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TWO-DERIVATIVE RUNGE-KUTTA TYPE METHOD WITH FSAL PROPERTY

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Abstract. We derive two-derivative Runge-Kutta type method of order four which has FSAL (First Same As Last) property for solving special third-order ordinary differential equations. Numerical examples are given to demonstrate the superiority of the the proposed method compared with the methods selected from literature on standard test problems.

Keywords: Special third-order ordinary differential equations, FSAL property, two-derivative Runge-Kutta type method.

AMS Subject Classification: 65L05, 65L06, 65L20.

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1 Introduction

In this paper, we focus on the initial value problem of special third-order ordinary differential equations in the following form

$$y'''(x) = f(x, y), \ \ y(x_0) = y_0, \ \ y'(x_0) = y_0', \ \ y''(x_0) = y_0'', \tag{1}$$

where $y \in \mathbb{R}^q$, $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ and the first and second derivatives are not appear in Equation (1). This type problems can be occur in physical problems such as thin film flow and electromagnetic waves. The Equation (1) can be solved by converting into a system of first order ordinary differential equations. But this way causes to increase the computational time. Many researchers have studied for solving (1) directly (Awoyemi & Idowu, 2005; Waeleh et al., 2011; You & Chen, 2013; Mechee et al., 2013). This gains to efficiency of the method.

Recently, to increase the accuracy of the Runge-Kutta type methods derived to directly solve special third-order ordinary differential equations (1) the fourth-order derivative of the solution has been used in the formulation of the method. These methods are called as two-derivative Runge-Kutta type (TDRKT) methods. Also, there are methods based on FSAL properties which directly solve higher order ordinary differential equations. Embedded explicit Runge-Kutta type methods for directly solving special third-order ordinary differential equations have been given by Senu et al. (Senu et al., 2014). Extended RKN methods with FSAL property for numerical integration of perturbed oscillators have been presented in (Fang et al., 2010). Exponentially fitted Runge–Kutta–Nyström methods for the numerical integration of second-order IVPs with oscillatory solutions have been derived in (Franco, 2004). In this study, we present a fourth-order explicit two-derivative Runge-Kutta type method with FSAL property for directly solving special third-order ordinary differential equations (1). The organization of the remainder of this paper is as follows. In Section 2, we give the preliminaries of TDRKT methods. In Section 3, we derive two-derivative Runge-Kutta type methods of order four in three stages has the property of FSAL. In Section 4, the stability of the proposed method is analyzed. In Section 5, numerical examples are given to demonstrate the superiority of the proposed method compared with the other Runge-Kutta methods. Lastly, conclusion is presented in Section 6.

2 The Explicit TDRKT Methods

We consider the s-stage explicit TDRKT methods derived by Lee et al. (Lee et al., 2020) in the following form

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 f(x_n, y_n) + h^4 \sum_{i=1}^{s} b''_i g(x_n + c_i h, Y_i, Y'_i), \\ y'_{n+1} &= y'_n + hy''_n + \frac{1}{2}h^2 f(x_n, y_n) + h^3 \sum_{i=1}^{s} b'_i g(x_n + c_i h, Y_i, Y'_i) \\ y''_{n+1} &= y''_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} b_i g(x_n + c_i h, Y_i, Y'_i), \end{aligned} \tag{2}$$

where $y^{iv}(x) = g(x, y, y') = f_x(x, y) + f_y(x, y)y'$ and

$$\begin{aligned} Y_1 &= y_n, \\ Y'_1 &= y'_n, \\ Y_i &= y_n + hc_i y'_n + \frac{1}{2}h^2 c_i^2 y''_n + \frac{1}{6}h^3 c_i^3 f(x_n, y_n) + h^4 \sum_{j=1}^{i-1} a_{ij} g(x_n + c_j h, Y_j, Y'_j), \ \ i = 2, \dots, s \\ Y'_i &= y'_n + hc_i y''_n + \frac{1}{2}h^2 c_i^2 f(x_n, y_n) + h^3 \sum_{j=1}^{i-1} \hat{a}_{ij} g(x_n + c_j h, Y_j, Y'_j), \ \ i = 2, \dots, s. \end{aligned} \tag{3}$$

The explicit TDRKT methods (2)-(3) include one evaluation of third derivative (f) and many evaluations of fourth derivative (g). The TDRKT methods can be expressed by Butcher tableau as follows

$$\begin{array}{c|c|c|c} c & A & \hat{A} & \\ \hline & b''^T & b'^T & b^T \end{array}$$

As given in (Lee et al., 2020), the order conditions for TDRKT methods up to order five are listed as in the following:
Order 2:

$$\sum_{i=1}^{s} b_i = \frac{1}{2}, \tag{4}$$

Order 3:

$$\sum_{i=2}^{s} b_i c_i = \frac{1}{6}, \qquad \sum_{i=1}^{s} b'_i = \frac{1}{6}, \tag{5}$$

Order 4:

$$\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{12}, \qquad \sum_{i=2}^{s} b'_i c_i = \frac{1}{24}, \qquad \sum_{i=1}^{s} b''_i = \frac{1}{24}, \tag{6}$$

Order 5:

$$\sum_{i=2}^{s} b_i c_i^3 = \frac{1}{20}, \qquad \sum_{i=2}^{s}\sum_{j=1}^{i-1} b_i \hat{a}_{ij} = \frac{1}{120}, \qquad \sum_{i=2}^{s} b_i' c_i^2 = \frac{1}{60}, \qquad \sum_{i=1}^{s} b_i'' c_i = \frac{1}{120}. \tag{7}$$

It is utilised determining coefficients of TDRKT methods the following assumption

$$\sum_{j=1}^{i-1} \hat{a}_{ij} = \frac{c_i^3}{6}, \quad \sum_{j=1}^{i-1} a_{ij} = \frac{c_i^4}{24}, \quad i = 2, \dots, s. \tag{8}$$

3 A fourth order TDRKT method with FSAL property

In this section, we derive an explicit three-stage ($s = 3$) fourth order TDRKT method based on FSAL property. This property means that the last evaluation at any integration step is the same as the first evaluation at the next integration step. New TDRKT method has effectively two stages cost per step except for three stages cost at the first step. The FSAL property plays important role on the efficiency of the schemes. That is, the usage of this property has effect to reduce computation cost. To obtain fourth order TDRKT method with FSAL property, we solve Eqs. (4)-(6) which are equations of order conditions up to order 4 together with FSAL conditions

$$c_3 = 1, \quad b_i'' = a_{3i}, \quad b_i' = \hat{a}_{3i}, \quad i = 1, 2, \quad b_3'' = 0 \quad \text{and} \quad b_3' = 0.$$

With simplifying assumption(8), we have two free parameters. Minimizing the fifth order error equations, we can chosen the free parameters as $c_2 = \frac{2}{5}$ and $a_{32} = \frac{13}{625}$. The fourth order TDRKT method with FSAL property can be given with Butcher tableau as in the following:

$$\begin{array}{c|ccc|ccc|ccc}
0 & 0 & & & 0 & & & & & \\
\frac{2}{5} & \frac{2}{1875} & 0 & & \frac{4}{375} & 0 & & & & \\
1 & \frac{313}{15000} & \frac{13}{625} & 0 & \frac{1}{16} & \frac{5}{48} & 0 & & & \\
\hline
 & \frac{313}{15000} & \frac{13}{625} & 0 & \frac{1}{16} & \frac{5}{48} & 0 & \frac{1}{8} & \frac{25}{72} & \frac{1}{36}
\end{array}$$

We denominate as TDRKT4FSAL the fourth order TDRKT method with FSAL property.

4 Numerical Results

In this section, we present numerical results obtained from studies carried out on some problems to show efficiency and accuracy of the new method. The new method with known methods emerged in the scientific literature are compared. It is used the maximum absolute error versus the number of function evaluations required by each method in logarithmic scale as criterion for comparisons. The methods used in comparisons are given in the following:

TDRKT4FSAL: a three-stage fourth order TDRKT method with FSAL property derived in Section 3.

TDRKT4: a two-stage fourth order TDRKT method derived in (Lee et al., 2020).

RK4: The classical four-stage fourth order Runge-Kutta method given in (Butcher, 2008).

Problem 4.1 We consider the nonhomogeneous linear initial value problem given in (You & Chen, 2013)

$$y''' = y + \cos(x), \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$$

The exact solution of the problem is $y(x) = \frac{1}{2}e^x - \frac{1}{2}(\cos(x) + \sin(x))$. The problem is integrated in the interval [0,10] with the step sizes $h = 1/2^i, i = 3, \dots, 7$.

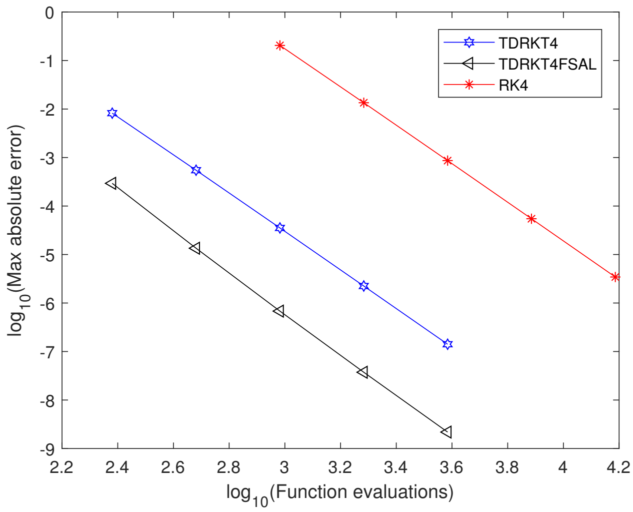

Figure 1: Efficiency curves for Problem 4.1

Figure 1 presents the efficiency curves.

Problem 4.2 We consider the nonlinear initial value problem given in (Lee et al., 2020)

$$y''' = -6y^4, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 2.$$

The exact solution of the problem is $y(x) = \frac{1}{1+x}$. The problem is integrated in the interval [0,5] with the step sizes $h = 1/2^i, i = 4, \ldots, 8$. Figure 2 presents the efficiency curves.

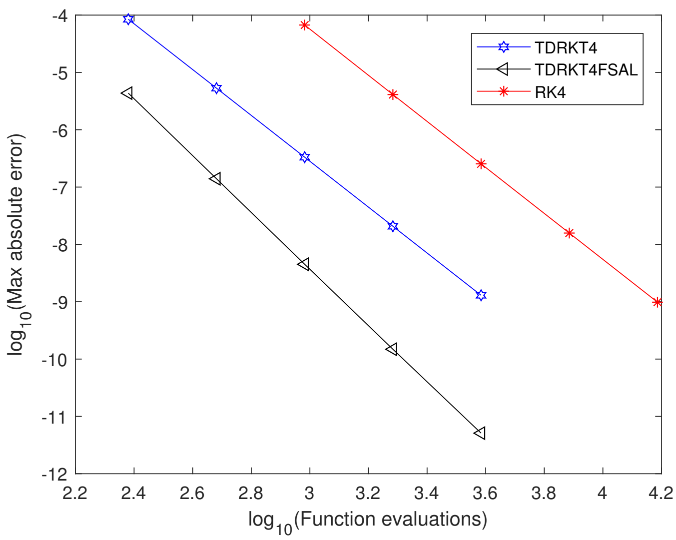

Figure 2: Efficiency curves for Problem 4.2

Problem 4.3 We consider the linear system given in (You & Chen, 2013)

$$\begin{aligned}
y_1''' &= \frac{1}{68}(817y_1 + 1393y_2 + 448y_3), \\
y_2''' &= -\frac{1}{68}(1141y_1 + 2837y_2 + 896y_3), \\
y_3''' &= \frac{1}{136}(3059y_1 + 4319y_2 + 1592y_3),
\end{aligned}$$

with the initial conditions

$$\begin{aligned} y_1(0) &= 2, \quad y_2(0) = -2, \quad y_3(0) = -12, \\ y_1'(0) &= -12, \quad y_2'(0) = 28, \quad y_3'(0) = -33, \\ y_1''(0) &= 20, \quad y_2''(0) = -52, \quad y_3''(0) = 5. \end{aligned}$$

The exact solution of the problem is

$$\begin{aligned} y_1(x) &= e^x - 2e^{2x} + 3e^{-3x}, \\ y_2(x) &= 3e^x + 2e^{2x} - 7e^{-3x}, \\ y_3(x) &= -11e^x - 5e^{2x} + 4e^{-3x}. \end{aligned}$$

The problem is integrated in the interval [0,2] with the step sizes $h = 1/2^i, i = 4, \ldots, 8$. Figure 3 presents the efficiency curves.

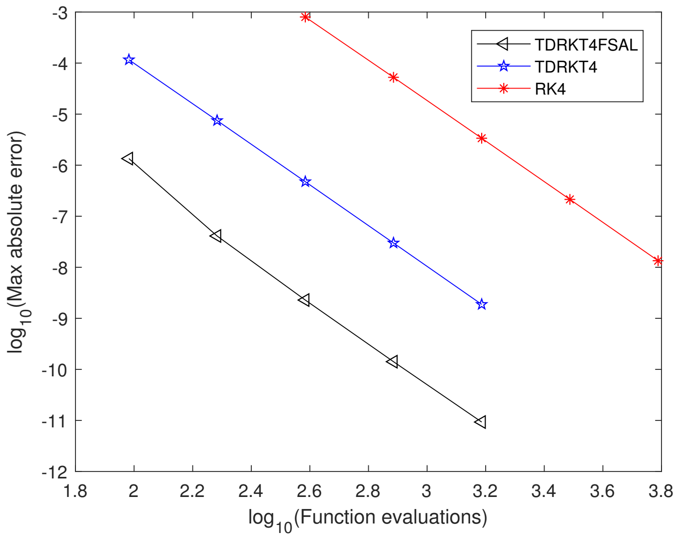

Figure 3: Efficiency curves for Problem 4.3

5 Conclusion

In this paper, we derived a three-stage fourth order TDRKT method with FSAL property (TDRKT4FSAL) for directly solving special third order ordinary differential equations. The new method has been compared with the classical Runge-Kutta and two-derivative Runge-Kutta type methods of the same order in terms of the number of function evaluations and the error. The numerical results are presented in Figures 1-3. Figures 1-3 demonstrated that the new method more efficient compared to the classical Runge-Kutta and two-derivative Runge-Kutta type methods.

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