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A NOTE ON THE SEMITOTAL DOMINATION NUMBER AND SEMITOTAL BONDAGE NUMBER OF WHEEL AND CYCLE RELATED GRAPHS

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Abstract. For a graph $G(V(G), E(G))$, a set S is semitotal dominating set if both the set S is a dominating set and every vertex in S must be within 2 distance of another vertex in S. The minimum cardinality of a semitotal dominating set is called semitotal domination number and is denoted by $\gamma_{t2}(G)$. The semitotal bondage number, denoted by $b_{t2}(G)$, of G is the minimum number of edges to be removed from the graph to increase the semitotal domination number. We determine the semitotal domination number and semitotal bondage number of some wheel and cycle related graphs such as friendship, gear, helm, fan and n-gon book.

Keywords: Graph theory, domination, bondage, semitotal domination number, semitotal bondage number.
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1 Introduction

Here, we are referring to readers to Chatrand & Lesniak (1996) for terminology and notations not specifically defined.

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ of order $|V(G)|$ and edge set $E(G)$ of size $|E(G)|$. Two vertices u and v in G are neighbors if they are adjacent, that is, if there is an edge e_{uv}. The distance $d_G(u, v)$ between two vertices u and v in G is the minimum number of edges of a path between them. The diameter $diam(G)$ of a graph G is the maximum distance between vertices of G. The open neighborhood of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v in G and close neighborhood of v is the set $N[v] = \{v\} \cup N(v)$. The degree of a vertex v in G is the cardinality of $N(v)$, denoted by $deg(v)$. If $deg(v) = 1$, the vertex $v \in V(G)$ is called a pendant vertex (or leaf). A support vertex is defined as a vertex adjacent to a leaf.

The one of the essential and classical conseption of graph theory is domination. It has also studied in interconnection network prevalently. The early results about domination have been surveyed in the two detailed books by Haynes et al. (1998a,b). In addition, enormous number of research papers on domination together with related topics appear in many scientific journals due to their applications in many fields such as networks, wireless communication and so on. A subset $D \subseteq V(G)$ is called a dominating set of G if every vertex not in D has at least one neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets.

A dominating set S is a semitotal dominating set, abbreviated a semi-TD-set, of graph G if every vertex in S is within distance 2 of another vertex of S. The semitotal domination number

$\gamma_{t2}(G)$ of G is minimum cardinality of a semitotal dominating set. A semitotal dominating set of G of cardinality $\gamma_{t2}(G)$ is called a $\gamma_{t2}(G)$-set (Goddard et al., 2014; Henning, 2017; Henning & Marcon, 2014, 2016a,b; Kartal Yıldız & Aytaç, 2021).

A measure of the efficiency of a domination in graphs was first given by (Bauer et al., 1983), who called this measure as domination line-stability, defined as the minimum number of lines (i.e. edges) which when removed from G increases γ.

Goal of the minimum dominating set of sites dominates the entire network with the minimum cost. Therefore, it should be considered whether it remains in good condition when the network is invaded. Assume that someone in the position of a saboteur does not know which sites in the network take part in the dominating role, however, does know that the set of these special sites corresponds to a minimum dominating set in the related graph. Then, how many connections should be attackted in order that the cost cannot remains the same to dominate the entire network? That minimum number of connections is simply the bondage number. Fink et al. (1990) officially introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network in case of connection collapse. The bondage number $b(G)$ of a nonempty undirected graph is the minimum number of edges whose removal from results in a graph with larger domination number. The precise definition of the bondage number is as follows:

$$b(G) = min\{|B| : B \subseteq E(G), \gamma(G - B) > \gamma(G)\}.$$

Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined.

We call such an edge-set B with $\gamma(G - B) > \gamma(G)$ the bondage set and the minimum one the minimum bondage set. In fact, if B is a minimum bondage set, then $\gamma(G - B) = \gamma(G) + 1$ because the removal of one single edge cannot increase the domination number by more than one. If $b(G)$ does not exist, for example, empty graphs, we define $b(G) = \infty$.

As a measure of the vulnerability of networks under link failures, the bondage number, total bondage number, double bondage number, strong-weak bondage numbers, paired bondage number, exponential domination bondage numbers, disjunctive bondage number and the disjunctive total bondage number have been attracted much attention (Aytaç & Turacı, 2016, 2019; Aytaç et al., 2011, 2013; Aytaç & Turacı, 2016, 2017, 2018; Kulli & Patwari, 1991; Raczek, 2008; Yi, 2015; Yogeesha & Soner, 2007). There are many research articles on these parameters in the literature. Inspired by these parameters, we defined new domination-related reliability parameter namely semitotal bondage number (Kartal & Aytaç, 2019).

The semitotal bondage number of G is the minimum number of edges to be removed from the graph to increase the semitotal domination number. The graph G here should not contain isolated vertices. Then the edge subset $F \subset E(G)$ is a semitotal bondage edge set satisfying the properties:

- there is no isolated vertex in $G - F$
- $\gamma_{t2}(G - F) > \gamma_{t2}(G)$.

If at least one semitotal bondage edge set can be found for the graph G we define semitotal bondage number, denoted by $b_{t2}(G)$, such that

$$b_{t2}(G) = min\{|F| : F \text{ is a semitotal bondage set of } G\}.$$

Otherwise the value of the semitotal bondage number of the graph is $b_{t2}(G) = \infty$ (Kartal & Aytaç, 2019).

In this study, semitotal domination number and semitotal bondage number on some wheel and cycle related graph structures are studied. Following theorems are very helpful in finding the semitotal domination number and semitotal bondage number of wheel and cycle related graphs.

Theorem 1. *(Kartal Yıldız & Aytaç, 2021) If $G_1 + G_2$ is a join graph of any two connected graphs G_1 and G_2 then semitotal domination number of $G_1 + G_2$ is $\gamma_{t2}(G1 + G2) = 2$.*

Theorem 2. *(Goddard et al., 2014) The semitotal domination number of the cycle and path graph with $n \geq 5$ vertices is $\gamma_{t2}(P_n) = \gamma_{t2}(C_n) = \lceil \frac{2n}{5} \rceil$.*

Theorem 3. *(Kartal & Aytaç, 2019; Kartal Yıldız, 2020) The semitotal bondage number of*
(a)the path graph with n vertices is $b_{t2}(P_n) = 1$,
(b)the cycle graph with n vertices is $b_{t2}(C_n) = 2$,
(c)the star graph with $n + 1$ vertices is $b_{t2}(K_{1,n}) = \infty$,
d)the wheel graph with $n + 1$ vertices is $b_{t2}(W_{1,n}) = 2$.

2 Semitotal Domination Number of Some Wheel Related Graphs

In this section, we investigate the exact values of a semitotal number and a semitotal bondage number of some wheel related graph such as friendship graph, gear graph and helm graph.

Definition 1. *(Gallian, 2012) The friendship graph F_n is the graph consisting of n edge-disjoint cycles of length 3, all meeting in a common vertex. In Figure 1, we display F_5.*

We will call this common vertex the central vertex and denote it with c. Let the vertex set be $V(F_n) = \{1, 2, 3, ..., 2n - 1, 2n, c\}$ and the edge set be $E(F_n) = E_1 \cup E_2$, where $E_1 = \{e_{i(i+1)} | i \in \{1, 3, 5, ..., 2n - 1\}$ and $E_2 = \{e_{cj} | j \in \{1, 2, ..., 2n\}\}$ of F_n.

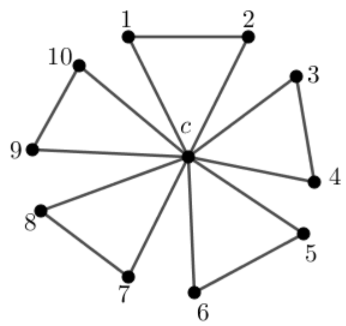

Figure 1: The graph F_5

Theorem 4. *Let F_n be a friendship graph with order $2n + 1$. Then the semitotal domination number of F_n is $\gamma_{t2}(F_n) = 2$.*

Proof. Let S be a semi-TD-set of F_n. Since the central vertex c is adjacent to other all vertices, the set $S = \{c\}$ is a dominating set of F_n. By the definition of semitotal dominating set, S must be included another vertex distnce in 2 with the vertex c. Since any vertex of $V(F_n) - \{c\}$ is adjacent to c, we can choose any vertex to add the set S. Thus, S is a γ_{t2}-set for F_n and $\gamma_{t2}(F_n) = 2$. □

Theorem 5. *Let F_n be a friendship graph with order $2n + 1$ for $n \geq 2$. Then the semitotal bondage number of F_n is $b_{t2}(F_n) = 2$.*

Proof. As you can see Fig 2 for the set $F = \{e_{ci}, e_{c(i+1)}\}$ where $i \in \{1, 3, ..., 2n - 1\}$, which two consecutive edges in the set E_2, the graph $H \cong F_n - F \cong P_2 \cup F_{n-1}$ is obtained. Thus we get,

$$\begin{aligned} \gamma_{t2}(H) &= \gamma_{t2}(F_{n-1}) + \gamma_{t2}(P_2) \\ &= 2 + 2 \\ &= 4 > \gamma_{t2}(F_n). \end{aligned}$$

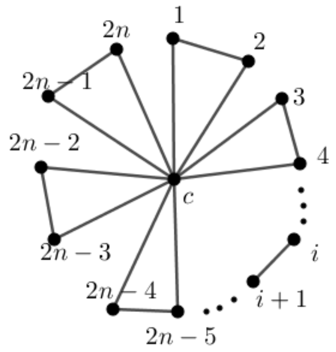

Figure 2: The graph $F_n - F$

It is easy to see that $b_{t2}(F_n) \leq 2$ because it increases the semitotal domination number value. Suppose $b_{t2}(F_n) < 2$. Let $b_{t2}(F_n) = 1$. In this case F is either one vertex in the set E_1 or one vertex in the set E_2. When any $e_{i(i+1)}$ edge for $i \in \{1, 3, ..., 2n-1\}$ is selected from the set E_1 or any e_{cj} edge for $j \in \{1, 2, ...2n\}$ is selected from E_2, one of the vertices sets $\{c, i\}$, $\{c, i+1\}$, $\{c, j\}$ or $\{c, j+1\}$ is γ_{t2}-set for F_n. Therefore in these cases, it is obtained that $\gamma_{t2}(F_n) = \gamma_{t2}(F_n - F) = 2$, a contradiction.

Thus, it is obtained that $b_{t2}(F_n) = 2$ and proof is completed. □

Definition 2. *(Gallian, 2012) The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outher cycle. The gear graph has* $2n+1$ *vertices and* $3n$ *edges. In Figure 3, we display* G_5.

We denote this added vertex with c. Let $V(G_n) = V_1 \cup V_2 \cup \{c\}$ be vertex set of G_n, where $V_1 = \{1, 3, ..., 2n-1\}$, $V_2 = \{2, 4, ..., 2n\}$ are odd and even vertices of the outher cycle, respectively. Let $E(G_n) = E_1 \cup E_2$, where $E_1 = \{e_{i(i+1)} | \ i \in \{1, 2, ..., 2n\} \ \}$ and $E_2 = \{e_{cj} | \ j \in \{1, 3, .., 2n-1\} \ \}$.

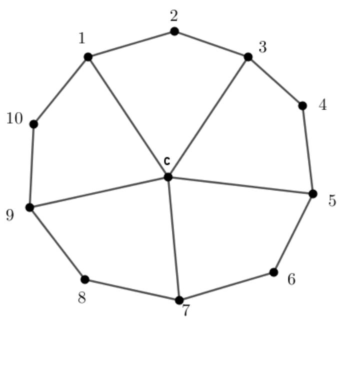

Figure 3: The graph G_5

Theorem 6. *Let* G_n *be a gear graph of order* $2n+1$. *Then the semitotal domination number of* G_n *is*

$$\gamma_{t2}(G_n) = \lceil \frac{n}{2} \rceil + 1.$$

Proof. In the graph G_n, any vertex i in V_2 is adjacent to two vertices in V_1, any vertex j in V_1 is adjacent to two vertices in V_2 and the vertex c. Let S be a semi-TD-set of G_n. Since the vertex c is adjacent to all odd vertices, the vertex c must be in S. Further we must add the new odd vertices to S to dominate all vertices in V_2. Since two even vertices are dominated by one odd vertex, $(4k-1)$th odd vertices where $k \in \{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$ must be added to S by definition semitotal dominating set. In this case, we get two cases according to the modulo 2.

Case 1. $n \equiv 0(mod2)$.
In this case, added last vertex is the $(2n-1)$th vertex to S. Hence, the set $S = \{c\} \cup \{4k-1 | k \in 1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$ is γ_{t2}-set for G_n and $\gamma_{t2}(G_n) = \lfloor \frac{n}{2} \rfloor + 1 = \frac{n}{2} + 1$ when $n \equiv 0(mod2)$.

Case 2. $n \equiv 1(mod2)$.
In this case, added last vertex is the $4k-1 = 4\lfloor \frac{n}{2} \rfloor - 1 = 4(\frac{n-1}{2}) - 1 = 2n-2-1 = (2n-3)$rd vertex to S. Since the vertex $2n$ is not dominated, we must be added the vertex $2n$ to S. Hence the set $S = \{c\} \cup \{4k-1 | k \in \{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}\} \cup \{2n\}$ is γ_{t2}-set for G_n and $\gamma_{t2}(G_n) = \lfloor \frac{n}{2} \rfloor + 1 + 1 = \frac{n-1}{2} + 1 + 1 = \frac{n+1}{2} + 1$ when $n \equiv 1(mod2)$.
Thus, it is obtained that $\gamma_{t2}(G_n) = \lceil \frac{n}{2} \rceil + 1$. □

Theorem 7. *Let G_n be a gear graph of order $2n+1$. Then the semitotal bondage number of G_n is $b_{t2}(G_n) = 2$.*

Proof. Let $F = \{e_{ic}, e_{jc}\}$, where $i, j \in V_1$ and $j - i = 4$. Let S' be semi-TD-set of $H \cong G_n - F$. By the definition of semitotal dominating set, the vertex c must be in S' to dominate all vertices in V_1 except for i and j. To dominate the undominated vertices i and j, we must take the vertices i, j or adjacent vertex of i, j in V_2. Since $d_H(i, c) > 2$, condition of semitotal dominating set is not provided. In other cases, since $d_H(i+1, c) = 2$ and $d_H(i-1, c) = 2$ this condition is provided. Suppose the vertex $i+1$ be in S'. In this case, when the undominated vertex j is also added to S', all vertices V_1 and the vertices $i+1$, $j-1$, $j+1$ in V_2 are dominated by S'. For the vertex j, by adding a vertex at distance 2 to the S', which is the vertex $j+2$, the even vertices $\{j+5, j+7, ..., j+2n-5\}_{mod2n}$, where $0 \equiv 2n(\text{mod2n})$ that are not dominated remain. We know that each added vertex dominate two even vertices. Similar to Teorem 6, the number of added vertices to S' is

$$
\begin{aligned}
&= \lfloor \frac{(j+2n-5-1)-(j+6)}{4} + 1 \rfloor \\
&= \lfloor \frac{2n-12}{4} + 1 \rfloor \\
&= \lfloor \frac{2n-8}{4} \rfloor \\
&= \lfloor \frac{n-4}{2} \rfloor.
\end{aligned}
$$

We have two cases similar to Teorem 6.

- when $n \equiv 0$, there is no need to add any vertex to S'. Hence, we get $\gamma_{t2}(H) = \lfloor \frac{n-4}{2} \rfloor + 4$.
- when $n \equiv 1$, one vertex should be added to S'. Hence, we get $\gamma_{t2}(H) = \lfloor \frac{n-4}{2} \rfloor + 1 + 4$.

For both cases, the semitotal domination number of H is $\gamma_{t2}(H) = \lceil \frac{n-4}{2} \rceil + 4$. Since when $n \equiv 0$, $\gamma_{t2}(H) = \lceil \frac{n-4}{2} \rceil + 4 = \frac{n-4}{2} + 4 = \frac{n+4}{2} > \lceil \frac{n}{2} \rceil + 1 = \gamma_{t2}(G_n)$, and when $n \equiv 1$, $\gamma_{t2}(H) = \lceil \frac{n-4}{2} \rceil + 4 = \frac{n-3}{2} + 4 = \frac{n+5}{2} > \lceil \frac{n}{2} \rceil + 1 = \gamma_{t2}(G_n)$, $b_{t2}(G_n) \leq |F| = 2$. Suppose $b_{t2}(G) < 2$. Let $b_{t2}(G_n) = 1$. We have two cases such that one edge of outher cycle or one edge between the vertices c and i, where $i \in V_1$. In both cases it is easy to see that $\gamma_{t2}(G_n) = \gamma_{t2}(G_n - F)$, a contradiction. Thus, it is obtained that $b_{t2}(G_n) = 2$ and proof is completed. □

Definition 3. *(Gallian, 2012) Helm H_n is a graph of order $2n+1$ obtained from a wheel W_{n+1} with cycle C_n having a pendant edge attached to each vertex of the cycle. In Figure 4, we display H_8.*

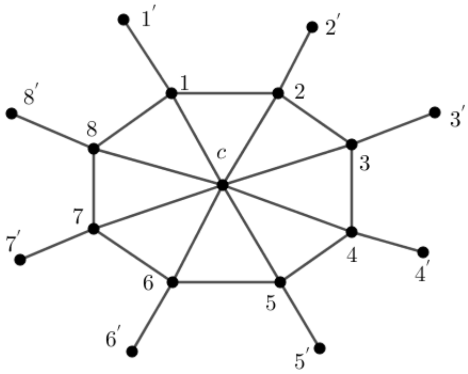

Figure 4: The helm graph H_8

We denote the central vertex of W_{n+1} with c. Let the vertex set $V(H_n) = \{1, 2, ..., n, c, 1', 2', ..., n'\}$ and the edge set $E(H_n) = \{e_{1c}, ..., e_{nc}, e_{12}, ..., e_{n1}, e_{11'}, ..., e_{nn'}\}$ of the Helm graph H_n, respectively.

Theorem 8. *Let H_n be a helm graph of order $2n + 1$. Then the semitotal domination number of helm is $\gamma_{t2}(H_n) = n$.*

Proof. Obviously, in any graph, the each support vertex must be added to dominating set to dominate the all leaves. Let S be a semi-TD-set of H_n. Thus, the set $S = \{i | i \in \{1, 2, ..., n\}\}$ formed by taking all support vertices is γ_{t2}-set of helm and then we get $\gamma_{t2}(H_n) = n$. □

Theorem 9. *Let H_n be a helm graph of order $2n + 1$. Then the semitotal bondage number of helm is $b_{t2}(H_n) = 3$.*

Proof. Let $F = \{e_{1n}, e_{12}, e_{1c}\}$ be a semitotal bondage set of helm. Since the graph $H = H_n - F$ contains H_{n-1} and P_2 graphs, we get

$$\begin{aligned}\gamma_{t2}(H) =&\gamma_{t2}(H_{n-1}) + \gamma_{t2}(P_2)\\ =&n - 1 + 2\\ =&n + 1.\end{aligned}$$

Since $\gamma_{t2}(H) > \gamma_{t2}(H_n)$, it is clear that $b_{t2}(H_n) \leq 3$. Suppose $b_{t2}(H_n) < 3$. Let $b_{t2}(H_n) = 2$. Since by definition there should be no isolated vertex in the graph, there must be $F = \{e_{i(i+1)}, e_{j(j+1)}\}$ or $F = \{e_{ic}, e_{jc}\}$ or $F = \{e_{i(i+1)}, e_{jc}\}$ where $i, j \in \{1, 2, ..., n - 1\}$ and $i \neq j$. In this cases, since $\gamma_{t2}(H_n - F) = \gamma_{t2}(H_n)$, a contradiction. Let $b_{t2} = 1$. Since the graph H_n contains a wheel graph, the semitotal bondage number cannot be less than the semitotal bondage number of the wheel graph. Thus, the semitotal bondage number of helm is $b_{t2}(H_n) = 3$.
□

3 Semitotal Domination Number of Some Cycle Related Graphs

In this section we investigate the exact values of a semitotal number and a semitotal bondage number of some cycle related graph such as fan graph and n-gon book graph.

Definition 4. *(Gallian, 2012) A fan graph $F_{1,n-1} = P_{n-1} + K_1$ is a graph obtained by joining the path P_{n-1} with the graph K_1. See Fig 5.*

We will call the vertex n on the graph K_1 as the central vertex. Let the vertex set and edge set be $V(F_{1,n-1}) = \{1, 2, ..., n - 1, n\}$, $E(F_{1,n-1}) = \{e_{12}, ..., e_{(n-2)(n-1)}, e_{1n}, ..., e_{(n-1)n}\}$ of $F_{1,n-1}$, respectively.

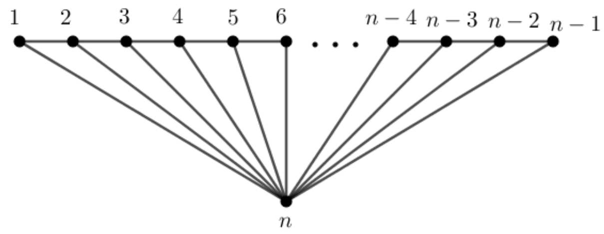

Figure 5: The fan fraph $F_{1,n-1}$

Theorem 10. *Let $F_{1,n-1}$ be a fan graph with order $n \geq 2$. Then the semitotal domination number of $F_{1,n-1}$ is*

$$\gamma_{t2}(F_{1,n-1}) = 2.$$

Proof. The fan graph is obtained by joining two graphs. By the Theorem 1, we know that the semitotal domination number of the graph obtained from the joining of two any graphs is equal to 2. Thus, the semitotal domination number of fan graph is $\gamma_{t2}(F_{1,n-1}) = 2$. □

Theorem 11. *Let $F_{1,n-1}$ be a fan graph with order $n \geq 8$. Then the semitotal bondage number of $F_{1,n-1}$ is*

$$b_{t2}(F_{1,n-1}) = 2.$$

Proof. As you can see Fig 6, let the semitotal bondage set be $F = \{e_{in}, e_{(i+3)n}\}$, where $i \in \{1, 2, ..., n-4\}$. When $n - 1 < 7$, all vertices dominated in $F_{1,n-1}$ with the vertices $i + 1$ and $i+4$. Otherwise, it is not dominated all vertices with two vertices with $n \geq 8$. In this case, either all vertices cannot be dominated by any two vertices or the feature of being within 2 distance cannot be achieved. Thus, when n is greater than 8, it is seen that the semitotal domination number increases. Hence, it is obtained that $b_{t2}(F_n) \leq 2$. Suppose $b_{t2}(F_n) < 2$. Let $b_{t2}(F_n) = 1$. In this case, F is either e_{in} for $i \in \{1, 2, ..., n\}$ or $e_{i(i+1)}$ for $i \in 1, 2, ..., n-2$. In these two cases, since $\gamma_{t2}(F_{1,n-1} - F)$=$\gamma_{t2}(F_{1,n-1})$, a contradiction. Hence, we get $b_{t2}(F_{1,n-1}) = 2$ and proof is completed. □

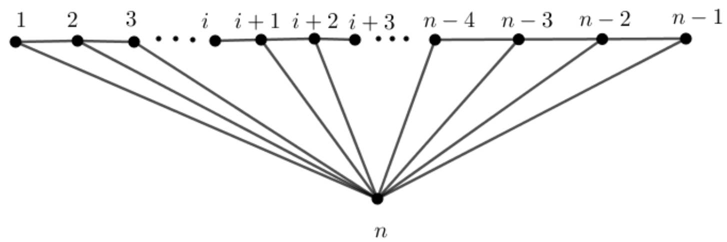

Figure 6: The graph $F_{1,n-1} - F$

Definition 5. *(Gallian, 2012) When k copies of C_n share a common edge, it will form an n-gon book of k pages, and it is denoted by $B(n, k)$ or $B_{n,k}$. See in Fig 7.*

Let the vertex set be $V(B(n, k)) = \{u, v\} \cup \{i^{(j)} | i \in \{1, 2, ..., n-2\}, j \in \{1, 2, ..., k\}\}$ of $B(n, k)$, where u and v are the end vertices of the common edge. Let the edge set be $E(B(n, k)) = \{e^{(j)}_{i(i+1)}\} \cup \{e^{(j)}_{u(n-2)}\} \cup \{e^{(j)}_{v1}\} \cup \{uv\}$, where $i \in \{1, 2, ..., n-3\}, j \in \{1, 2, ..., k\}$.

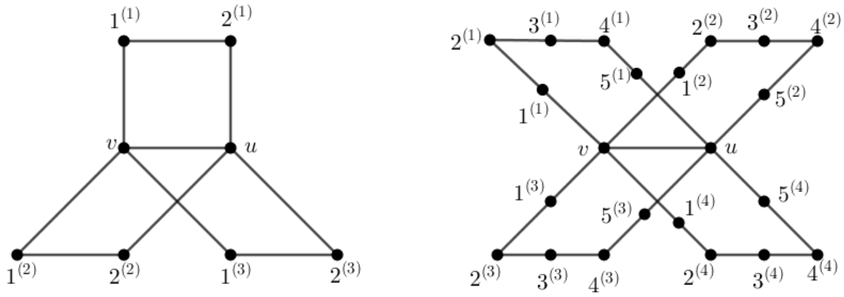

Figure 7: The graphs $B(4,3)$ and $B(7,4)$

Theorem 12. *Let $B(n,k)$ be a n-gon book with k pages and $k(n-2)+2$ vertices for $n \geq 3$. Then the semitotal domination number of $B(n,k)$ is*

$$\gamma_{t2}(B(n,k)) = \begin{cases} k(\lceil \frac{2n}{5} \rceil - 1) + 1, & n \equiv 0, 2(mod 5) \\ k(\lceil \frac{2n}{5} \rceil - 2) + 2, & otherwise. \end{cases}$$

Proof. Let the set S be a semi-TD-set of $B_{n,k}$. Further, let the set S_i be γ_{t2}-set of any C_n^i for $i \in \{1, 2, ..., k\}$. For this proof, we divide the proof into two cases as follows.
Case 1. Let $n \equiv 0, 2(mod5)$.
We have three subcases depending on the vertices u and v.

Case 1.1. Let $u \in S_i$ (or $v \in S_i$) for $i \in \{1, 2, ..., k\}$.
We know that $\gamma_{t2}(C_n^i)$=$|S_i| = \lceil \frac{2n}{5} \rceil$ by Theorem 2. In this case, it is obtained that $S = \bigcup_{i=1}^{k} S_i$. Since any S_i has common vertex u, we get

$$\begin{aligned} |S| &= \sum_{i=1}^{k} |S_i| - (k-1) \\ &= k\lceil \frac{2n}{5} \rceil - (k-1) \\ &= k(\lceil \frac{2n}{5} \rceil - 1) + 1. \end{aligned}$$

Case 1.2. Let $u, v \in S_i$ for $i \in \{1, 2, ..., k\}$.
When a set S_i is created with u and v, it is obtained that $|S_i| = \lceil \frac{2n}{5} \rceil + 1$. However this is a contradiction.

Case 1.3. Let $u, v \notin S$ for $i \in \{1, 2, ..., k\}$.
When a set S_i is created without u and v, it is ıbtained that $|S_i| = \lceil \frac{2n}{5} \rceil$ by Theorem 2. Thus, $S = \bigcup_{i=1}^{k} S_i$ and then $|S| = \sum_{i=1}^{k} |S_i| = k(\lceil \frac{2n}{5} \rceil)$.

By subcases 1.1, 1.2 and 1.3, we get

$$\gamma_{t2}(B(n,k)) = k(\lceil \frac{2n}{5} - 1) + 1. \tag{1}$$

Case 2. Let $n \equiv 1, 3, 4(mod5)$.
We have three subcases depending on the vertices u and v.

Case 2.1. Let $u \in S_i$ (or $v \in S_i$) for $i \in \{1, 2, ..., k\}$.
Similar to Case 1.1., we have $|S| = k(\lceil \frac{2n}{5} \rceil - 1) + 1$.

Case 2.2. Let $u, v \in S_i$ for $i \in \{1, 2, ..., k\}$.
When a set S_i is created with u and v, it is obtained that $|S_i| = \lceil \frac{2n}{5} \rceil$ by Theorem 2. Thus,

$S = \bigcup_{i=1}^{k} S_i$. Since any S_i has common vertices u and v, we get

$$\begin{aligned} |S| &= \sum_{i=1}^{k} |S_i| - 2(k-1) \\ &= k\lceil \frac{2n}{5} \rceil - 2k + 2 \\ &= k(\lceil \frac{2n}{5} \rceil - 2) + 2. \end{aligned}$$

Case 2.3. Let $u, v \notin S_i$ for $i \in \{1, 2, ..., k\}$.
Similar to Case 1.3., we have $|S| = k\lceil \frac{2n}{5} \rceil$.

By subcases 2.1, 2.2 and 2.3, we get

$$\gamma_{t2}(B(n,k)) = k(\lceil \frac{2n}{5} \rceil - 2) + 2. \tag{2}$$

The proof is completed by (1) and (2). □

Theorem 13. *Let $B(n,k)$ be a n-gon book with k-pages and $k(n-2)+2$ vertices for $n \geq 4$. Then the semitotal bondage number of $B(n,k)$ is*

$$b_{t2}(B(n,k)) = \begin{cases} 2, & n \equiv 0, 2, 3(mod 5) \\ 1, & n \equiv 1, 4(mod 5). \end{cases}$$

Proof. Let $e^{(1)}_{u(n-2)}$,$e^{(2)}_{u(n-2)}$,...,$e^{(k)}_{u(n-2)}$ and uv be indicent to the vertex u and $e^{(1)}_{v1}$, $e^{(2)}_{v1}$,...,$e^{(k)}_{v1}$ and uv be indicent to the vertex v. We have three cases depending on n.

Case 1. Let $n \equiv 1, 4(mod 5)$.
Let $F = \{e^{(1)}_{v1}\}$ be a semitotal bondage set of $B(n,k)$ and D be a semi-TD-set of the graph $H \cong B(n,k) - F$. In this case, the graph H contains the subgraphs $B(n,k-1)$ and P_n and also the intersection of these graph is the vertex u. Let D_1 and D_2 be γ_{t2}-set of $B(n,k-1)$ and P_n, respectively. Since D_1 and D_2 have common vertex u, we get

$$\begin{aligned} |D| &= |D_1| + |D_2| - 1 \\ &= (k-1)(\lceil \frac{2n}{5} \rceil - 2) + 2 + \lceil \frac{2n}{5} \rceil - 1 \\ &= k\lceil \frac{2n}{5} \rceil - 2(k-1) + 2 - 1 \\ &= k\lceil \frac{2n}{5} \rceil - 2k + 2 + 2 - 1 \\ &= k(\lceil \frac{2n}{5} \rceil - 2) + 3 > k(\lceil \frac{2n}{5} \rceil - 2) + 2 = \gamma_{t2}(B(n,k)). \end{aligned}$$

Thus, when $n \equiv 1, 4(mod 5)$ we get $b_{t2}(B(n,k)) = 1$.

Case 2. Let $n \equiv 0, 2(mod 5)$.
Let $F = \{e^{(1)}_{v1}\}$ and D be a semi-TD-set of the graph $H \cong B(n,k) - F$. We have also two

subgraphs as $B(n,k-1)$ and P_n. When proof is done similar to Case 1, it is obtained that

$$\begin{aligned}|D| &= |D_1| + |D_2| - 1\\ &= (k-1)(\lceil\frac{2n}{5}\rceil - 1) + 1 + \lceil\frac{2n}{5}\rceil - 1\\ &= k\lceil\frac{2n}{5}\rceil - k + 1 + 1 - 1\\ &= k\lceil\frac{2n}{5}\rceil - 2k + 2 + 2 - 1\\ &= k(\lceil\frac{2n}{5}\rceil - 1) + 1 = \gamma_{t2}(B(n,k)).\end{aligned}$$

Thus, when $n \equiv 0, 2(mod5)$ it is seen that $b_{t2}(B(n,k)) \geq 2$. Assume that $b_{t2}(B(n,k)) = 2$. Let $F = \{e^{(1)}_{v1}, e^{(1)}_{23}\}$. In this case, the graph $H \cong B(n,k) - F$ contains P_2, $B(n,k-1)$ and P_{n-2} graphs. Let D_1, D_2, D_3 be γ_{t2}-set of these graphs, respectively. Since $D_2 \cap D_3 = \{u\}$,

$$\begin{aligned}|D| &= |D_1| + |D_2| + |D_3| - 1\\ &= 2 + (k-1)(\lceil\frac{2n}{5}\rceil - 1) - 1 - 1 + 1 + \lceil\frac{2(n-2)}{5}\rceil - 1.\end{aligned}$$

When $n \equiv 0(mod5)$,

$$\begin{aligned}|D| =&2 + (k-1)(\frac{2n}{5} - 1) + 1 + \frac{2n}{5} - 1\\ =&k(\frac{2n}{5} - 1) + 3 > k(\frac{2n}{5} - 1) + 1 = \gamma_{t2}(B(n,k)).\end{aligned}$$

When $n \equiv 2(mod5)$,

$$\begin{aligned}|D| =&2 + (k-1)(\frac{2n+1}{5} - 1) + 1 + \frac{2n-4}{5} - 1\\ =&2 + k(\frac{2n-4}{5}) > k(\frac{2n-4}{5}) + 1 = \gamma_{t2}(B(n,k)).\end{aligned}$$

Thus, when $n \equiv 0, 2(mod5)$ we get, $b_{t2}(B(n,k)) = 2$.

Case 3. Let $n \equiv 3(mod5)$.
Let $F = \{e^{(1)}_{v1}, e^{(1)}_{23}\}$. In this case, the graph $H \cong B(n,k) - F$ contains P_2, $B(n,k-1)$ and P_{n-2} graphs similar to Case 2. Let D_1, D_2, D_3 be γ_{t2}-set of these graphs, respectively. Since $D_2 \cap D_3 = \{u\}$,

$$\begin{aligned}|D| &= |D_1| + |D_2| + |D_3| - 1\\ &= 2 + (k-1)(\lceil\frac{2n}{5}\rceil - 2) + 2 + \lceil\frac{2(n-2)}{5}\rceil - 1\\ &= 2 + (k-1)(\frac{2n+4}{5} - 2) + 2 + \frac{2n-1}{5} - 1\\ &= 2 + k(\frac{2n+4}{5} - 2) - \frac{2n+4}{5} + 2 + 2 + \frac{2n-1}{5} - 1\\ &= 2 + k(\frac{2n-6}{5}) - 1 + 4 - 1\\ &= k(\frac{2n-6}{5}) + 4 > k(\frac{2n}{5} - 2) + 2 = \gamma_{t2}(B(n,k)).\end{aligned}$$

Thus, when $n \equiv 3(mod5)$ we get $b_{t2}(B(n,k)) = 2$.

By the Cases 1,2 and 3, the proof is completed. □

4 Conclusion

Semitotal bondage number is a new vulnerability parameter defined by us. Since vulnerability is an important concept for network design, we think that this parameter will also be helpful for network designers. Further, semitotal bondage number can be examined in different graph structures such as graph operations.

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