Journal of Modern Technology and Engineering
Vol.7, No.1, 2022, pp.30-39

TWO ANALYTIC METHODS FOR SOLVING THE VOLTERRA INTEGRAL EQUATIONS WITH A WEAKLY SINGULAR KERNEL

Arkan Sh. Hasan*, Sizar A. Mohammed

Department of Mathematics, College of Basic Education, University of Duhok, Kurdistan Region, Iraq

Abstract. In this work, two analytic methods called the Adomian decomposition method (ADM) and the series solution method with new employment (SSM) will be used for the solution of the Volterra integral equation with a weakly singular kernel in the reproducing kernel space. Both methods give the exact solution for this equation in a facility way. The ADM method gives a sequence of components of the solution, which composes a sequence of approximations, whereas the SSM more directly gives the exact solution with the help of the Maclaurin series; both exhibit high accuracy. Four numerical problems are examined to confirm these two methods' validity and power.

Keywords: Volterra integral equation, Adomian decomposition method, Series solution method and weakly singular kernel.

Corresponding author: Arkan Sh. Hasan, Department of Mathematics, College of Basic Education, University of Duhok, Kurdistan Region, Iraq, e-mail: *arkan.hasan@uod.ac*

Received: 26 January 2022; Revised: 18 February 2022; Accepted: 4 April 2022; Published: 29 April 2022.

1 Introduction

We consider the Volterra integral equations with a weakly singular kernel in the reproducing kernel space as given by Wazwaz & Rach (2016)

$$\varphi(t) = f(t) + \int_0^t \frac{s^{\mu-1}}{t^\mu}\varphi(s), t \in [0,T], \tag{1}$$

where $\mu > 1$ and f is a given function. The exponent μ plays a major role in solving this equation. In the case when $\mu < 1$, an infinite set of solutions exists, one of which is continuous whenever $f(t)$ is continuously differentiable, and all other solutions have an infinite gradient at the origin. This infinity of solutions arises because, for $\mu < 1$, the singularity persists at $t = 0$ for all $t > 0$ (Diogo et al., 2007). In the case of $\mu > 1$, the kernel is singular only at $t = 0$.

Lemma 1.

(a) If $\mu \in]0,1[$ and $f \in C^1[0,T]$ (with $f(0) = 0$ if $\mu = 1$), then equation (1) has a family of solutions $\varphi \in C[0,T]$ given by the formula (Diogo et al., 2006)

$$\varphi(t) = c_0 t^{1-\mu} + f(t) + \gamma + t^{1-\mu}\int_0^t s^{\mu-2}(f(s) - f(0))ds, t \in [0,T], \tag{2}$$

where

$$\gamma = \begin{cases} \frac{1}{\mu-1} f(0) & \text{if } \mu < 1 \\ 0 & \text{if } \mu = 1 \end{cases}. \tag{3}$$

and c_0 *is an arbitrary constant. Out of the family of solutions, there is one particular solution* $\varphi \in C[0,T]$*. Such a solution is unique and can be obtained from equation (2) by taking* $c_0 = 0$*.*
(b) If $\mu > 1$ *and* $f \in C^m[0,T], m \geq 0$*, then the unique solution* $\varphi \in C^m[0,T]$ *of (1) is*

$$\varphi(t) = f(t) + t^{1-\mu} + \int_0^t s^{\mu-2} f(s)ds. \tag{4}$$

We note that (4) can be obtained from (2) with $c_0 = 0$*. Indeed, it follows from (2) that*

$$c_0 = \lim_{t \to 0^+} t^{1-\mu}\varphi(t) \tag{5}$$

and this limit is zero when $\mu > 1$*. In principle, if we know the value of* c_0 *we may use (2) to obtain the numerical approximations of the solution.*

The two credible methods, ADM and SSM; were formerly used by Wazwaz for solving linear and non-linear VIE (Wazwaz, 2011). Regarding the Volterra equations with weakly singular kernel, Wazwaz used ADM for solving it (Wazwaz & Rach, 2016). In this paper, we will present the new employment of the SSM for the solution of VIEWSK with the help of the Maclaurin series. The two methods will be applied in the case when $\mu \in]0,1[$, because of the non-uniqueness solution. Some known and new numerical problems will be checked to highlight the analysis and to confirm the efficiency of the two methods.

2 Reviews of past work

The equation(1) with a non-compact kernel has been the subject of several works for example, in Wazwaz & Rach (2016), Wazwaz solved this type of equation using ADM and VIM, and in Diogo et al. (2004), the author presented high order product integration methods with the logarithmic singular kernel. In Tang & Li (2008), solution of a class of Volterra integral equations with singular and weakly singular kernels is presented. A numerical solution of VIE with WSK using the SCW method is studied in Zhu & Wang (2015). In Lima & Diogo (2002), a numerical solution of a non-uniquely solvable using extrapolation methods. A split-interval method for the solution of singular IE was studied in Chen & Jiang (2011). Chen, Z., & Jiang, W presented the exact solution of this equation in Chen & Jiang (2011). In Mokhtary et al. (2020), a computational approach was given for non-linear VIEWSK.

3 Discussions of ADM and SSM

Consider the Volterra integral equation of weakly singular kernel (Diogo et al., 2006)

$$\varphi(t) = 1 + t + t^2 + \int_0^t \frac{s^{\mu-1}}{t^\mu}\varphi(s), t \in [0,T] \tag{6}$$

That has a singularity at $t = 0$ and $s = 0$ for any positive value of t.

3.1 The Adomian Decomposition Method (ADM)(Wazwaz & Rach, 2016)

The Adomian decomposition method (ADM) was introduced and developed first by George Adomian. The ADM can be easily applied on (6), it mainly comprises of decomposing the unknown function $\varphi(t)$ of any equation into a sum of an endless number of components defined by the decomposition series

$$\varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t) \tag{7}$$

Substituting eq. (7) into both sides of eq. (6) yields

$$\sum_{n=0}^{\infty}\varphi_n(t) = f(t) + \int_0^t t^{-\mu}s^{\mu-1}(\sum_{n=0}^{\infty}\varphi_n(s))ds. \tag{8}$$

Specifying the zero solution term $\varphi_0(t) = f(t)$, and the other terms $\varphi_J(t), j \geq 1$, can be achieved by setting the recurrence relation

$$\begin{cases}\varphi_0(t) = f(t)\\ \varphi_{k+1}(t) = \int_0^t t^{-\mu}s^{\mu-1}\varphi_k(s)ds, k \geq 0,\end{cases} \tag{9}$$

where the approximation can be used as

$$\psi_{n+1}(t) = \sum_{m=0}^{n}\varphi_m(t) \tag{10}$$

which will lead to the complete determination of the components $\varphi_{n+1}(t)$ of $\varphi(t)$. The series solution of $\varphi(t)$ follows immediately. The series solution will converge to the exact solution if such a solution exists. Consequently, the sequence of solution approximations converges to the exact solution as

$$\varphi(t) = \lim_{n\to\infty}\psi_{n+1}(t). \tag{11}$$

3.2 The Series Solution Method (SSM) (Wazwaz, 2011)

In this section, we use the series solution method, which stems mainly from the Maclaurin series

$$\varphi(t) = \sum_{n=0}^{\infty}a_n t^n \tag{12}$$

or equivalently

$$\varphi(t) = a_0 + a_1t^2 + a_3t^2 + ... \tag{13}$$

for analytic functions, for solving (6). We assume that the solution $\varphi(t)$ of (6) is analytic and possesses a Maclaurin series given in (13), where the coefficients a_n should be determined.

Substituting (13) into both sides of (6) gives

$$\sum_{n=0}^{\infty}a_nt^n = M(f(t)) + \int_0^t\sum_{n=0}^{\infty}a_nt^{-\mu}s^{n+\mu-1}ds, \tag{14}$$

where M is the Maclaurin series for $f(t)$. Evaluating the integral in the right side, gives

$$\sum_{n=0}^{\infty}a_nt^n = M(f(t)) + \sum_{n=0}^{\infty}\frac{1}{n+\mu}a_nt^n. \tag{15}$$

Equation(15) can be written as

$$a_0(\frac{\mu-1}{\mu}) + a_1(\frac{\mu}{\mu+1})t + a_1(\frac{\mu+1}{\mu+2})t^2 + \ldots = M(f(t)). \tag{16}$$

We use equation (16) in order to evaluate the terms $a_n, n \geq 0$ by equating the coefficients of like powers of t, and then substituting the recurrence relation that resulted in (13) to obtain the exact solution:

4 Numerical Problems

In this section, we illustrate the both mentioned methods known and new problems:

Problem 1: Consider the second kind Volterra integral equation with a weakly singular kernel (Wazwaz & Rach, 2016)

$$\varphi(t) = 1 + t + t^2 + \int_0^t \frac{s^{\mu-1}}{t^{\mu}}\varphi(s) \quad t \in [0, T]. \tag{17}$$

i. **Using ADM**

Applying relations (9) gives

$$\varphi_0(t) = 1 + t + t^2,$$
$$\varphi_{n+1}(t) = \int_0^t t^{-\mu} s^{\mu-1} \varphi_n(s) ds, n \geq 0$$

which will lead to the determination of the components $\varphi_{n+1}(t)$ of $\varphi(t)$ as follows

$$\varphi_1(t) = t^{-\mu} \int_0^t s^{\mu-1} \varphi_0(s) ds = \frac{1}{\mu} + \frac{1}{\mu+1} t + \frac{1}{\mu+2} t^2,$$

$$\varphi_2(t) = t^{-\mu} \int_0^t s^{\mu-1} \varphi_1(s) ds = \frac{1}{\mu^2} + \frac{1}{(\mu+1)^2} t + \frac{1}{(\mu+2)^2} t^2,$$

$$\varphi_3(t) = t^{-\mu} \int_0^t s^{\mu-1} \varphi_2(s) ds = \frac{1}{\mu^3} + \frac{1}{(\mu+1)^3} t + \frac{1}{(\mu+2)^3} t^2$$

and so on.
This leads to the series

$$\begin{aligned} \varphi(t) = {} & (1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{1}{\mu^3} + \ldots) + \\ & (\frac{1}{\mu+1} + \frac{1}{(\mu+1)^2} + \frac{1}{(\mu+1)^3} + \ldots) t + \\ & (\frac{1}{\mu+2} + \frac{1}{(\mu+2)^2} + \frac{1}{(\mu+2)^3} + \ldots) t^2. \end{aligned} \tag{18}$$

It is clear that (18) represents a geometric series which in turn gives the exact solution as

$$\varphi(t) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t + \frac{\mu+2}{\mu+1} t^2. \tag{19}$$

Notice that all of the geometric series converge if $|\frac{1}{\mu}| < 1$,$|\frac{1}{\mu+1}| < 1$,$|\frac{1}{\mu+2}| < 1$ respectively, and this is justified because $\mu > 1$ as given in eq. (1). Thus the restriction is on the value of the parameter μ in the specified domain. As stated earlier, although this equation is linear, an infinite set of solutions may exist. It was shown in Wazwaz & Rach (2016) that other solutions for this equation exist in the form

$$\varphi(t) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t + \frac{\mu+2}{\mu+1} t^2 + \alpha, t^{1-\mu} \tag{20}$$

where α is an arbitrary constant. It is obvious that solution (26) is contained in the generalized solution (20) for $\alpha = 0$.

ii. **Using SSM**

The Maclaurin series is given as

$$\varphi(t)=\sum_{n=0}^{\infty}a_nt^n=a_0+a_1t+a_2t^2+a_3t^3+\ldots \tag{21}$$

Substituting (21) into (17) yields

$$\sum_{n=0}^{\infty}a_nt^n=1+t+t^2+\int_0^t\sum_{n=0}^{\infty}a_nt^{-\mu}s^{n+\mu-1}ds. \tag{22}$$

Evaluating the integral in the right side, gives

$$\sum_{n=0}^{\infty}a_nt^n=1+t+t^2+\sum_{n=0}^{\infty}\frac{1}{n+\mu}a_nt^n. \tag{23}$$

Equation(23) can be written as

$$a_0(\frac{\mu-1}{\mu})+a_1(\frac{\mu}{\mu+1})t+a_1(\frac{\mu+1}{\mu+2})t^2+\ldots=1+t+t^2. \tag{24}$$

Equating the coefficient of like powers of t in both sides of (24) gives

$$a_0=\frac{\mu}{\mu-1}\quad,\quad a_1=\frac{\mu+1}{\mu}t\quad,\quad a_2=\frac{\mu+2}{\mu+1}t^2\quad;\quad a_n=0,n\geq 3. \tag{25}$$

Substituting (25) into (21) gives

$$\varphi(t)=\frac{\mu}{\mu-1}+\frac{\mu+1}{\mu}t+\frac{\mu+2}{\mu+1}t^2 \tag{26}$$

which is the exact solution.
As stated above, the restriction is on the value of the parameter μ and there is no corresponding restriction on values of t in the specified domain.

Problem 2: Consider the second kind Volterra integral equation with a weakly singular kernel

$$\varphi(t)=1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\frac{t^4}{5!}+\frac{t^5}{6!}+\int_0^t\frac{s^{\mu-1}}{t^{\mu}}\varphi(s)\quad t\in[0,T]. \tag{27}$$

i. **Using ADM**

Applying relations (9) gives the recurrence relations

$$\varphi_0(t)=1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\frac{t^4}{5!}+\frac{t^5}{6!},$$

$$\varphi_1(t)=\frac{1}{\mu}+\frac{1}{2!(\mu+1)}t+\frac{1}{3!(\mu+2)}t^2+\frac{1}{4!(\mu+3)}t^3+\frac{1}{5!(\mu+4)}t^4+\frac{1}{6!(\mu+5)}t^5,$$

$$\varphi_2(t)=\frac{1}{\mu^2}+\frac{1}{2!(\mu+1)^2}t+\frac{1}{3!(\mu+2)^2}t^2+\frac{1}{4!(\mu+3)^2}t^3+\frac{1}{5!(\mu+4)^2}t^4+\frac{1}{6!(\mu+5)^2}t^5,$$

$$\varphi_3(t)=\frac{1}{\mu^3}+\frac{1}{2!(\mu+1)^3}t+\frac{1}{3!(\mu+2)^3}t^2+\frac{1}{4!(\mu+3)^3}t^3+\frac{1}{5!(\mu+4)^3}t^4+\frac{1}{6!(\mu+5)^3}t^5$$

and so on.
This leads to the series solution:

$$\begin{aligned}\varphi(t) = (1+\frac{1}{\mu}+1+\frac{1}{\mu^2}+1+\frac{1}{\mu^3}+\ldots)+\\ (\frac{1}{\mu+1}+\frac{1}{(\mu+1)^2}+\frac{1}{(\mu+1)^3}+\ldots)\frac{t}{2!}+\\ (\frac{1}{\mu+2}+\frac{1}{(\mu+2)^2}+\frac{1}{(\mu+2)^3}+\ldots)\frac{t^2}{3!}+\\ (\frac{1}{\mu+3}+\frac{1}{(\mu+3)^2}+\frac{1}{(\mu+3)^3}+\ldots)\frac{t^3}{4!}+\\ (\frac{1}{\mu+4}+\frac{1}{(\mu+4)^2}+\frac{1}{(\mu+4)^3}+\ldots)\frac{t^4}{5!}+\\ (\frac{1}{\mu+5}+\frac{1}{(\mu+5)^2}+\frac{1}{(\mu+5)^3}+\ldots)\frac{t^5}{6!}\end{aligned} \tag{28}$$

from which we can recognize the respective geometric series as the exact solution:

$$\varphi(t) = \frac{\mu}{\mu-1}+\frac{\mu+1}{2!\mu}t+\frac{\mu+2}{3!(\mu+1)}t^2+\frac{\mu+3}{4!(\mu+2)}t^3+\frac{\mu+4}{5!(\mu+3)}t^4+\frac{\mu+5}{6!(\mu+4)}t^5. \tag{29}$$

Note that the geometric series $|\frac{1}{\mu+k}|<1, k=\{0,1,\ldots,5\}$, converge and this is justified because $\mu>1$ as given in eq.(1). Thus the restriction is on the value of the parameter μ and there is no corresponding restriction on values of t in the specified domain. As stated earlier, although this equation is linear, an infinite set of solutions may exist for such equations. It was shown in Wazwaz & Rach (2016) that other solutions for this equation exist in the form

$$\varphi(t) = \frac{\mu}{\mu-1}+\frac{\mu+1}{2!\mu}t+\frac{\mu+2}{3!(\mu+1)}t^2+\frac{\mu+3}{4!(\mu+2)}t^3+\frac{\mu+4}{5!(\mu+3)}t^4+\frac{\mu+5}{6!(\mu+4)}t^5+\alpha t^{1-\mu}. \tag{30}$$

where α is an arbitrary constant. It is obvious that solution (42) is contained in the generalized solution (30) for $\alpha=0$.

ii. **Using SSM**

Substituting the Maclaurin series (12) in (27) yields

$$\sum_{n=0}^{\infty} a_n t^n = 1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\frac{t^4}{5!}+\frac{t^5}{6!}+\int_0^t \sum_{n=0}^{\infty} a_n t^{-\mu} s^{n+\mu-1} ds. \tag{31}$$

Evaluating the integral in the right side, and using (16) we get

$$a_0=\frac{\mu}{\mu-1}, a_1=\frac{\mu+1}{2!\mu}, a_2=\frac{\mu+2}{3!(\mu+1)}, a_3=\frac{\mu+3}{4!(\mu+2)},$$

$$a_4=\frac{\mu+4}{5!(\mu+3)}, a_5=\frac{\mu+5}{6!(\mu+4)}, a_n=0, n\geq 6. \tag{32}$$

Substituting the components a_n from (32) into (13) to obtain

$$\varphi(t) = \frac{\mu}{\mu-1}+\frac{\mu+1}{2!\mu}t+\frac{\mu+2}{3!(\mu+1)}t^2+\frac{\mu+3}{4!(\mu+2)}t^3+\frac{\mu+4}{5!(\mu+3)}t^4+\frac{\mu+5}{6!(\mu+4)}t^5. \tag{33}$$

which is the exact solution.

As stated earlier, the restriction is on the value of the parameter μ and there is no corresponding restriction on values of t in the specified domain.

Problem 3: Consider the second kind Volterra integral equation with a weakly singular kernel

$$\varphi(t) = f(t) + \int_0^t t^{-\mu} s^{\mu-1} ds, t \in [0, T], \tag{34}$$

where

$$f(t) = \frac{1}{\mu} + \frac{1}{\mu^2(\mu^2+1)}t + \frac{1}{\mu}t^3 + \frac{1}{\mu^3(\mu^3+1)^2}t^5 + \frac{1}{\mu^2(\mu^2+1)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2}t^9.$$

i. **Using ADM**

Applying relations (9) gives the recurrence relations

$$\varphi_0(t) = \frac{1}{\mu} + \frac{1}{\mu^2(\mu^2+1)}t + \frac{1}{\mu}t^3 + \frac{1}{\mu^3(\mu^3+1)^2}t^5 + \frac{1}{\mu^2(\mu^2+1)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2}t^9$$

$$\varphi_1(t) = \frac{1}{\mu^2} + \frac{1}{\mu^2(\mu^2+1)(\mu+1)}t + \frac{1}{\mu(\mu+3)}t^3 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+5)}t^5 + \frac{1}{\mu^2(\mu^2+1)(\mu+7)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+9)}t^9$$

$$\varphi_2(t) = \frac{1}{\mu^3} + \frac{1}{\mu^2(\mu^2+1)(\mu+1)^2}t + \frac{1}{\mu(\mu+3)^2}t^3 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+5)^2}t^5 + \frac{1}{\mu^2(\mu^2+1)(\mu+7^2)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+9)^2}t^9$$

$$\varphi_2(t) = \frac{1}{\mu^3} + \frac{1}{\mu^2(\mu^2+1)(\mu+1)^3}t + \frac{1}{\mu(\mu+3)^3}t^3 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+5)^3}t^5 + \frac{1}{\mu^2(\mu^2+1)(\mu+7^3)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2(\mu+9)^3}t^9$$

...

and so on.

This leads to the series solution:

$$\begin{aligned}\varphi(t) = {} & (1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{1}{\mu^3} + \ldots)\frac{1}{\mu} + \\ & (1 + \frac{1}{\mu+1} + \frac{1}{(\mu+1)^2} + \frac{1}{(\mu+1)^3} + \ldots)\frac{1}{\mu^2(\mu^2+1)}t + \\ & (1 + \frac{1}{\mu+3} + \frac{1}{(\mu+3)^2} + \frac{1}{(\mu+3)^3} + \ldots)\frac{1}{\mu}t^3 + \\ & (1 + \frac{1}{\mu+5} + \frac{1}{(\mu+5)^2} + \frac{1}{(\mu+5)^3} + \ldots)\frac{1}{\mu^3(\mu^3+1)^2}t^5 + \\ & (1 + \frac{1}{\mu+7} + \frac{1}{(\mu+7)^2} + \frac{1}{(\mu+7)^3} + \ldots)\frac{1}{\mu^2(\mu^2+1)}t^7 + \\ & (1 + \frac{1}{\mu+9} + \frac{1}{(\mu+9)^2} + \frac{1}{(\mu+9)^3} + \ldots)\frac{1}{\mu^3(\mu^3+1)^2}t^9\end{aligned} \tag{35}$$

from which we can recognize the respective geometric series as the exact solution:

$$\begin{aligned}\varphi(t) = {} & \frac{1}{\mu-1} + \frac{\mu+1}{\mu^2(\mu^2+1)}t + \frac{\mu+3}{\mu}t^3 + \frac{\mu+5}{\mu^3(\mu^3+1)^2(\mu+4)}t^5 + \frac{\mu+7}{\mu^2(\mu^2+1)(\mu+6)}t^7 + \\ & \frac{\mu+9}{\mu^3(\mu^3+1)^2(\mu+8)}t^9\end{aligned} \tag{36}$$

As before, all of the geometric series converge. Thus the restriction is on the value of the parameter μ and there is no corresponding restriction on values of t in the specified domain. As stated earlier, although this equation is linear, an infinite set of solutions may exist for such equations. It was shown in Wazwaz & Rach (2016) that other solutions for this equation exist.

ii. **Using SSM**

Substituting the Maclaurin series (12) in (34) yields

$$\sum_{n=0}^{\infty} a_n t^n = f(t) + \int_0^t \sum_{n=0}^{\infty} a_n t^{-\mu} s^{n+\mu-1} ds, \tag{37}$$

where

$$f(t) = \frac{1}{\mu} + \frac{1}{\mu^2(\mu^2+1)}t + \frac{1}{\mu}t^3 + \frac{1}{\mu^3(\mu^3+1)^2}t^5 + \frac{1}{\mu^2(\mu^2+1)}t^7 + \frac{1}{\mu^3(\mu^3+1)^2}t^9.$$

Evaluating the integral in the right side of (37), and using (16), gives the recurrence relations

$$\begin{aligned} a_0 &= \frac{1}{\mu-1}, a_1 = \frac{\mu+1}{\mu^2(\mu^2+1)}, \\ a_3 &= \frac{\mu+3}{\mu(\mu+2)}, a_5 = \frac{\mu+5}{\mu^3(\mu^3+1)^2(\mu+4)}, \\ a_7 &= \frac{\mu+7}{\mu^2(\mu^2+1)(\mu+6)}, a_9 = \frac{\mu+9}{\mu^3(\mu^3+1)^2(\mu+8)}, \\ a_{2n+1} &= 0, n \geq 5, a_{2n} = 0, n \geq 1. \end{aligned} \tag{38}$$

Substituting the components a_n from (38) into (13) to obtain

$$\begin{aligned} \varphi(t) = &\frac{1}{\mu-1} + \frac{\mu+1}{\mu^2(\mu^2+1)}t + \frac{\mu+3}{\mu}t^3 + \frac{\mu+5}{\mu^3(\mu^3+1)^2(\mu+4)}t^5 + \\ &\frac{\mu+7}{\mu^2(\mu^2+1)(\mu+6)}t^7 + \frac{\mu+9}{\mu^3(\mu^3+1)^2(\mu+8)}t^9 \end{aligned} \tag{39}$$

which is the exact solution.
As introduced earlier, the convergence concept can be followed as presented earlier in the other problems.

Problem 4: Consider the second kind Volterra integral equation with a weakly singular kernel (Wazwaz & Rach, 2016)

$$\varphi(t) = 1 + \frac{(\mu+1)^2}{\mu(\mu+2)} + \int_0^t t^{-\mu} s^{\mu-1} \varphi(s) \quad t \in [0, T]. \tag{40}$$

i. **Using ADM**

Applying relations (9) gives the recurrence relations

$$\varphi_0(t) = 1 + \frac{(\mu+1)^2}{\mu(\mu+2)}t^2$$

$$\varphi_1(t) = \frac{1}{\mu} + \frac{(\mu+1)^2}{\mu(\mu+2)^2}t^2$$

$$\varphi_2(t) = \frac{1}{\mu^2} + \frac{(\mu+1)^2}{\mu(\mu+2)^3}t^2$$

$$\varphi_3(t) = \frac{1}{\mu^3} + \frac{(\mu+1)^2}{\mu(\mu+2)^4} t^2$$

...
and so on. And this leads to the series solution

$$\varphi(t) = (1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{1}{\mu^3} + \ldots) + \\ (1 + \frac{1}{\mu+2} + \frac{1}{(\mu+2)^2} + \frac{1}{(\mu+2)^3} + \ldots) \frac{(\mu+1)^2}{\mu(\mu+2)} t^2 \tag{41}$$

from which we can recognize the respective geometric series as the exact solution

$$\varphi(t) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t^2. \tag{42}$$

As before, all of the geometric series converge. Thus the restriction is on the value of the parameter μ and there is no corresponding restriction on values of t in the specified domain. As stated earlier, although this equation is linear, an infinite set of solutions may exist for such equations. It was shown in Wazwaz & Rach (2016) that other solutions for this equation exist.

ii. **Using SSM**

Substituting the Maclaurin series (12) in (40) yields

$$\sum_{n=0}^{\infty} a_n t^n = \frac{(\mu+1)^2}{\mu(\mu+2)} t^2 + \int_0^t \sum_{n=0}^{\infty} a_n t^{-\mu} s^{n+\mu-1} ds. \tag{43}$$

Evaluating the integral in the right side of (43),and using (16), gives

$$a_0 = \frac{\mu}{\mu-1}, a_1 = \frac{\mu+1}{\mu}, a_n = 0, n \geq 3. \tag{44}$$

Substituting the components a_n from (44) in (13) to obtain

$$\varphi(t) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t^2 \tag{45}$$

which is the exact solution.

As introduced earlier, the convergence concept can be followed as presented earlier in the other problems.

5 Conclusion

In this paper, we deal the Volterra integral equation with a weakly singular kernel. Many numerical methods such as method of non-polynomial spline function, the split interval method and the collocation method have been used formally in order to investigate such equation. We used in this paper two analytic methods, called the Adomian decomposition method which had employed formally in Wazwaz & Rach (2016) and the series solution method. Both methods show to be effective and powerful, having its own unique characteristics. We demonstrated that, a convergent power series can be used to achieve the exact solutions in ADM through applying variety of problems; whereas in SSM, the Maclaurin series has its own significant in solving these problems and Speeding up the process of getting to the exact solution. We tested four numerical problems utilizing these two ways for examining each model to confirm the power of each strategy.

References

Chen, Z., Jiang, W. (2011). The exact solution of a class of Volterra integral equation with weakly singular kernel. *Applied Mathematics and Computation, 217*(18), 7515-7519.

Diogo, T., Ford, N.J., Lima, P., & Valtchev, S. (2006). Numerical methods for a Volterra integral equation with non-smooth solutions. *Journal of Computational and Applied Mathematics, 189*(1-2), 412-423

Diogo, T., Franco, N.B., & Lima, P. (2004). High order product integration methods for a Volterra integral equation with logarithmic singular kernel. *Communications on Pure & Applied Analysis, 3*(2), 217.

Diogo, T., Ford, N.J., Lima, P.M., & Thomas, S.M. (2007). Solution of a singular integral equation by a split-interval method. *Int. Numer. Anal. Modeling*, 4, 63-73.

Lima, P., Diogo, T. (2002). Numerical solution of a non-uniquely solvable Volterra integral equation using extrapolation methods. *Journal of Computational and Applied Mathematics, 140*(1-2), 537-557.

Mokhtary, P., Moghaddam, B.P., Lopes, A.M., & Machado, J.T. (2020). A computational approach for the non-smooth solution of non-linear weakly singular Volterra integral equation with proportional delay. *Numerical Algorithms, 83*(3), 987-1006.

Tang, B.Q., Li, X.F. (2008). Solution of a class of Volterra integral equations with singular and weakly singular kernels. *Applied Mathematics and Computation, 199*(2), 406-413.

Wazwaz, A.M., Rach, R. (2016). Two reliable methods for solving the Volterra integral equation with a weakly singular kernel. *Journal of Computational and Applied Mathematics*, 302, 71-80.

Wazwaz, A.M. (2011). *Linear and Nonlinear Integral Equations* (Vol. 639, pp. 35-36). Berlin: Springer.

Zhu, L., Wang, Y. (2015). Numerical solutions of Volterra integral equation with weakly singular kernel using SCW method. *Applied Mathematics and Computation*, 260, 63-70.