eta_i > 0, \alpha_i > 0, \beta_i > 0, \gamma_i > 0$ for $i = 1, 2$. In physical terms η is the elastic density, $\alpha(x)$ is the elastic modulus, $\beta(x)$ is the transverse section area, $\gamma(x)$ is the moment of inertia of the transverse section. We define the Green's function as a generalized $\Omega(x,t;x_0)$ function that satisfies the following equations for the transverse vibration of the composite material:

$$\eta(x)\alpha(x)\frac{\partial^2\Omega}{\partial t^2} + \frac{\partial^2}{\partial x^2}(\beta(x)\gamma(x)\frac{\partial^2\Omega}{\partial x^2}) = \delta(x - x_0)\delta(t), \tag{1}$$

$$x \in (0,p) \cup (p,q), \quad t \in R,$$

and the following initial data, boundary and interface conditions

$$\Omega(x,t;x_0)|_{t<0} = 0, \tag{2}$$

$$\Omega(0,t;x_0) = \frac{\partial}{\partial x}\Omega(0,t;x_0) = 0, \ \ \Omega(q,t;x_0) = \frac{\partial}{\partial x}\Omega(q,t;x_0) = 0, \tag{3}$$

$$\begin{gathered} \Omega(p-0,t;x_0) = \Omega(p+0,t;x_0), \ \ \tfrac{\partial}{\partial x}\Omega(p-0,t;x_0) = \tfrac{\partial}{\partial x}\Omega(p+0,t;x_0), \\ \psi_1\tfrac{\partial^2}{\partial x^2}\Omega(p-0,t;x_0) = \psi_2\tfrac{\partial^2}{\partial x^2}\Omega(p+0,t;x_0), \\ \psi_1\tfrac{\partial^3}{\partial x^3}\Omega(p-0,t;x_0) = \psi_2\tfrac{\partial^3}{\partial x^3}\Omega(p+0,t;x_0), \end{gathered} \tag{4}$$

where x_0 is a fixed real parameter from $(0,p) \cup (p,q)$, $\psi_1 = \beta_1\gamma_1, \psi_2 = \beta_2\gamma_2, x \in (0,p) \cup (p,q)$ are constants and $\delta(x - x_0)\delta(t)$ is the Dirac delta functions with the support at $x = x_0$ and $t = 0$.

The following lemma holds in generalized functions theory (see, for example, Vladimirov, 1971).

Lemma 1. *Let $x_0 \in (0,p)\cup(p,q)$ be a parameter, $\Gamma(t)$ be the Heaviside function ($\Gamma(t)=1, t\geq 0$ and $\Gamma(t)=0, t<0$ and $\omega(x,t;x_0)$ be a generalized function satisfying*

$$\eta(x)\alpha(x)\frac{\partial^2\omega}{\partial t^2}+\frac{\partial^2}{\partial x^2}(\beta(x)\gamma(x)\frac{\partial^2\omega}{\partial x^2})=0, \ \ (x\in(0,p)\cup(p,q), \ \ t>0), \tag{5}$$

the boundary conditions at the end faces $x=0$ and $x=q$

$$\omega(0,t;x_0)=\frac{\partial}{\partial x}\omega(0,t;x_0)=0, \ \ \omega(q,t;x_0)=\frac{\partial}{\partial x}\omega(q,t;x_0)=0, \ t>0; \tag{6}$$

and the interface conditions at $x=p$ for $t>0$

$$\begin{gathered}\omega(p-0,t;x_0)=\omega(p+0,t;x_0), \ \ \tfrac{\partial}{\partial x}\omega(p-0,t;x_0)=\tfrac{\partial}{\partial x}\omega(p+0,t;x_0),\\ \psi_1\tfrac{\partial^2}{\partial x^2}\omega(p-0,t;x_0)=\psi_2\tfrac{\partial^2}{\partial x^2}\omega(p+0,t;x_0),\\ \psi_1\tfrac{\partial^3}{\partial x^3}\omega(p-0,t;x_0)=\psi_2\tfrac{\partial^3}{\partial x^3}\omega(p+0,t;x_0);\end{gathered} \tag{7}$$

and the initial conditions

$$\omega(x,0;x_0)=0, \ \ \frac{\partial\omega}{\partial t}(x,0;x_0)=\frac{1}{\eta(x_0)\alpha(x_0)}\delta(x-x_0). \tag{8}$$

Then $\Omega(x,t;x_0)=\Gamma(t)\omega(x,t;x_0)$ is the Green's function of the transverse vibration of composite material.

Proof. Let $\Omega(x,t;x_0)$ be the generalized function which is equal to $\omega(x,t;x_0)$ for $t>0$ and is equal to 0 for $t<0$, i.e. $\Omega(x,t;x_0)=\Gamma(t)\omega(x,t;x_0)$. Applying operator of differentiations $\frac{\partial}{\partial t}, \frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^2}$ to $\Omega(x,t;x_0)$ and using (8) we find

$$\begin{gathered}\frac{\partial}{\partial t}\Omega(x,t;x_0)=\delta(t)\omega(x,t;x_0)+\Gamma(t)\frac{\partial}{\partial t}\omega(x,t;x_0)=\\ =\delta(t)\omega(x,0;x_0)+\Gamma(t)\frac{\partial}{\partial t}\omega(x,t;x_0)=\Gamma(t)\frac{\partial}{\partial t}\omega(x,t;x_0),\end{gathered}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Omega(x,t;x_0)=\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\Omega(x,t;x_0))=\frac{\partial}{\partial t}(\Gamma(t)\frac{\partial}{\partial t}\omega(x,t;x_0))=\\ =\delta(t)\frac{\partial}{\partial t}\omega(x,t;x_0)+\Gamma(t)\frac{\partial^2}{\partial t^2}\omega(x,t;x_0)=\\ =\delta(t)\frac{\partial}{\partial t}\omega(x,t;x_0)|_{t=0}+\Gamma(t)\frac{\partial^2}{\partial t^2}\omega(x,t;x_0)=\\ =\frac{1}{\rho(x_0)\alpha(x_0)}\delta(x-x_0)\delta(t)+\Gamma(t)\frac{\partial^2}{\partial t^2}\omega(x,t;x_0)\end{aligned}$$

$$\frac{\partial^2}{\partial x^2}\Omega(x,t;x_0)=\Gamma(t)\frac{\partial^2}{\partial x^2}\omega(x,t;x_0).$$

Here we use equalities (see, for example, Vladimirov, 1971, p. 77-78)

$$\frac{\partial\omega}{\partial t}\delta(t)=\frac{\partial\omega}{\partial t}|_{t=0}\delta(t), \quad \eta(x)\alpha(x)\delta(x-x_0)=\eta(x_0)\alpha(x_0)\delta(x-x_0).$$

Moreover, we have

$$\frac{\partial^2}{\partial x^2}(\beta(x)\gamma(x)\frac{\partial^2}{\partial x^2}\Omega(x,t;x_0)) = \Gamma(t)\frac{\partial^2}{\partial x^2}(\beta(x)\gamma(x)\frac{\partial^2}{\partial x^2}\omega(x,t;x_0)).$$

It follows from above mentioned equalities that the function $\Omega(x,t;x_0) = \Gamma(t)\omega(x,t;x_0)$ satisfies (1) if $\omega(x,t;x_0)$ satisfies (5)-(8). The interface conditions (3), (4) follow from conditions (6), (7). □

Remark 1. *Equation (5) can be written in the form*

$$\mu(x)\frac{\partial^2\omega}{\partial t^2} + \frac{\partial^4\omega}{\partial x^4} = 0, \qquad x \in (0,p)\cup(p,q), \quad t > 0,$$

where $\mu(x)$ is define by

$$\mu(x) = \begin{cases} \mu_1, & 0 \le x < p \\ \mu_2,, & p < x \le q \end{cases}, \quad \mu_1 = \frac{\eta_1\alpha_1}{\beta_1\gamma_1}, \ \ \mu_2 = \frac{\eta_2\alpha_2}{\beta_2\gamma_2}.$$

3 Computing of the Regularized Green's Function

In our paper we replace the Dirac delta function $\delta(x-x_0)$ by a regularized (approximate) classical function $\delta_N(x,x_0)$ (see section 3.4) and describe a method of computation of an approximate Green's function which is an exact solution of (5)-(8), where instead of $\delta(x-x_0)$ we have $\delta_N(x,x_0)$.

3.1 Computing of Eigenvalues and Eigenfunctions of Singular Sturm-Liouville Problem

We consider the following ordinary differential equation of the fourth order

$$z^{(4)}(x) = \lambda\mu(x)z(x), \qquad x \in (0,p)\cup(p,q), \tag{9}$$

subject to the matching conditions

$$\begin{aligned} &z(p-0) = z(p+0), \ \ z'(p-0) = z'(p+0), \\ &\psi_1 z''(p-0) = \psi_2 z''(p+0), \ \ \psi_1 z'''(p-0) = \psi_2 z'''(p+0), \end{aligned} \tag{10}$$

and boundary conditions

$$z(0) = z(q) = 0, \ \ z'(0) = z'(q) = 0. \tag{11}$$

Here the function $\mu(x)$ is defined in Remark 1.

The number λ, for which there exists a non-zero function $z(x)$ satisfying (9)-(11), is called an eigenvalue of the boundary value problem (BVP) (9)-(11) and this non-zero function $z(x)$ is called an eigenfunction of BVP (9)-(11) corresponding to the eigenvalue λ. The main problem of this section is to find all eigenvalues and associated eigenfunctions of BVP (9)-(11).

We note that the solutions of (9) under the conditions (11) can be presented as:

$$z(x) = \begin{cases} c_1[\cosh(sx) - \cos(sx)] + c_2[\sinh(sx) - \sin(sx)], \quad x \in [0,p), \\ a_1[\cosh(s_1(q-x)) - \cos(s_1(q-x))] + a_2[\sinh(s_1(q-x)) \\ \qquad\qquad - \sin(s_1(q-x))], \quad x \in (p,q], \end{cases} \tag{12}$$

where c_1, c_2, a_1, a_2 are arbitrary constants; s and s_1 are parameters such that $s^4 = \lambda\mu_1$, $s_1^4 = s^4(\mu_2/\mu_1)$.

The eigenfunctions of the problem (9)-(11) in the form (12) are nonzero and satisfy the interface conditions (10). Using (10), (12) we obtain the homogeneous system of the following linear algebraic equations for finding c_1, c_2, a_1, a_2 such that $c_1^2 + c_2^2 + a_1^2 + a_2^2 > 0$:

$$c_1[\cosh(sp) - \cos(sp)] + c_2[\sinh(sp) - \sin(sp)] - a_1[\cosh(s_1(q-p)) - \cos(s_1(q-p))] - a_2[\sinh(s_1(q-p)) - \sin(s_1(q-p))] = 0, \tag{13}$$

$$c_1 s[\sinh(sp) + \sin(sp)] + c_2 s[\cosh(sp) - \cos(sp)] + a_1 s_1[\sinh(s_1(q-p)) + \sin(s_1(q-p))] + a_2 s_1[\cosh(s_1(q-p)) - \cos(s_1(q-p))] = 0, \tag{14}$$

$$c_1 s^2[\cosh(sp) + \cos(sp)] + c_2 s^2[\sinh(sp) + \sin(sp)] - a_1\psi s_1^2[\cosh(s_1(q-p)) + \cos(s_1(q-p))] - a_2\psi s_1^2[\sinh(s_1(q-p)) + \sin(s_1(q-p))] = 0, \tag{15}$$

$$c_1 s^3[\sinh(sp) - \sin(sp)] + c_2 s^3[\cosh(sp) + \cos(sp)] + a_1\psi s_1^3[\sinh(s_1(q-p)) - \sin(s_1(q-p))] + a_2\psi s_1^3[\cosh(s_1(q-p)) + \cos(s_1(q-p))] = 0. \tag{16}$$

Here ψ is a given constant defined by $\psi = \psi_2/\psi_1$, where ψ_1, ψ_2 are constants given after formula (4). We know that the homogeneous system (13)-(16) has non-zero solutions if the determinant of this system equal to zero. We denote this determinant as $\Delta(s)$. The roots of $\Delta(s) = 0$ can be computed numerically.

Example 1. *Computation of eigenvalues: Let us consider the composite rod with the following characteristics*

$$\alpha_1 = 0.1(m^2), \quad \alpha_2 = 0.1(m^2), \quad \beta_1 = 10^6(N/m^2),$$
$$\beta_2 = 0.9x10^6(N/m^2), \quad \eta_1 = 4x10^3(kg/m^3), \quad \eta_2 = 3x10^3(kg/m^3),$$
$$\gamma_1 = 10^{-2}(m^2), \quad \gamma_2 = 0.3x10^{-2}(m^2), \quad q = 10(m), \quad p = 4(m).$$

Applying Maple tools the roots we compute roots of $\Delta(s) = 0$ as follows:

$$s_1 = 0.4004802408,\ s_2 = 0.6740200856,\ s_3 = 0.9420222890,\ s_4 = 1.192368928,$$
$$s_5 = 1.476182727,\ s_6 = 1.743724014,\ s_7 = 1.995220148,\ s_8 = 2.279817702,$$
$$s_9 = 2.544695328,\ s_{10} = 2.797873532,\ s_{11} = 3.083294900,\ s_{12} = 3.345615782,$$
$$s_{13} = 3.600730249,\ s_{14} = 3.886648011\, s_{15} = 4.146486921,\ s_{16} = 4.403762698,$$
$$s_{17} = 4.689856810,\ s_{18} = 4.947340975,\ s_{19} = 5.206957379,\ s_{20} = 5.492904816,$$
$$s_{21} = 5.748209291,\ s_{22} = 6.010297860,\ s_{23} = 6.295778953,\ s_{24} = 6.549118274, ...$$

Using these computed values we determine eigenvalues of BVP (9)-(11) by the formula $\lambda_n = s_n^4/\mu_1$. Here $n = 1, 2, 3, ...$ and the constant μ_1 is introduced in Remark 1.

An eigenfunction associated to λ_n is found by taking $a_2 = 1$ and considering the system of linear algebraic equations (13), (14), (15) with unknown c_1, c_2, a_1 for any fixed natural number n. The determinant of this system is different from zero for $s = (\lambda_n\mu_1)^{\frac{1}{4}}$ and $s = s_1 \equiv (\lambda_n\mu_2)^{\frac{1}{4}}$ (we have checked it by Maple tools). Here constants μ_1, μ_2 are defined in Remark 1.

Using the Cramer's method we obtain that the system (13), (14), (15) has a unique solution c_1, c_2, a_1 for any fixed natural number n and for $s = (\lambda_n\mu_1)^{\frac{1}{4}}$ and $s = s_1$. Considering solution (12) for $s = (\lambda_n\mu_1)^{\frac{1}{4}}$ and $s_1 \equiv (\lambda_n\mu_2)^{\frac{1}{4}}$ we find eigenfunctions $y_n(x)$ of BVP (9)-(11) corresponding to λ_n for any natural number n.

3.2 Some Properties of Eigenfunctions and Eigenvalues of Singular Sturm-Liouville Problem

Proposition 1. *Eigenvalues of (9)-(11) are real and positive.*

Proof. The proof is given in the work Faydaoglu, 2019 (see the proof of the theorem 8, p.2519). □

Proposition 2. *Eigenfunctions z_n and z_m of (9)-(11) associated with the distinct λ_n and λ_m provides the following orthogonality relation:*

$$\int_0^q \mu(x)\, z_n(x) z_m(x) dx = 0, \;\; n \neq m, \;\; x_0 \in (0,p) \cup (p,q). \tag{17}$$

Proof. The proof is given in the work Faydaoglu, 2019 (see the proof of the theorem 9, p.2519). □

Proposition 3. *Let $f(x)$ be from the class of functions defined on $C^1[0,q] \cap C^4([0,p) \cup (p,q])$ such that*

$$f(0) = f(q) = 0,$$

$$\psi_1 f''(p-0) = \psi_2 f''(p+0), \;\; \psi_1 f'''(p-0) = \psi_2 f'''(p+0),$$

$$\int_0^q |f^{(4)}(x)| dx < \infty.$$

Then the Fourier series of $f(x)$ is uniformly absolutely-convergent in $[0,q]$ and

$$f(x) = \sum_{m=1}^{\infty} f_m X_m(x), \tag{18}$$

where

$$f_m = \int_0^q \mu(x)\, f(x) X_m(x) dx, \tag{19}$$

$$X_m(x) = \frac{z_m(x)}{\|z_m\|}, \qquad \|z_m\|^2 = \int_0^q \mu(x)\, z_m^2(x) dx, m = 1, 2, 3.... \tag{20}$$

Proof. The proof is described in the work Faydaoglu, 2018. □

3.3 Fourier Series of the Dirac Delta Function $\delta(x - x_0)$

The class of functions $f(x)$ from $C^1[0,q] \cap C^4([0,p) \cup (p,q])$ such that

$$f(0) = f(q) = 0,$$

$$\psi_1 f''(p-0) = \psi_2 f''(p+0), \;\; \psi_1 f'''(p-0) = \psi_2 f'''(p+0),$$

$$\int_0^q |f^{(4)}(x)| dx < \infty$$

is denoted as $f = F(p, \psi_1, \psi_2, q)$.

We define the inner product $< U(x), V(x) >_F$ of two arbitrary functions $U(x)$, $V(x)$ from F by

$$< U(x), V(x) >_F = \int_0^q \mu(x)\, U(x)\, V(x) dx.$$

For this inner product space F we consider the dual space F' which is the set of all linear functionals from F into R.

The Dirac delta function with the support at $x = x_0$, where x_0 is a fixed point from $(0, p) \cup (p, q)$, is defined by the formula

$$< \delta(x - x_0), \tau(x) >_F = \mu(x_0)\tau(x_0) \tag{21}$$

for any test function $\tau(x) \in F$ and $x \in (0, q)$. Moreover, the function $\tau(x_0)$ can be presented in the form of the uniformly convergent series (see Proposition 3)

$$\tau(x_0) = \sum_{m=1}^{\infty} \tau_m X_m(x_0), \tag{22}$$

where

$$\tau_m = \int_0^q \mu(x)\tau(x)X_m(x)dx, \quad m = 1, 2, 3, \ldots \tag{23}$$

Hence, using (21), (22), we have

$$< \delta(x - x_0), \tau(x) >_F = \mu(x_0) \sum_{m=1}^{\infty} \tau_m X_m(x_0). \tag{24}$$

Using formula (23) and uniformly convergence of the series in the right side of (22), we have

$$< \delta(x - x_0), \tau(x) >_F = \int_0^q \mu(x) \left[\mu(x_0) \sum_{m=1}^{\infty} X_m(x_0)X_m(x) \right] \tau(x)dx$$

$$=< \mu(x_0) \sum_{m=1}^{\infty} X_m(x)X_m(x_0), \tau(x) >_F . \tag{25}$$

Therefore, the series

$$\mu(x_0) \sum_{m=1}^{\infty} X_m(x_0)X_m(x) \tag{26}$$

is the formal Fourier series of the Dirac delta function.

3.4 A Regularization for the Dirac Delta Function

Let us consider the partial sums of the series (26)

$$\delta_N(x; x_0) = \mu(x_0) \sum_{m=1}^{N} X_m(x_0)X_m(x), \tag{27}$$

where N runs natural numbers.

Theorem 1. *Let N be a natural number; p, q be given real constants such that $0 < p < q$; $x_0 \in (0, p) \cup (p, q)$; $\delta(x - x_0)$, $\delta_N(x, x_0)$ be the Dirac delta function and the function defined by (21), (27). Then*

$$\lim_{N\to\infty} < \delta_N(x; x_0), \tau(x) >_F =< \delta(x - x_0), \tau(x) >_F,$$

for any $\tau(x) \in F$.

Proof. Using formula (27) and the uniformly convergence of $\sum_{m=1}^{\infty} \tau_m X_m(x)$ to $\tau(x)$ on $[0, q]$ we have

$$< \delta_N(x; x_0), \tau(x) >_F = < \mu(x_0) \sum_{m=1}^{\infty} X_m(x_0) X_m(x), \tau(x) >_F$$

$$= \mu(x_0) \sum_{m=1}^{N} \left[\int_0^q \mu(x) X_m(x) \tau(x) dx \right] X_m(x_0) = \mu(x_0) \sum_{m=1}^{N} \tau_m X_m(x_0). \tag{28}$$

Using formulas (24), (28) we find

$$< \delta(x - x_0) - \delta_N(x; x_0), \tau(x) >_F = \mu(x_0) \sum_{m=N}^{\infty} \tau_m X_m(x_0)$$

for any $\tau(x) \in F$. Using the convergence of $\sum_{m=1}^{\infty} \tau_m X_m(x_0)$ we find that $\sum_{m=N}^{\infty} \tau_m X_m(x_0)$ tends to 0 for any $\tau \in F$ and $x_0 \in (0, p) \cup (p, q)$. This means that

$$\lim_{N \to \infty} < \delta(x - x_0) - \delta_N(x; x_0), \tau(x) >_F = 0$$

or

$$\lim_{N \to \infty} < \delta_N(x; x_0), \tau(x) >_F = < \delta(x - x_0), \tau(x) >_F,$$

for any $\tau(x) \in F$ and $x_0 \in (0, p) \cup (p, q)$. □

Definition 1. *The function $\delta_N(x; x_0)$ with the parameter N, defined by (27), is called the regularization of the Dirac delta function $\delta(x - x_0)$ and N is called the parameter of the regularization.*

Example 2. *Computation of the regularization of the Dirac delta function: Let us consider the rod with characteristics described in Example 1. Using formula (27) for $x_0 = 2$, $N = 5$ and $N = 19$ and applying Maple tools we have computed the regularized Dirac delta function for two different regularized parameters. The results of this computation are presented in Figs. 1, 2.*

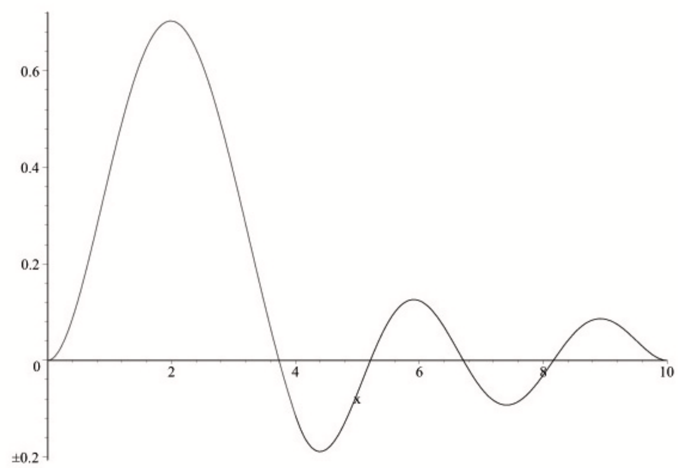

Figure 1: The graph of $\delta_N(x, x_0)$ for $N = 5$, $x_0 = 2$

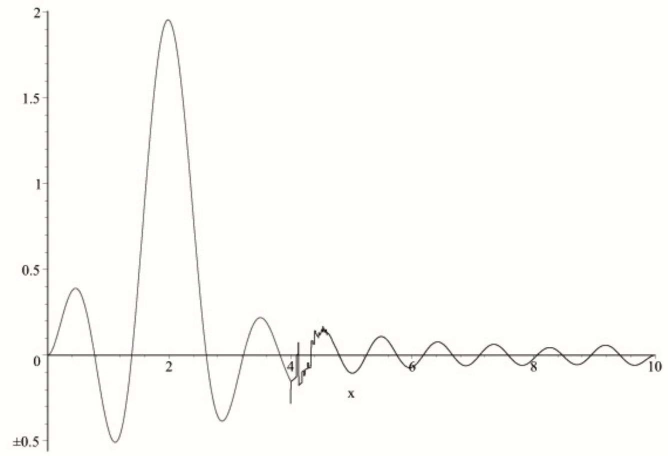

Figure 2: The graph of $\delta_N(x, x_0)$ for $N = 19$, $x_0 = 2$

3.5 An Explicit Formula for Regularized Solution for Singular Boundary Value Problem (5) - (8)

Let us consider the problem of finding a function satisfying (5) - (8), when the Dirac delta function $\delta(x - x_0)$ is replaced by its regularization $\delta_N(x, x_0)$. This problem is stated as follows. Let p, q be positive constants, $p < q$; x_0 be parameter from $x_0 \in (0, p) \cup (p, q)$; N be the natural number; $\delta_N(x; x_0)$ be function, defined by (27); $\eta_i, \alpha_i, \beta_i, \gamma_i$ be given real constants such that $\eta_i > 0, \alpha_i > 0, \beta_i > 0, \gamma_i > 0$, (i=1,2); $\psi_i = \beta_i \gamma_i$ (i=1,2); $\alpha(x) = \alpha_1$, $\eta(x) = \eta_1$ for $x \in (0, p)$ and $\alpha(x) = \alpha_2$, $\eta(x) = \eta_2$ for $x \in (p, q)$; $\mu(x)$ be the function defined in Remark 1. The problem is to find a function $\omega_N(x, t; x_0)$ satisfying the following differential equation

$$\mu(x)\frac{\partial^2 \omega_N}{\partial t^2} + \frac{\partial^4 \omega_N}{\partial x^4} = 0, \quad (x \in (0, p) \cup (p, q), \quad t > 0), \tag{29}$$

with boundary conditions at the end faces $x = 0$ and $x = q$:

$$\omega_N(0, t; x_0) = \frac{\partial}{\partial x}\omega_N(0, t; x_0) = 0, \ \ \omega_N(q, t; x_0) = \frac{\partial}{\partial x}\omega_N(q, t; x_0) = 0, \ t > 0; \tag{30}$$

and interface conditions at $x = p$ for $t > 0$:

$$\begin{gathered} \omega_N(p - 0, t; x_0) = \omega_N(p + 0, t; x_0), \ \ \tfrac{\partial}{\partial x}\omega_N(p - 0, t; x_0) = \tfrac{\partial}{\partial x}\omega_N(p + 0, t; x_0), \\ \psi_1 \tfrac{\partial^2}{\partial x^2}\omega_N(p - 0, t; x_0) = \psi_2 \tfrac{\partial^2}{\partial x^2}\omega_N(p + 0, t; x_0), \\ \psi_1 \tfrac{\partial^3}{\partial x^3}\omega_N(p - 0, t; x_0) = \psi_2 \tfrac{\partial^3}{\partial x^3}\omega_N(p + 0, t; x_0); \end{gathered} \tag{31}$$

and initial conditions:

$$\omega_N(x, 0; x_0) = 0, \ \ \frac{\partial \omega_N}{\partial t}(x, 0; x_0) = \frac{1}{\eta(x_0)\alpha(x_0)}\delta_N(x; x_0). \tag{32}$$

Theorem 2. *Let constants p, q, ψ_1, ψ_2; parameter x_0 and functions $\alpha(x)$, $\eta(x)$, $\mu(x)$, $\delta_N(x; x_0)$ satisfy above mentioned assumptions. Then a solution of (29) - (32) is defined by the following formula*

$$\omega_N(x, t; x_0) = \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)} \sum_{m=1}^{N} X_m(x_0) \frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} X_m(x), \tag{33}$$

where λ_m ($m = 1, 2, 3, ..., N$) are eigenvalues of boundary value problem (9)-(11), $X_m(x)$ is the eigenfunction of boundary value problem (9)-(11) corresponding to λ_m and defined by (20) for any natural number m.

Proof. A solution of initial boundary value problem (29) - (32) we search in the form

$$\omega_N(x,t;x_0) = \sum_{k=1}^{N} T_k(t;x_0)\, X_k(x), \quad x \in (0,p)\cup(p,q),\ t>0, \tag{34}$$

where x_0 is a parameter from the set $(0,p)\cup(p,q)$; $T_k(t;x_0)$, $k = 1,2,3,...,N$ are unknown functions; $X_k(x)$ is the eigenfunction of (9)-(11) associated to λ_k and defined by (20) for any natural number k. Substituting (34) into (29) and using the equality (property of eigenvalues and eigenfunctions of (9)-(11)) $X_k(x) = \lambda_k \mu(x) X_k(x)$ $(x \in (0,p)\cup(p,q))$, we find

$$\sum_{k=1}^{N} \mu(x)\,\left[T_k''(t;x_0) + \lambda_k\, T_k(t;x_0)\right] X_k(x) = 0, \quad x \in (0,p)\cup(p,q),\ t>0. \tag{35}$$

Multiplying the right and left sides of (35) by $X_m(x)$ and then integrating with respect to x from 0 to q, we find

$$\sum_{k=1}^{N} \left[T_k''(t;x_0) + \lambda_k\, T_k(t;x_0)\right] \int_0^q \mu(x)\, X_k(x)\, X_m(x)\, dx = 0, \quad x \in (0,p)\cup(p,q),\ t>0. \tag{36}$$

Applying the property of orthogonality of eigenfunctions of (9)-(11) (see Proposition 2), we find the following equality

$$T_k''(t;x_0) + \lambda_k\, T_k(t;x_0) = 0, \quad t>0, \tag{37}$$

for any fixed natural m from the set $\{1,2,3,...,N\}$.

Substituting (34) into (32) we find

$$\sum_{k=1}^{N} T_k(0;x_0)\, X_k(x) = 0, \qquad \sum_{k=1}^{N} \left[T_k'(0;x_0) - \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)}\, X_k(x_0)\right] X_k(x) = 0. \tag{38}$$

Multiplying the right and left sides of (38) by $X_m(x)$ and then integrating with respect to x from 0 to q, we find the following two equalities

$$\sum_{k=1}^{N} T_k(t;x_0) \int_0^q \mu(x)\, X_k(x)\, X_m(x)\, dx = 0, \tag{39}$$

$$\sum_{k=1}^{N} \left[T_k'(0;x_0) - \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)}\, X_k(x_0)\right] \int_0^q \mu(x)\, X_k(x)\, X_m(x)\, dx = 0. \tag{40}$$

Applying Proposition 2 to (39), (40) we have

$$T_m(0;x_0) = 0, \qquad T_m'(0;x_0) = 0 \tag{41}$$

for any fixed natural m from the set $\{1,2,3,...,N\}$. As a result we obtain that the function $\omega_N(x,t;x_0)$, defined by (34), is a solution of initial boundary value problem (29) - (32) if and only if each function $T_m'(t;x_0)$ is a solution of the initial value problem (37), (41) for any $m = 1,2,3,...,N$. A solution of initial value problem (37), (41) is defined by the following formula

$$T_m(t;x_0) = \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)}\, \frac{\sin\left(\sqrt{\lambda_m}\ t\right)}{\sqrt{\lambda_m}} \tag{42}$$

for any $m = 1,2,3,...,N$.

Substituting (42) into (34) we find that the function $\omega_N(x,t;x_0)$, defined by (33), is a solution of initial boundary value problem (29) - (32). □

Remark 2. *Let $\omega(x,t;x_0)$ be generalized function satisfying (5)-(8). We consider the formal Fourier series (26) instead of the Dirac delta function $\delta(x-x_0)$ in (8). Applying to (5)-(8) reasoning of the proof of Theorem 2 we find the presentation*

$$\omega(x,t;x_0) = \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)} \sum_{m=1}^{\infty} X_m(x_0) \frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} X_m(x). \tag{43}$$

Theorem 3. *Let $\omega(x,t;x_0)$ be generalized function satisfying (5)-(8), $\omega_N(x,t;x_0)$ be the function, defined by (33). Then*

$$\lim_{N\to\infty} < \omega(x,t;x_0) - \omega_N(x,t;x_0), \tau(x) >_F = 0 \tag{44}$$

for any fixed $x_0 \in (0,p)\cup(p,q)$, $t>0$ and any $\tau(x)\in F$.

Proof. Using formulas (33), (43) we have

$$< \omega(x,t;x_0) - \omega_N(x,t;x_0), \tau(x) >_F$$

$$= \sum_{m=1}^{\infty} \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)} X_m(x_0) \frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} \left[\int_0^q \mu(x)X_m(x)\tau(x)dx\right]$$

$$= \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)} \sum_{m=1}^{\infty} \frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} \tau_m X_m(x_0).$$

Applying the following inequality

$$\left|\frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}\, t}\right| \le 1 \quad (t>0)$$

we have

$$\left|\frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} \tau_m X_m(x_0)\right| \le t\,|\tau_m X_m(x_0)| \quad (t>0)$$

Using uniformly absolutely convergence of the series $\sum_{m=1}^{\infty} \tau_m X_m(x_0)$ (see Proposition 3) we find that

$$\lim_{N\to\infty} \frac{\mu(x_0)}{\eta(x_0)\alpha(x_0)} \sum_{m=N}^{\infty} \frac{\sin\left(\sqrt{\lambda_m}\, t\right)}{\sqrt{\lambda_m}} \tau_m X_m(x_0) = 0$$

for any fixed $x_0 \in (0,p)\cup(p,q)$, $t>0$ and any $\tau(x)\in F$. □

Definition 2. *The function $\omega_N(x,t;x_0)$ with the parameter N is called the regularization of the generalized function $\omega(x,t;x_0)$ if equality (44) is satisfied for for any fixed $x_0 \in (0,p)\cup(p,q)$, $t>0$ and any $\tau(x)\in F$ and N is called the parameter of the regularization.*

Remark 3. *Let $\omega(x,t;x_0)$ be generalized function satisfying (5)-(8) , $\omega_N(x,t;x_0)$ be the function, defined by (33). Applying Lemma 1 and Theorem 3 we find that the function $\Omega_N(x,t;x_0) = \Gamma(t)\omega_N(x,t;x_0)$ is the regularization of the Green's function of the transverse vibration of the two-layered rod $\Omega(x,t;x_0) = \Gamma(t)\omega(x,t;x_0)$.*

Example 3. *Computation of the regularized Green's function $\Omega_N(x,t;x_0)$: Let us consider the rod with characteristics described in Example 1. Using formula (43), Remark 3 for $x_0 = 2$, $t=1$, $N=5$ and $N=19$ and applying Maple tools we have computed the regularization of the Green's function of of the transverse vibration of the two-layered rod for two different regularized parameters. The results of this computation are presented in Fig.3 and Fig.4.*

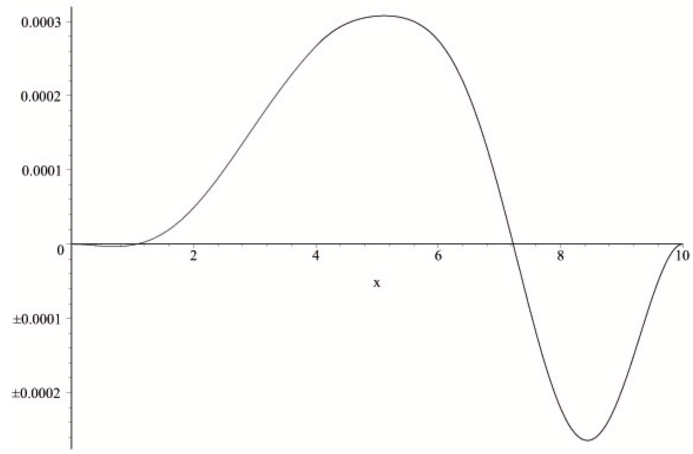

Figure 3: The graph of $\Omega_N(x,t,x_0)$ for $N = 5$, $t = 1$, $x_0 = 2$

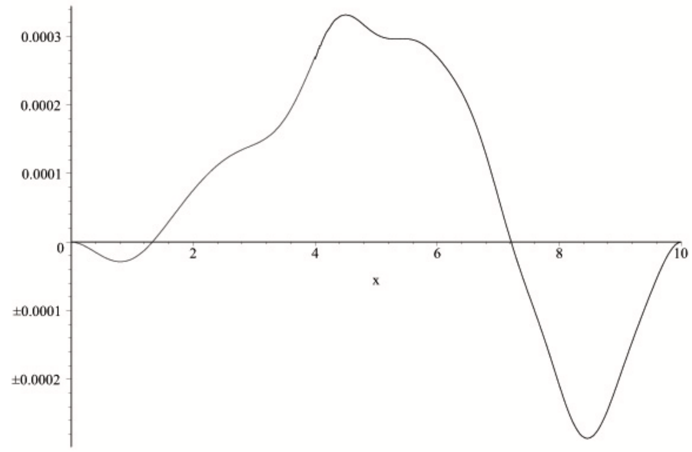

Figure 4: The graph of $\Omega_N(x,t,x_0)$ for $N = 19$, $t = 1$, $x_0 = 2$

4 Conclusion

We have proposed a new analytical method for the approximate computation of the Green's function for a non-stationary partial differential equation for the transverse vibration of two stepped rods. Applying this method, a formula for the computation of the Green's function was obtained in the regularized form. This formula is the form the finite Fourier series. The number of terms in the series is a regularization parameter. This parameter is chosen as a suitable approximation of the Dirac delta function, which appears in the dynamic equation that defines the Green's function. Computational experiments have confirmed the reliability of the claimed method.

References

Allahverdiev, B.P. & Tuna, H. (2019). Indices Defect Theory of Singular Hahn-Sturm-Liouville Operators, *Journal of Applied Analysis and Computation*, 9(5), 1719-1730.

Allahverdiev, B.P. & Tuna, H. (2019). Nonlinear Singular Sturm-Liouville Problems with Impulsive Conditions. *Facta Universitatis, Series: Mathematics and Informatics*, *34*(3), 439-457.

Ba, Z., Wu, M. & Liang, J.(2020). 3D dynamic responses of a multi-layered transversely isotropic saturated half-space under concentrated forces and pore pressure. *Applied Mathematical Modelling*, *80*, 859-878. https://doi.org/10.1016/j.apm.2019.11.014

Faydaoglu, S. & Guseinov, G.Sh. (2003). Eigenfunction expansion for a Sturm-Liouville boundary value problem with impulse. *International Journal of Pure and Applied Mathematics*, *8*(2), 137-170.

Faydaoglu, S. & Guseinov, G.Sh. (2010). An expansion result for a Sturm-Liouville eigenvalue problem with impulse, *Turkish Journal of Mathematics*, *34*(3), 355-366.

Faydaoglu, S. & Yakhno, V.G. (2016). Approximate computation of the Green's function of transverse vibration of the composite rods. *AIP Conference Proceedings* 1776, 090047. https://doi.org/10.1063/1.4965411

Faydaoglu, S. (2018). An expansion result for the equation of transverse vibration of two-layered composite. *AIP Conference Proceedings* 1997, 020015. https://doi.org/10.1063/1.5049009.

Faydaoglu, S. (2019). Properties of the fourth-order boundary value problem with transmission conditions. *Iranian Journal of Science and Technology, Transactions A: Science*, *43*:2515-2522. https://doi.org/10.1007/s40995-019-00732-2

Ghadi, M.E., Pak, R.Y.S. & Behrestaghi, A.A. (2009). Elastostatic Green's functions for an arbitrary internal load in a transversely isotropic bi-material full-space. *International Journal of Engineering Science*, *47*(4), 631-641.

Inceoglu, S. & Gurgoze, M. (2000). Longitudinal vibrations of rods coupled by several spring-mass system. *Journal of Sound and Vibration*, *234*(5), 895-905.

Iraniparast, N., Nguyen, L. & Khenner, M. (2017). Asymptotic Behavior of Waves in a Nonuniform Medium. *Applications and Applied Mathematics: An International Journal (AAM)*, *12*(1), 217-229.

Kulaev, R.Ch. (2016). On the oscillation property of Green's function of a fourth-order discontinuous boundary-value problem. *Mathematical Notes*, *100*, 391–402. https://doi.org/10.1134/S0001434616090054

Mukhtarov, O.Sh., Olgar, H., Aydemir, K. & Jabbarov, I.Sh. (2018). Operator-Pencil Realization of one Sturm-Liouville Problem with Transmission Conditions. *Applied and Computational Mathematics*, *17*(3), 284-294.

Mukhtarov, O.Sh. & Yucel, M. (2020). A Study of the Eigenfunctions of the Singular Sturm-Liouville Problem Using the Analytical Method and the Decomposition Technique. *Mathematics*, *8*(3), 415. doi:10.3390/math8030415

Mukhtarov, O., Olğar, H. & Aydemir, K. (2020). Eigenvalue Problems with Interface Conditions. *Konuralp Journal of Mathematics*, *8*(2), 284-286.

Polyanin, A.D. (2002). *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. Chapman and Hall/CRC Press, Boca Raton-London.

Pan, E. (2019). Green's functions for geophysics: a review. *Reports on Progress in Physics*, *82*(10) 106801. DOI: 10.1088/1361-6633/ab1877

Vladimirov, V.S. (1971). *Equations of Mathematical Physics*. Dekker, New York.

Yakhno, V.G. (2018). The Well-Posedness of Dynamical Equations of Magneto-electro-elasticity. *Mediterranean Journal of Mathematics*, *15*(21). https://doi.org/10.1007/s00009-018-1065-4

Yakhno, V.G. (2018). An explicit formula for modeling wave propagation in magneto-electro-elastic materials. *Journal of Electromagnetic Waves and Applications*, *32*(7), 899-912. DOI: 10.1080/09205071.2017.1410076

Yakhno, V.G. & Altunkaynak, M. (2018). Symbolic computation of the time-dependent electric and magnetic fields in bi-anisotropic media with polynomial inputs. *International Journal of Numerical Modelling Electronic Networks Devices and Fields*, *31*(2). DOI: 10.1002/jnm.2339

Yakhno, V.G. (2020). A new analytical approach for solving equations of elasto-hydrodynamics in quasicrystals. *Journal of Mechanics of materials and Structures*, *15*(1), 135-157.

Zhanm, Q., Zhuang, M., Fang, Y., Liu, J-G. & Liu, Q.H. (2019). Green's function for anisotropic dispersive poroelastic media based on the Radon transform and eigenvector diagonalization. *Proceedings Royal Society A*, *475*: 20180610. http://dx.doi.org/10.1098/rspa.2018.0610