Advanced Mathematical Models & Applications
Vol.7, No.3, 2022, pp.275-291

# GALERKIN METHOD WITH B-SPLINES FOR COUPLED SYSTEM OF NONLINEAR BOUNDARY VALUE PROBLEMS*

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**Abstract.** The work's main contribution is the innovation of two proposed methods for resolving a system of nonlinear boundary value problems (BVPs) to enhance the method's computational efficiency. The work's novelty is the comparative study between the Galerkin method (GM) with B-splines-based quartic (Q4BS) and the GM with B-splines-based quintic (Q5BS) for solving a coupled system of fourth-order nonlinear BVPs. Its numerical results demonstrate the advantage and efficiency of the technique, and they accept well with the published results previously. To determine which method is better, we also compare the residual error. The accuracy and quick convergence of the numerical technique in this research make it enjoyable to study.

**Keywords**: Galerkin method, quasilinearization technique, quartic B-splines, quintic B-splines, residual error.
**AMS Subject Classification:** 65L60, 65D07.
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*Received: 14 June 2022; Revised: 12 August 2022; Accepted: 23 September 2022;*
*Published: 15 December 2022.*

## 1 Introduction

Several problems in engineering and science, including problems in heat transfer and mass transfer, are mathematically modeled by a coupled system of nonlinear BVPs. Numerous techniques have been developed by researchers for BVPs. Particularly in fluid dynamics, the study of specific effects on various fluids, such as nanofluid, MHD, micropolar, etc., over the geometry of infinite dimension results in the modeling of the coupled system of nonlinear BVPs. Sheikholeslami et al. (2014), Reddy & Ali (2016), Kasi & Murali (2009) revealed the finite element method (FEM) based on the GM with B-splines-based cubic method and so on for a coupled system of BVPs. Dhivya & Murali (2018) have presented a collocation approach with B-splines for a system of nonlinear BVPs with four unknowns and four equations. We described a GM procedure with cubic and quartic B-splines for the solution of highly coupled system of nonlinear BVPs in Murali & Dhivya (2019). Kasi (2019) solved the system of BVPs along with different possibly boundary conditions. FEM based technique is to solved the eigth order BVPs in Murali (2016). Sai et al. (2020) have obtained the solutions of the fluid mechanics problem using wavelet Galerkin approach. Siva & Anwar (2019) evaluated the governing BVPs with the FEM based on weighted residual technique. In Mechee & Mussa (2020), the RK-type direct explicit integrators for resolving eighth order ordinary differential equations (ODEs) were presented. In ReddyS (2016), Reddy (2016), authors found the FEM for solving different order linear BVPs

***How to cite (APA)**: Dhivya, C., Krishna, P.M. (2022). Galerkin method with B-splines for coupled system of nonlinear boundary value problems. *Advanced Mathematical Models & Applications*, *7*(3), 275-291.

and nonlinear BVPs. Lodhi et al. (2022) have developed the quintic B-spline collocation method (QBSCM) for solving the non-linear Bratu-type BVPs. Khalid et al. (2019) have discussed the Cubic B-splines to solve eighth-order linear BVPs and non-linear BVPs.
The current work aims to study the coupled system of nonlinear BVPs related to all the above applications. We consider the coupled system of fourth-order nonlinear BVPs, given by:

$$NLO_i(\mathfrak{T},\mathfrak{A},\mathfrak{A}',\mathfrak{A}'',\mathfrak{A}''',\mathfrak{A}^{iv},\mathfrak{B},\mathfrak{B}',\mathfrak{B}'',\mathfrak{C},\mathfrak{C}',\mathfrak{C}'',\mathfrak{D},\mathfrak{D}',\mathfrak{D}'')=0 \qquad (1)$$

with corresponding boundary conditions (bc's)

$$\mathfrak{A}(\mathfrak{T}_{lb})=\mathfrak{A}_0,\ \mathfrak{A}(\mathfrak{T}_{rb})=\mathfrak{A}_1,\ \mathfrak{A}'(\mathfrak{T}_{lb})=\mathfrak{A}_2,\ \mathfrak{A}'(\mathfrak{T}_{rb})=\mathfrak{A}_3,\ \mathfrak{B}(\mathfrak{T}_{lb})=\mathfrak{B}_0,\ \mathfrak{B}(\mathfrak{T}_{rb})=\mathfrak{B}_1,$$
$$\mathfrak{C}(\mathfrak{T}_{lb})=\mathfrak{C}_0,\ \mathfrak{C}(\mathfrak{T}_{rb})=\mathfrak{C}_1,\ \mathfrak{D}(\mathfrak{T}_{lb})=\mathfrak{D}_0,\ \mathfrak{D}(\mathfrak{T}_{rb})=\mathfrak{D}_1. \qquad (2)$$

Where $NLO$ is denoted as the nonlinear operator and $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ and $\mathfrak{D}$ are the unknown variables. We used the quasilinearization technique Bellman & Kalaba (1965) to linearize the considered nonlinear BVPs as given below

$$\sum_{\mathscr{k}_*=1}^{5} \mathscr{u}_{i\mathscr{k}_*}(\mathfrak{T})\mathfrak{A}^{(5-\mathscr{k}_*)} + \sum_{\mathscr{k}_*=3}^{5} \mathscr{v}_{i\mathscr{k}_*}(\mathfrak{T})\mathfrak{B}^{(5-\mathscr{k}_*)} + \sum_{\mathscr{k}_*=3}^{5} \mathscr{w}_{i\mathscr{k}_*}(\mathfrak{T})\mathfrak{C}^{(5-\mathscr{k}_*)}$$
$$+\sum_{\mathscr{k}_*=3}^{5} \mathscr{y}_{i\mathscr{k}_*}(\mathfrak{T})\mathfrak{D}^{(5-\mathscr{k}_*)} = \mathfrak{F}_i(\mathfrak{T}), \quad i=1,2,3,4 \qquad (3)$$

together with the bc's

$$\mathfrak{A}(\mathfrak{T}_{lb})=\mathfrak{A}_0,\ \mathfrak{A}(\mathfrak{T}_{rb})=\mathfrak{A}_1,\ \mathfrak{A}'(\mathfrak{T}_{lb})=\mathfrak{A}_2,\ \mathfrak{A}'(\mathfrak{T}_{rb})=\mathfrak{A}_3,\ \mathfrak{B}(\mathfrak{T}_{lb})=\mathfrak{B}_0,\ \mathfrak{B}(\mathfrak{T}_{rb})=\mathfrak{B}_1,$$
$$\mathfrak{C}(\mathfrak{T}_{lb})=\mathfrak{C}_0,\ \mathfrak{C}(\mathfrak{T}_{rb})=\mathfrak{C}_1,\ \mathfrak{D}(\mathfrak{T}_{lb})=\mathfrak{D}_0,\ \mathfrak{D}(\mathfrak{T}_{rb})=\mathfrak{D}_1. \qquad (4)$$

Where $\mathfrak{A}_i$ $(i=0,..,3)$ is a real constant and $\mathfrak{B}_i,\mathfrak{C}_i,\mathfrak{D}_i$ $(i=0,1)$ is a real constant. Also $\mathfrak{T}_{lb}$ denotes the boundary point to the left side and $\mathfrak{T}_{rb}$ denotes the boundary point to the right side. Here $\mathscr{u}_{i\mathscr{k}_*},\mathscr{v}_{i\mathscr{k}_*},\mathscr{w}_{i\mathscr{k}_*},\mathscr{y}_{i\mathscr{k}_*}$ are continuous functions on the interval $[\mathfrak{T}_{lb},\mathfrak{T}_{rb}]$. On the interval $[\mathfrak{T}_{lb},\mathfrak{T}_{rb}]$, $\mathfrak{F}_i$ is assumed to be continuous.

The structure of this work is described below. Section 2 presents the justification for implementing GM. In Section explains the definitions of B-splines, accompanied by Galerkin approach. Section 4 tests the proposed method on a coupled system of a fourth-order nonlinear BVPs. Furthermore, finding the residual graphs. Section 5 described the findings and discussions. Ultimately, Section 6 summarizes the conclusions.

## 2 Justification for implementing the Galerkin method

The FEM has become a potent and valuable tool for solving the BVPs in complex dynamic systems in last decades. In Reddy (2010), the approximate solution in FEM may be written as the basis function linear combination which provide basis for the specified approximation space.

In GM, the approximation residual is formed orthogonal to the basis functions. When GM is used, weak form of such an approximate solution toward a specified differential equations exists and thus is unique Bers et al. (1964), Lions & Magenes (1972) under suitable conditions, independent of such properties of such a specified differential operator. Furthermore, considering the boundary conditions (bc's), a weak solution will become a classical solution of a specified differential equation in Mitchel (1977), Reddy (2010). That is, at the boundary where Dirichlet bc's are specified, and the basis functions should vanish. Therefore, inside this paper, Galerkin methodology with the basis functions of B-splines is used to approximate a solution of a coupled system of fourth-order BVPs.

## 3 Methodology explanation

The Q4BS are described in Cox (1972), Carl (1978), and Prenter (1975). Introduce the additional knots eight $\mathscr{G}_{4_*,-4},...,\mathscr{G}_{4_*,-1},\mathscr{G}_{4_*,\varpi+1},...,\mathscr{G}_{4_*,\varpi+4}$ so that $\mathfrak{T}_{-4} < ... < \mathfrak{T}_0$ and $\mathfrak{T}_\varpi < ... < \mathfrak{T}_{\varpi+4}$. The quartic B-splines $\mathscr{G}_{4_*,i}(\mathfrak{T})$, Murali & Dhivya (2019) now defined as follows

$$\mathscr{G}_{4_*,i}(\mathfrak{T}) = \begin{cases} \sum_{r=i-2}^{r=i+3} \left[\frac{(\mathfrak{T}_r-\mathfrak{T})_+^3}{\Pi'(\mathfrak{T}_r)}\right] & \text{if } \mathfrak{T} \in [\mathfrak{T}_{i-2}, \mathfrak{T}_{i+3}] \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

where $\Pi(\mathfrak{T}) = (\mathfrak{T} - \mathfrak{T}_{i-2})...(\mathfrak{T} - \mathfrak{T}_{i+3})$ and $(\mathfrak{T}_r - \mathfrak{T})_+^3$ is the positive part function.

And $\mathscr{G}_{4_*,-2}(\mathfrak{T}),...,\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T})$ forms the basis for quartic polynomial spline space $\mathscr{S}_4(\Pi)$. According to Schoenberg (1966) we must add eight extra knots from the outside interval at $\mathfrak{T}_{-4} < ... < \mathfrak{T}_0 < ... < \mathfrak{T}_\varpi < ... < \mathfrak{T}_{\varpi+4}$ in quartic B-splines.
Similarly, quintic B-splines are defined by $\mathscr{G}_{5_*,i}(\mathfrak{T})$,

$$\mathscr{G}_{5_*,i}(\mathfrak{T}) = \begin{cases} \sum_{r=i-3}^{r=i+3} \left[\frac{(\mathfrak{T}_r-\mathfrak{T})_+^5}{\Pi'(\mathfrak{T}_r)}\right] & \mathfrak{T} \in [\mathfrak{T}_{i-3}, \mathfrak{T}_{i+3}] \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

where $\Pi(\mathfrak{T}) = (\mathfrak{T} - \mathfrak{T}_{i-3})...(\mathfrak{T} - \mathfrak{T}_{i+3})$ , $(\mathfrak{T}_r - \mathfrak{T})_+^5$ is the positive part function.

And $\mathscr{G}_{5_*,-2}(\mathfrak{T}),...,\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T})$ forms the space's basis $\mathscr{S}_5(\Pi)$ of quintic polynomial splines and also the addition of two knots are $\mathfrak{T}_{-5}, \mathfrak{T}_{\varpi+5}$. Schoenberg (1966) demonstrated that quintic B-splines have the smallest and most compact support and knots in $\mathfrak{T}_{-5} < ... < \mathfrak{T}_0 < ... < \mathfrak{T}_\varpi < ... < \mathfrak{T}_{\varpi+5}$.

We used GM with the B-splines is to resolve a system of fourth-order BVPs with two possible cases. The first case is the GM, which uses Q4BS to approximate a solution of a system of fourth-order BVPs. In the second case, the GM with Q5BS is used to approximate a solution of a system of fourth-order BVPs. Furthermore, we compared the results of the two cases.

In the first case, we define the approximation for "$\mathfrak{A}(\mathfrak{T}), \mathfrak{B}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T}), \mathfrak{D}(\mathfrak{T})$", by the GM with Q4BS as basis functions to resolve the system of BVPs of this type (1) - (2).

$$\mathfrak{A}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+1} \mathfrak{A}_{k_*}\mathscr{G}_{4_*,k_*}(\mathfrak{T}) \tag{7}$$

$$\mathfrak{B}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+1} \mathfrak{B}_{k_*}\mathscr{G}_{4_*,k_*}(\mathfrak{T}) \tag{8}$$

$$\mathfrak{C}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+1} \mathfrak{C}_{k_*}\mathscr{G}_{4_*,k_*}(\mathfrak{T}) \tag{9}$$

$$\mathfrak{D}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+1} \mathfrak{D}_{k_*}\mathscr{G}_{4_*,k_*}(\mathfrak{T}) \tag{10}$$

At the boundary where Dirichlet bc's are defined, the basis functions in the GM must vanish. As a consequence, Kasi (2019) the basis functions must be redefined to form a recent set of the basis functions that vanish at boundary, where Dirichlet bc's are given. Since quartic B-splines polynomial were used to approximate the fourth-order BVPs, we redefine the basis functions into the recently defined set of basis functions that vanish at the boundary in which Dirichlet bc's are given.

We obtain approximate solution at boundary points by quartic B-splines definition and Dirichlet bc's of (2). Now eliminate "$\mathfrak{A}_{-2}, \mathfrak{B}_{-2}, \mathfrak{C}_{-2}, \mathfrak{D}_{-2}$" and "$\mathfrak{A}_{\varpi+1}, \mathfrak{B}_{\varpi+1}, \mathfrak{C}_{\varpi+1}, \mathfrak{D}_{\varpi+1}$", from equations (7) - (10). We get the below recent approximations for our unknown variables as follows.

$$\mathfrak{A}(\mathfrak{T}) = \mathfrak{Aw}(\mathfrak{T}) + \sum_{\mathscr{k}_*=-1}^{\varpi} \mathfrak{A}_{\mathscr{k}_*}\mathfrak{Q}_{4_*,\mathscr{k}_*}(\mathfrak{T}), \tag{11}$$

$$\mathfrak{B}(\mathfrak{T}) = \mathfrak{Bw}(\mathfrak{T}) + \sum_{\mathscr{k}_*=-1}^{\varpi} \mathfrak{B}_{\mathscr{k}_*}\mathfrak{Q}_{4_*,\mathscr{k}_*}(\mathfrak{T}), \tag{12}$$

$$\mathfrak{C}(\mathfrak{T}) = \mathfrak{Cw}(\mathfrak{T}) + \sum_{\mathscr{k}_*=-1}^{\varpi} \mathfrak{C}_{\mathscr{k}_*}\mathfrak{Q}_{4_*,\mathscr{k}_*}(\mathfrak{T}), \tag{13}$$

$$\text{and } \mathfrak{D}(\mathfrak{T}) = \mathfrak{Dw}(\mathfrak{T}) + \sum_{\mathscr{k}_*=-1}^{\varpi} \mathfrak{D}_{\mathscr{k}_*}\mathfrak{Q}_{4_*,\mathscr{k}_*}(\mathfrak{T}) \tag{14}$$

where

$$\mathfrak{Aw}(\mathfrak{T}) = \frac{\mathfrak{A}_0}{\mathscr{G}_{4_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{4_*,-2}(\mathfrak{T}) + \frac{\mathfrak{A}_1}{\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}_{rb})}\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}) \tag{15}$$

$$\mathfrak{Bw}(\mathfrak{T}) = \frac{\mathfrak{B}_0}{\mathscr{G}_{4_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{4_*,-2}(\mathfrak{T}) + \frac{\mathfrak{B}_1}{\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}_{rb})}\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}) \tag{16}$$

$$\mathfrak{Cw}(\mathfrak{T}) = \frac{\mathfrak{C}_0}{\mathscr{G}_{4_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{4_*,-2}(\mathfrak{T}) + \frac{\mathfrak{C}_1}{\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}_{rb})}\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}) \tag{17}$$

$$\mathfrak{Dw}(\mathfrak{T}) = \frac{\mathfrak{D}_0}{\mathscr{G}_{4_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{4_*,-2}(\mathfrak{T}) + \frac{\mathfrak{D}_1}{\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}_{rb})}\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}) \tag{18}$$

and

$$\mathfrak{Q}_{4_*,\mathscr{k}_*}(\mathfrak{T}) = \begin{cases} \mathscr{G}_{4_*,\mathscr{k}_*}(\mathfrak{T}) - \left[\frac{\mathscr{G}_{4_*,\mathscr{k}_*}(\mathfrak{T}_{lb})}{\mathscr{G}_{4_*,-2}(\mathfrak{T}_{lb})}\right]\mathscr{G}_{4_*,-2}(\mathfrak{T}), & \text{for } \mathscr{k}_* = -1, 0, 1; \\ \mathscr{G}_{4_*,\mathscr{k}_*}(\mathfrak{T}), & \text{for } \mathscr{k}_* = 2, ..., \varpi - 3; \\ \mathscr{G}_{4_*,\mathscr{k}_*}(\mathfrak{T}) - \left[\frac{\mathscr{G}_{4_*,\mathscr{k}_*}(\mathfrak{T}_{rb})}{\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}_{rb})}\right]\mathscr{G}_{4_*,\varpi+1}(\mathfrak{T}), & \text{for } \mathscr{k}_* = \varpi - 2, \varpi - 1, \varpi. \end{cases} \tag{19}$$

We now have $n+2$ basis functions in $\mathfrak{A}, \mathfrak{B}, \mathfrak{C},$ and $\mathfrak{D}$. By employing the GM to the system of equations (3) with the new basis functions, and we obtain the following system of algebraic equations with unknown parameters $\mathfrak{A}_{\mathscr{k}_*}, \mathfrak{B}_{\mathscr{k}_*}, \mathfrak{C}_{\mathscr{k}_*}$ and $\mathfrak{D}_{\mathscr{k}_*}$.

$$\begin{aligned} \mathscr{H}_{11}\mathfrak{A} + \mathscr{H}_{12}\mathfrak{B} + \mathscr{H}_{13}\mathfrak{C} + \mathscr{H}_{14}\mathfrak{D} &= \mathscr{K}_1 \\ \mathscr{H}_{21}\mathfrak{A} + \mathscr{H}_{22}\mathfrak{B} + \mathscr{H}_{23}\mathfrak{C} + \mathscr{H}_{24}\mathfrak{D} &= \mathscr{K}_2 \\ \mathscr{H}_{31}\mathfrak{A} + \mathscr{H}_{32}\mathfrak{B} + \mathscr{H}_{33}\mathfrak{C} + \mathscr{H}_{34}\mathfrak{D} &= \mathscr{K}_3 \\ \mathscr{H}_{41}\mathfrak{A} + \mathscr{H}_{42}\mathfrak{B} + \mathscr{H}_{43}\mathfrak{C} + \mathscr{H}_{44}\mathfrak{D} &= \mathscr{K}_4 \end{aligned} \tag{20}$$

where "$\mathfrak{A} = [\mathfrak{A}_{-1} \ ... \ \mathfrak{A}_{\varpi}]^T$, $\mathfrak{B} = [\mathfrak{B}_{-1} \ ... \ \mathfrak{B}_{\varpi}]^T$, $\mathfrak{C} = [\mathfrak{C}_{-1} \ ... \ \mathfrak{C}_{\varpi}]^T$ and $\mathfrak{D} = [\mathfrak{D}_{-1} \ ... \ \mathfrak{D}_{\varpi}]^T$". Each entry of matrices $\mathscr{H}_{\mathfrak{d}_*\mathfrak{q}_*}$ is shown below. We have the following expression for the first row. And each matrix entry is given below for $m = 1$. We denote $\mathfrak{Q}'''_{4_*,\mathfrak{d}_*}(\mathfrak{T}) = \lambda^{(3)}$, $\mathfrak{Q}''_{4_*,\mathfrak{d}_*}(\mathfrak{T}) = \lambda^{(2)}$, $\mathfrak{Q}'_{4_*,\mathfrak{d}_*}(\mathfrak{T}) = \lambda^{(1)}$, $\mathfrak{Q}_{4_*,\mathfrak{d}_*}(\mathfrak{T}) = \lambda$, $\mathfrak{Q}'_{4_*,\mathfrak{q}_*}(\mathfrak{T}) = \mu^{(1)}$ and $\mathfrak{Q}_{4_*,\mathfrak{q}_*}(\mathfrak{T}) = \mu$ for details in Murali &

Dhivya (2019)

$$(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ - \Big[u_{m1}(\mathfrak{T})\lambda^{(3)}\Big] - 3\Big[{u_{m1}}'(\mathfrak{T})\lambda^{(2)}\Big] - 3\Big[{u_{m1}}''(\mathfrak{T})\lambda^{(1)}\Big] - \Big[{u_{m1}}'''(\mathfrak{T})\lambda\Big]$$
$$+ \Big[u_{m2}(\mathfrak{T})\lambda^{(2)}\Big] + 2\Big[{u_{m2}}'(\mathfrak{T})\lambda^{(1)}\Big] + \Big[{u_{m2}}''(\mathfrak{T})\lambda\Big] - \Big[u_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[{u_{m3}}'(\mathfrak{T})\lambda\Big]$$
$$+ [u_{m4}(\mathfrak{T})\lambda]\Bigg\}\mu^{(1)} + [u_{m5}(\mathfrak{T})\lambda]\Bigg\}\mu d\mathfrak{T} - \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}'_{4_*,\mathfrak{d}_*}(\mathfrak{T}_{rb})\Big]\mathfrak{Q}''_{4_*,\mathfrak{q}_*}(\mathfrak{T}_{rb})$$
$$+ \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}'_{4_*,\mathfrak{d}_*}(\mathfrak{T}_{lb})\Big]\mathfrak{Q}''_{4_*,\mathfrak{q}_*}(\mathfrak{T}_{lb}), \qquad \mathfrak{d}_* = -1,...,\varpi;\ \mathfrak{q}_* = -1,...,\varpi.$$

$$(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ - \Big[v_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[{v_{m3}}'(\mathfrak{T})\lambda\Big] + [v_{m4}(\mathfrak{T})\lambda]\Bigg\}$$
$$\mu^{(1)} + [v_{m5}(\mathfrak{T})\lambda]\,\mu\Bigg\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1,...,\varpi;\ \mathfrak{q}_* = -1,...,\varpi.$$

$$(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ - \Big[w_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[{w_{m3}}'(\mathfrak{T})\lambda\Big] + [w_{m4}(\mathfrak{T})\lambda]\Bigg\}$$
$$\mu^{(1)} + [w_{m5}(\mathfrak{T})\lambda]\,\mu\Bigg\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1,...,\varpi;\ \mathfrak{q}_* = -1,...,\varpi.$$

$$(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ - \Big[y_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[{y_{m3}}'(\mathfrak{T})\lambda\Big] + [y_{m4}(\mathfrak{T})\lambda]\Bigg\}$$
$$\mu^{(1)} + [y_{m5}(\mathfrak{T})\lambda]\,\mu\Bigg\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1,...,\varpi;\ \mathfrak{q}_* = -1,...,\varpi.$$

and

$$(k_m)_{\mathfrak{d}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{(k_{*m})(\mathfrak{T})\lambda + u_{m1}(\mathfrak{T})\lambda^{(3)} + 3{u_{m1}}'(\mathfrak{T})\lambda^{(2)} + 3{u_{m1}}''(\mathfrak{T})\lambda^{(1)} + {u_{m1}}'''(\mathfrak{T})\lambda$$
$$- u_{m2}(\mathfrak{T})\lambda^{(2)} - 2{u_{m2}}'(\mathfrak{T})\lambda^{(1)} - {u_{m2}}''(\mathfrak{T})\lambda + u_{m3}(\mathfrak{T})\lambda^{(1)} + {u_{m3}}'(\mathfrak{T})\lambda - u_{m4}(\mathfrak{T})\lambda\Bigg\}\mathfrak{Aw}'(\mathfrak{T})$$
$$- u_{m5}(\mathfrak{T})\lambda\mathfrak{Aw}(\mathfrak{T}) + \Bigg\{v_{m3}(\mathfrak{T})\lambda^{(1)} + {v_{m3}}'(\mathfrak{T})\lambda - v_{m4}(\mathfrak{T})\lambda\Bigg\}\mathfrak{Bw}'(\mathfrak{T}) - v_{m5}(\mathfrak{T})\lambda\mathfrak{Bw}(\mathfrak{T})$$
$$+ \Bigg\{w_{m3}(\mathfrak{T})\lambda^{(1)} + {w_{m3}}'(\mathfrak{T})\lambda - w_{m4}(\mathfrak{T})\lambda\Bigg\}\mathfrak{Cw}'(\mathfrak{T}) - w_{m5}(\mathfrak{T})\lambda\mathfrak{Cw}(\mathfrak{T}) + \Bigg\{y_{m3}(\mathfrak{T})\lambda^{(1)}$$
$$+ {y_{m3}}'(\mathfrak{T})\lambda - y_{m4}(\mathfrak{T})\lambda\Bigg\}\mathfrak{Dw}'(\mathfrak{T}) - y_{m5}(\mathfrak{T})\lambda\mathfrak{Dw}(\mathfrak{T})\Bigg\}d\mathfrak{T} + \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]$$
$$\mathfrak{Aw}''(\mathfrak{T}_{rb}) - \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{Aw}''(\mathfrak{T}_{lb}) - \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3$$
$$+ \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2 - 2\Big[{u_{m1}}'(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 + 2\Big[{u_{m1}}'(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]$$
$$\mathfrak{A}_2 + \Big[u_{m2}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 - \Big[u_{m2}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2, \qquad \mathfrak{d}_* = -1,...,\varpi;$$

The following are the expressions for the second through fourth rows. And each matrix entry for $m = 2,3,4$ is given below.

$$(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Big\{\Big\{ - \Big[u_{m1}(\mathfrak{T})\lambda^{(3)}\Big] - 3\Big[u_{m1}{}'(\mathfrak{T})\lambda^{(2)}\Big] - 3\Big[u_{m1}{}''(\mathfrak{T})\lambda^{(1)}\Big] - \Big[u_{m1}{}'''(t)\lambda\Big]$$
$$+ \Big[u_{m2}(\mathfrak{T})\lambda^{(2)}\Big] + 2\Big[u_{m2}{}'(\mathfrak{T})\lambda^{(1)}\Big] + \Big[u_{m2}{}''(\mathfrak{T})\lambda\Big] - \Big[u_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[u_{m3}{}'(\mathfrak{T})\lambda\Big]$$
$$+ [u_{m4}(\mathfrak{T})\lambda]\Big\}\mu^{(1)} + [u_{m5}(\mathfrak{T})\lambda]\Big\}\mu d\mathfrak{T} - \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}'_{4_*,\mathfrak{d}_*}(\mathfrak{T}_{rb})\Big]\mathfrak{Q}''_{4_*,\mathfrak{q}_*}(\mathfrak{T}_{rb})$$
$$+ \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}'_{4_*,\mathfrak{d}_*}(\mathfrak{T}_{lb})\Big]\mathfrak{Q}''_{4_*,\mathfrak{q}_*}(\mathfrak{T}_{lb}), \qquad \mathfrak{d}_* = -1, ..., \varpi;\ \mathfrak{q}_* = -1, ..., \varpi.$$

$$(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Big\{\Big\{ - \Big[v_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[v_{m3}{}'(\mathfrak{T})\lambda\Big] + [v_{m4}(\mathfrak{T})\lambda]\Big\}$$
$$\mu^{(1)} + [v_{m5}(t)\lambda]\mu\Big\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi;\ \mathfrak{q}_* = -1, ..., \varpi.$$

$$(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Big\{\Big\{ - \Big[w_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[w_{m3}{}'(\mathfrak{T})\lambda\Big] + [w_{m4}(\mathfrak{T})\lambda]\Big\}$$
$$\mu^{(1)} + [w_{m5}(\mathfrak{T})\lambda]\mu\Big\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi;\ \mathfrak{q}_* = -1, ..., \varpi.$$

$$(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Big\{\Big\{ - \Big[y_{m3}(\mathfrak{T})\lambda^{(1)}\Big] - \Big[y_{m3}{}'(\mathfrak{T})\lambda\Big] + [y_{m4}(\mathfrak{T})\lambda]\Big\}$$
$$\mu^{(1)} + [y_{m5}(\mathfrak{T})\lambda]\mu\Big\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi;\ \mathfrak{q}_* = -1, ..., \varpi.$$

and

$$(\hbar_m)_{\mathfrak{d}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Big\{\Big\{(\hbar_{*m})(\mathfrak{T})\lambda + u_{m1}(\mathfrak{T})\lambda^{(3)} + 3u_{m1}{}'(\mathfrak{T})\lambda^{(2)} + 3u_{m1}{}''(\mathfrak{T})\lambda^{(1)} + u_{m1}{}'''(\mathfrak{T})\lambda$$
$$- u_{m2}(\mathfrak{T})\lambda^{(2)} - 2u_{m2}{}'(\mathfrak{T})\lambda^{(1)} - u_{m2}{}''(\mathfrak{T})\lambda + u_{m3}(\mathfrak{T})\lambda^{(1)} + u_{m3}{}'(\mathfrak{T})\lambda - u_{m4}(\mathfrak{T})\lambda\Big\}\mathfrak{A}\mathfrak{w}'(\mathfrak{T})$$
$$- u_{m5}(\mathfrak{T})\lambda\mathfrak{A}\mathfrak{w}(\mathfrak{T}) + \Big\{v_{m3}(\mathfrak{T})\lambda^{(1)} + v_{m3}{}'(\mathfrak{T})\lambda - v_{m4}(\mathfrak{T})\lambda\Big\}\mathfrak{B}\mathfrak{w}'(\mathfrak{T}) - v_{m5}(\mathfrak{T})\lambda\mathfrak{B}\mathfrak{w}(\mathfrak{T})$$
$$+ \Big\{w_{m3}(\mathfrak{T})\lambda^{(1)} + w_{m3}{}'(\mathfrak{T})\lambda - w_{m4}(\mathfrak{T})\lambda\Big\}\mathfrak{C}\mathfrak{w}'(\mathfrak{T}) - w_{m5}(\mathfrak{T})\lambda\mathfrak{C}\mathfrak{w}(\mathfrak{T}) + \Big\{y_{m3}(\mathfrak{T})\lambda^{(1)}$$
$$+ y_{m3}{}'(\mathfrak{T})\lambda - y_{m4}(\mathfrak{T})\lambda\Big\}\mathfrak{D}\mathfrak{w}'(\mathfrak{T}) - y_{m5}(\mathfrak{T})\lambda\mathfrak{D}\mathfrak{w}(\mathfrak{T})\Big\}d\mathfrak{T} + \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]$$
$$\mathfrak{A}w''(\mathfrak{T}_{rb}) - \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{A}\mathfrak{w}''(\mathfrak{T}_{lb}) - \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3$$
$$+ \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2 - 2\Big[u_{m1}{}'(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 + 2\Big[u_{m1}{}'(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]$$
$$\mathfrak{A}_2 + \Big[u_{m2}(\mathfrak{T}_{rb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 - \Big[u_{m2}(\mathfrak{T}_{lb})\mathfrak{Q}_{4_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2, \qquad \mathfrak{d}_* = -1, ..., \varpi;$$

The evaluation of the each integration from $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$, and $(\hbar_m)_{\mathfrak{d}_*}$ for $m = 1$, as well as for $m = 2$ to 4 the integration $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$ and $(\hbar_m)_{\mathfrak{d}_*}$. The Gauss-Legendre quadrature formula was used to calculate the nodal parameter vectors $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar}, \mathfrak{C}_{\hbar_*}$ and $\mathfrak{D}_{\hbar_*}$. We can use the approximation formula to approximate each unknown variable after determining the nodal parameters $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}$ and $\mathfrak{D}_{\hbar_*}$. We also developed the residual error to test for accuracy. This is defined as the error obtained after approximate the solutions of a proposed method. Using a computer programme written in MATLAB code, the proposed method was used to solve the system of BVPs (3)-(4).

**Theorem 1.** *The quartic B-spline approximation of (7)-(10) by Galerkin to the generalised solution of the problem (3)-(4) exists. Further, if equation (20) then satisfies the approximate solution;*

$$\mathfrak{A}(\mathfrak{T}) = \mathfrak{A}\mathfrak{w}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi} \mathfrak{A}_{\hbar_*}\mathfrak{Q}_{4_*,k_*}(\mathfrak{T}),$$

$$\mathfrak{B}(\mathfrak{T}) = \mathfrak{B}\mathfrak{w}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi} \mathfrak{B}_{\hbar_*}\mathfrak{Q}_{4_*,k_*}(\mathfrak{T}),$$

$$\mathfrak{C}(\mathfrak{T}) = \mathfrak{C}\mathfrak{w}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi} \mathfrak{C}_{\hbar_*}\mathfrak{Q}_{4_*,k_*}(\mathfrak{T}),$$

$$\text{and} \quad \mathfrak{D}(\mathfrak{T}) = \mathfrak{D}\mathfrak{w}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi} \mathfrak{D}_{\hbar_*}\mathfrak{Q}_{4_*,k_*}(\mathfrak{T}).$$

**Proof:** Taking the problem (3)-(4) and applying GM with Q4BS. We obtain $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$, and $(\hbar_m)_{\mathfrak{d}_*}$ for $m = 1$, and for $m = 2$ to 4, $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$ and $(\hbar_m)_{\mathfrak{d}_*}$.
And hence $\mathscr{U} = \mathscr{H}^{-1}\mathscr{K}$ where $\mathscr{U} = [\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{D}]^T$;

$$\mathscr{H} = \begin{bmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} & \mathscr{H}_{13} & \mathscr{H}_{14} \\ \mathscr{H}_{21} & \mathscr{H}_{22} & \mathscr{H}_{23} & \mathscr{H}_{24} \\ \mathscr{H}_{31} & \mathscr{H}_{32} & \mathscr{H}_{33} & \mathscr{H}_{34} \\ \mathscr{H}_{41} & \mathscr{H}_{42} & \mathscr{H}_{43} & \mathscr{H}_{44} \end{bmatrix} \quad \text{and } \mathscr{K} = \begin{bmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \\ \mathscr{K}_4 \end{bmatrix}.$$

Thus, we obtain the parameters $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}$, and $\mathfrak{D}_{\hbar_*}$ in the programming part.
Hence, using an approximate solution to approximate each unknown variables $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, and $\mathfrak{D}$.
Therefore equation (20) satisfies the approximate solution.
This completes the theorem proof.

In second case, we define the approximation for "$\mathfrak{A}(\mathfrak{T}), \mathfrak{B}(\mathfrak{T}), \mathfrak{C}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{T})$" by the GM with Q5BS to resolve the system of BVPs of (1)-(2).

$$\mathfrak{A}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+2} \mathfrak{A}_{\hbar_*}\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) \tag{21}$$

$$\mathfrak{B}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+2} \mathfrak{B}_{\hbar_*}\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) \tag{22}$$

$$\mathfrak{C}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+2} \mathfrak{C}_{\hbar_*}\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) \tag{23}$$

$$\mathfrak{D}(\mathfrak{T}) = \sum_{k_*=-2}^{\varpi+2} \mathfrak{D}_{\hbar_*}\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) \tag{24}$$

where the parameters to be determined are $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}, \mathfrak{D}_{\hbar_*}$. The basis functions in the Galerkin method must vanish at the boundary where Dirichlet bc's are defined. The quintic B-splines definition and Dirichlet bc's produce an approximate solution only at boundary points (2). Now eliminate $\mathfrak{A}_{-2}, \mathfrak{A}_{\varpi+2}, \mathfrak{B}_{-2}, \mathfrak{B}_{\varpi+2}, \mathfrak{C}_{-2}, \mathfrak{C}_{\varpi+2}, \mathfrak{D}_{-2}, \mathfrak{D}_{\varpi+2}$ from the equations (21) -

(24). As a result, we get the recent approximations for our unknown variables as follows.

$$\mathfrak{A}(\mathfrak{T}) = \mathfrak{Aw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{A}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}), \tag{25}$$

$$\mathfrak{B}(\mathfrak{T}) = \mathfrak{Bw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{B}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}), \tag{26}$$

$$\mathfrak{C}(\mathfrak{T}) = \mathfrak{Cw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{C}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}), \tag{27}$$

$$\text{and } \mathfrak{D}(\mathfrak{T}) = \mathfrak{Dw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{D}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}) \tag{28}$$

where

$$\mathfrak{Aw}(\mathfrak{T}) = \frac{\mathfrak{A}_0}{\mathscr{G}_{5_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{5_*,-2}(\mathfrak{T}) + \frac{\mathfrak{A}_1}{\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}_{rb})}\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}) \tag{29}$$

$$\mathfrak{Bw}(\mathfrak{T}) = \frac{\mathfrak{B}_0}{\mathscr{G}_{5_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{5_*,-2}(\mathfrak{T}) + \frac{\mathfrak{B}_1}{\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}_{rb})}\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}) \tag{30}$$

$$\mathfrak{Cw}(\mathfrak{T}) = \frac{\mathfrak{C}_0}{\mathscr{G}_{5_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{5_*,-2}(\mathfrak{T}) + \frac{\mathfrak{C}_1}{\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}_{rb})}\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}) \tag{31}$$

$$\mathfrak{Dw}(\mathfrak{T}) = \frac{\mathfrak{D}_0}{\mathscr{G}_{5_*,-2}(\mathfrak{T}_{lb})}\mathscr{G}_{5_*,-2}(\mathfrak{T}) + \frac{\mathfrak{D}_1}{\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}_{rb})}\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}) \tag{32}$$

and

$$\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}) \quad = \quad \begin{cases} \mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) - \left[\frac{\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}_{lb})}{\mathscr{G}_{5_*,-2}(\mathfrak{T}_{lb})}\right]\mathscr{G}_{5_*,-2}(\mathfrak{T}), & \text{for } \hbar_* = -1, ..., 2; \\ \mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}), & \text{for } \hbar_* = 3, ..., \varpi - 3; \\ \mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}) - \left[\frac{\mathscr{G}_{5_*,\hbar_*}(\mathfrak{T}_{rb})}{\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}_{rb})}\right]\mathscr{G}_{5_*,\varpi+2}(\mathfrak{T}), & \text{for } \hbar_* = \varpi - 2, ..., \varpi + 1. \end{cases} \tag{33}$$

In $\mathfrak{A}, \mathfrak{B}, \mathfrak{C},$ and $\mathfrak{D}$, we now have $n+3$ basis functions. We attain the following system of equations which is algebraic with unknown parameters $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}$ and $\mathfrak{D}_{\hbar_*}$ by employing the GM to the system of equations (3) with the recent basis functions.

$$\begin{aligned} \mathscr{H}_{11}\mathfrak{A} + \mathscr{H}_{12}\mathfrak{B} + \mathscr{H}_{13}\mathfrak{C} + \mathscr{H}_{14}\mathfrak{D} &= \mathscr{K}_1 \\ \mathscr{H}_{21}\mathfrak{A} + \mathscr{H}_{22}\mathfrak{B} + \mathscr{H}_{23}\mathfrak{C} + \mathscr{H}_{24}\mathfrak{D} &= \mathscr{K}_2 \\ \mathscr{H}_{31}\mathfrak{A} + \mathscr{H}_{32}\mathfrak{B} + \mathscr{H}_{33}\mathfrak{C} + \mathscr{H}_{34}\mathfrak{D} &= \mathscr{K}_3 \\ \mathscr{H}_{41}\mathfrak{A} + \mathscr{H}_{42}\mathfrak{B} + \mathscr{H}_{43}\mathfrak{C} + \mathscr{H}_{44}\mathfrak{D} &= \mathscr{K}_4 \end{aligned} \tag{34}$$

where "$\mathfrak{A} = [\mathfrak{A}_{-1} \;\; ... \;\; \mathfrak{A}_{\varpi+1}]^T$, $\mathfrak{B} = [\mathfrak{B}_{-1} \;\; ... \;\; \mathfrak{B}_{\varpi+1}]^T$, $\mathfrak{C} = [\mathfrak{C}_{-1} \;\; ... \;\; \mathfrak{C}_{\varpi+1}]^T$ and $\mathfrak{D} = [\mathfrak{D}_{-1} \;\; ... \;\; \mathfrak{D}_{\varpi+1}]^T$". Also, the matrices $\mathscr{H}_{\mathfrak{d}_*\mathfrak{q}_*}$ and their entries are shown below.
The following expression applies to the first row. Each matrix entry is given below for $m = 1$. We denote $\mathfrak{Q}'''_{5_*,\mathfrak{d}_*}(\mathfrak{T}) = \zeta^{(3)}$, $\mathfrak{Q}''_{5_*,\mathfrak{d}_*}(\mathfrak{T}) = \zeta^{(2)}$, $\mathfrak{Q}'_{5_*,\mathfrak{d}_*}(\mathfrak{T}) = \zeta^{(1)}$, $\mathfrak{Q}_{5_*,\mathfrak{d}_*}(\mathfrak{T}) = \zeta$, $\mathfrak{Q}'_{5_*,\mathfrak{q}_*}(\mathfrak{T}) = \delta^{(1)}$ and $\mathfrak{Q}_{5_*,\mathfrak{q}_*}(\mathfrak{T}) = \delta$ for details in Murali & Dhivya (2019)

$$\begin{aligned} (\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ & -\left[u_{m1}(\mathfrak{T})\zeta^{(3)}\right] - 3\left[{u_{m1}}'(\mathfrak{T})\zeta^{(2)}\right] - 3\left[{u_{m1}}''(\mathfrak{T})\zeta^{(1)}\right] - \left[{u_{m1}}'''(\mathfrak{T})\zeta\right] \\ & + \left[u_{m2}(\mathfrak{T})\zeta^{(2)}\right] + 2\left[{u_{m2}}'(\mathfrak{T})\zeta^{(1)}\right] + \left[{u_{m2}}''(\mathfrak{T})\zeta\right] - \left[u_{m3}(\mathfrak{T})\zeta^{(1)}\right] - \left[{u_{m3}}'(\mathfrak{T})\zeta\right] \\ & + \left[u_{m4}(\mathfrak{T})\zeta\right]\Bigg\}\delta^{(1)} + \left[u_{m5}(\mathfrak{T})\zeta\right]\Bigg\}\delta d\mathfrak{T} - \left[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}'_{5_*,\mathfrak{d}_*}(\mathfrak{T}_{rb})\right]\mathfrak{Q}''_{5_*,\mathfrak{q}_*}(\mathfrak{T}_{rb}) \\ & + \left[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}'_{5_*,\mathfrak{d}_*}(\mathfrak{T}_{lb})\right]\mathfrak{Q}''_{5_*,\mathfrak{q}_*}(\mathfrak{T}_{lb}), \qquad \mathfrak{d}_* = -1, ..., \varpi + 1;\; \mathfrak{q}_* = -1, 0, ..., \varpi + 1. \end{aligned}$$

$$(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ -\left[ v_{m3}(\mathfrak{T})\zeta^{(1)} \right] - \left[ {v_{m3}}'(\mathfrak{T})\zeta \right] + \left[ v_{m4}(\mathfrak{T})\zeta \right] \Bigg\}$$
$$\delta^{(1)} + \left[ v_{m5}(\mathfrak{T})\zeta \right] \delta \Bigg\} d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi + 1; \; \mathfrak{q}_* = 0, ..., \varpi + 1.$$

$$(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ -\left[ w_{m3}(\mathfrak{T})\zeta^{(1)} \right] - \left[ {w_{m3}}'(\mathfrak{T})\zeta \right] + \left[ w_{m4}(\mathfrak{T})\zeta \right] \Bigg\}$$
$$\delta^{(1)} + \left[ w_{m5}(\mathfrak{T})\zeta \right] \delta \Bigg\} d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi + 1; \; \mathfrak{q}_* = 0, ..., \varpi + 1.$$

$$(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ -\left[ y_{m3}(\mathfrak{T})\zeta^{(1)} \right] - \left[ {y_{m3}}'(\mathfrak{T})\zeta \right] + \left[ y_{m4}(\mathfrak{T})\zeta \right] \Bigg\}$$
$$\delta^{(1)} + \left[ y_{m5}(\mathfrak{T})\zeta \right] \delta \Bigg\} d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi + 1; \; \mathfrak{q}_* = 0, ..., \varpi + 1.$$

and

$$(\hbar_m)_{\mathfrak{d}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ (\hbar_m)(\mathfrak{T})\zeta + u_{m1}(\mathfrak{T})\zeta^{(3)} + 3{u_{m1}}'(\mathfrak{T})\zeta^{(2)} + 3{u_{m1}}''(\mathfrak{T})\zeta^{(1)} + {u_{m1}}'''(\mathfrak{T})\zeta$$
$$- u_{m2}(\mathfrak{T})\zeta^{(2)} - 2{u_{m2}}'(\mathfrak{T})\zeta^{(1)} - {u_{m2}}''(\mathfrak{T})\zeta + u_{m3}(\mathfrak{T})\zeta^{(1)} + {u_{m3}}'(\mathfrak{T})\zeta - u_{m4}(\mathfrak{T})\zeta \Bigg\} \mathfrak{A}\mathfrak{w}'(\mathfrak{T})$$
$$- u_{m5}(\mathfrak{T})\zeta\mathfrak{A}\mathfrak{w}(\mathfrak{T}) + \Bigg\{ v_{m3}(\mathfrak{T})\zeta^{(1)} + {v_{m3}}'(\mathfrak{T})\zeta - v_{m4}(\mathfrak{T})\zeta \Bigg\} \mathfrak{B}\mathfrak{w}'(\mathfrak{T}) - v_{m5}(\mathfrak{T})\zeta\mathfrak{B}\mathfrak{w}(\mathfrak{T})$$
$$+ \Bigg\{ w_{m3}(\mathfrak{T})\zeta^{(1)} + {w_{m3}}'(\mathfrak{T})\zeta - w_{m4}(\mathfrak{T})\zeta \Bigg\} \mathfrak{C}\mathfrak{w}'(\mathfrak{T}) - w_{m5}(\mathfrak{T})\zeta\mathfrak{C}\mathfrak{w}(\mathfrak{T}) + \Bigg\{ y_{m3}(\mathfrak{T})\zeta^{(1)}$$
$$+ {y_{m3}}'(\mathfrak{T})\zeta - y_{m4}(\mathfrak{T})\zeta \Bigg\} \mathfrak{D}\mathfrak{w}'(\mathfrak{T}) - y_{m5}(\mathfrak{T})\zeta\mathfrak{D}\mathfrak{w}(\mathfrak{T}) \Bigg\} d\mathfrak{T} + \left[ u_{m1}(\mathfrak{T}_{rb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{rb}) \right]$$
$$\mathfrak{A}\mathfrak{w}''(\mathfrak{T}_{rb}) - \left[ u_{m1}(\mathfrak{T}_{lb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{lb}) \right] \mathfrak{A}\mathfrak{w}''(\mathfrak{T}_{lb}) - \left[ u_{m1}(\mathfrak{T}_{rb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}''(\mathfrak{T}_{rb}) \right] \mathfrak{A}_3$$
$$+ \left[ u_{m1}(\mathfrak{T}_{lb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}''(\mathfrak{T}_{lb}) \right] \mathfrak{A}_2 - 2\left[ {u_{m1}}'(\mathfrak{T}_{rb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{rb}) \right] \mathfrak{A}_3 + 2\left[ {u_{m1}}'(\mathfrak{T}_{lb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{lb}) \right]$$
$$\mathfrak{A}_2 + \left[ u_{m2}(\mathfrak{T}_{rb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{rb}) \right] \mathfrak{A}_3 - \left[ u_{m2}(\mathfrak{T}_{lb}){\mathfrak{Q}_{5_*,\mathfrak{d}_*}}'(\mathfrak{T}_{lb}) \right] \mathfrak{A}_2, \qquad \mathfrak{d}_* = -1, ..., \varpi + 1;$$

The following are the expressions for the second through fourth rows. And each matrix entry for $m$ = 2,3,4 is given below.

$$(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ -\left[ u_{m1}(\mathfrak{T})\zeta^{(3)} \right] - 3\left[ {u_{m1}}'(\mathfrak{T})\zeta^{(2)} \right] - 3\left[ {u_{m1}}''(\mathfrak{T})\zeta^{(1)} \right] - \left[ {u_{m1}}'''(t)\zeta \right]$$
$$+ \left[ u_{m2}(\mathfrak{T})\zeta^{(2)} \right] + 2\left[ {u_{m2}}'(\mathfrak{T})\zeta^{(1)} \right] + \left[ {u_{m2}}''(\mathfrak{T})\zeta \right] - \left[ u_{m3}(\mathfrak{T})\zeta^{(1)} \right] - \left[ {u_{m3}}'(\mathfrak{T})\zeta \right]$$
$$+ \left[ u_{m4}(\mathfrak{T})\zeta \right] \Bigg\} \delta^{(1)} + \left[ u_{m5}(\mathfrak{T})\zeta \right] \Bigg\} \delta d\mathfrak{T} - \left[ u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}'_{5_*,\mathfrak{d}_*}(\mathfrak{T}_{rb}) \right] \mathfrak{Q}''_{5_*,\mathfrak{q}_*}(\mathfrak{T}_{rb})$$
$$+ \left[ u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}'_{5_*,\mathfrak{d}_*}(\mathfrak{T}_{lb}) \right] \mathfrak{Q}''_{5_*,\mathfrak{q}_*}(\mathfrak{T}_{lb}), \qquad \mathfrak{d}_* = -1, ..., \varpi + 1; \; \mathfrak{q}_* = -1, ..., \varpi + 1.$$

$$(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{ \Bigg\{ -\left[ v_{m3}(\mathfrak{T})\zeta^{(1)} \right] - \left[ {v_{m3}}'(\mathfrak{T})\zeta \right] + \left[ v_{m4}(\mathfrak{T})\zeta \right] \Bigg\}$$
$$\delta^{(1)} + \left[ v_{m5}(\mathfrak{T})\zeta \right] \delta \Bigg\} d\mathfrak{T}, \qquad \mathfrak{d}_* = -1, ..., \varpi + 1; \; \mathfrak{q}_* = -1, ..., \varpi + 1.$$

$$(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ -\Big[w_{m3}(\mathfrak{T})\zeta^{(1)}\Big] - \Big[w_{m3}{}'(\mathfrak{T})\zeta\Big] + [w_{m4}(\mathfrak{T})\zeta]\Bigg\}$$
$$\delta^{(1)} + [w_{m5}(\mathfrak{T})\zeta]\,\delta\Bigg\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1,...,\varpi+1;\ \mathfrak{q}_* = -1,...,\varpi+1.$$

$$(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{ -\Big[y_{m3}(\mathfrak{T})\zeta^{(1)}\Big] - \Big[y_{m3}{}'(\mathfrak{T})\zeta\Big] + [y_{m4}(\mathfrak{T})\zeta]\Bigg\}$$
$$\delta^{(1)} + [y_{m5}(\mathfrak{T})\zeta]\,\delta\Bigg\}d\mathfrak{T}, \qquad \mathfrak{d}_* = -1,...,\varpi+1;\ \mathfrak{q}_* = -1,...,\varpi+1.$$

and

$$(\hbar_m)_{\mathfrak{d}_*} = \int_{\mathfrak{T}_{lb}}^{\mathfrak{T}_{rb}} \Bigg\{\Bigg\{(\hbar_m)(\mathfrak{T})\zeta + u_{m1}(\mathfrak{T})\zeta^{(3)} + 3u_{m1}{}'(\mathfrak{T})\zeta^{(2)} + 3u_{m1}{}''(\mathfrak{T})\zeta^{(1)} + u_{m1}{}'''(\mathfrak{T})\zeta$$
$$- u_{m2}(\mathfrak{T})\zeta^{(2)} - 2u_{m2}{}'(\mathfrak{T})\zeta^{(1)} - u_{m2}{}''(\mathfrak{T})\zeta + u_{m3}(\mathfrak{T})\zeta^{(1)} + u_{m3}{}'(\mathfrak{T})\zeta - u_{m4}(\mathfrak{T})\zeta\Bigg\}\mathfrak{Aw}'(\mathfrak{T})$$
$$- u_{m5}(\mathfrak{T})\zeta\mathfrak{Aw}(\mathfrak{T}) + \Bigg\{v_{m3}(\mathfrak{T})\zeta^{(1)} + v_{m3}{}'(\mathfrak{T})\zeta - v_{m4}(\mathfrak{T})\zeta\Bigg\}\mathfrak{Bw}'(\mathfrak{T}) - v_{m5}(\mathfrak{T})\zeta\mathfrak{Bw}(\mathfrak{T})$$
$$+ \Bigg\{w_{m3}(\mathfrak{T})\zeta^{(1)} + w_{m3}{}'(\mathfrak{T})\zeta - w_{m4}(\mathfrak{T})\zeta\Bigg\}\mathfrak{Cw}'(\mathfrak{T}) - w_{m5}(\mathfrak{T})\zeta\mathfrak{Cw}(\mathfrak{T}) + \Bigg\{y_{m3}(\mathfrak{T})\zeta^{(1)}$$
$$+ y_{m3}{}'(\mathfrak{T})\zeta - y_{m4}(\mathfrak{T})\zeta\Bigg\}\mathfrak{Dw}'(\mathfrak{T}) - y_{m5}(\mathfrak{T})\zeta\mathfrak{Dw}(\mathfrak{T})\Bigg\}d\mathfrak{T} + \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]$$
$$\mathfrak{Aw}''(\mathfrak{T}_{rb}) - \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{Aw}''(\mathfrak{T}_{lb}) - \Big[u_{m1}(\mathfrak{T}_{rb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3$$
$$+ \Big[u_{m1}(\mathfrak{T}_{lb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}''(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2 - 2\Big[u_{m1}{}'(\mathfrak{T}_{rb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 + 2\Big[u_{m1}{}'(\mathfrak{T}_{lb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]$$
$$\mathfrak{A}_2 + \Big[u_{m2}(\mathfrak{T}_{rb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{rb})\Big]\mathfrak{A}_3 - \Big[u_{m2}(\mathfrak{T}_{lb})\mathfrak{Q}_{5_*,\mathfrak{d}_*}{}'(\mathfrak{T}_{lb})\Big]\mathfrak{A}_2, \qquad \mathfrak{d}_* = -1,...,\varpi+1;$$

The evaluation of each integration of $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}, (h_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}, (h_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}, (h_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$, and $(k_m)_{\mathfrak{d}_*}$ for $m = 1$, as well as for $m = 2$ to 4 the integration $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}, (\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$ and $(\hbar_m)_{\mathfrak{d}_*}$. The Gauss-Legendre quadrature formula was used to calculate the nodal parameter vectors $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}$ and $\mathfrak{D}_{\hbar_*}$. We can use the approximation formula to approximate any unknown variable after determining the nodal parameters $\mathfrak{A}_{\hbar_*}, \mathfrak{B}_{\hbar_*}, \mathfrak{C}_{\hbar_*}$ and $\mathfrak{D}_{\hbar_*}$. We also formed the residual error to test the accuracy. This is defined as the error obtained after approximating the solutions of a proposed method. Using a computer program written in MATLAB code, the proposed technique was used to solve the system of BVPs (3) and (4).

**Theorem 2.** *The quintic B-spline approximation of (21)-(24) by Galerkin to the generalised solution of the problem (3)-(4) exists. Further, if equation (34) then satisfies the approximate solution;*

$$\mathfrak{A}(\mathfrak{T}) = \mathfrak{Aw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{A}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}),$$
$$\mathfrak{B}(\mathfrak{T}) = \mathfrak{Bw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{B}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}),$$
$$\mathfrak{C}(\mathfrak{T}) = \mathfrak{Cw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{C}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}),$$
$$\text{and}\quad \mathfrak{D}(\mathfrak{T}) = \mathfrak{Dw}(\mathfrak{T}) + \sum_{k_*=-1}^{\varpi+1} \mathfrak{D}_{\hbar_*}\mathfrak{Q}_{5_*,\hbar_*}(\mathfrak{T}).$$


**Figure 1:** Variation in $f$, $g$ with $Re$ in Sheikholeslami et al. (2014).

**Proof:** Taking the problem (3)-(4) and applying GM with Q5BS. We obtain $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$, and $(\not{k}_{m})_{\mathfrak{d}_*}$ for $m = 1$, and for $m = 2$ to 4, $(\hbar_{m1})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m2})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m3})_{\mathfrak{d}_*\mathfrak{q}_*}$, $(\hbar_{m4})_{\mathfrak{d}_*\mathfrak{q}_*}$ and $(\not{k}_{m})_{\mathfrak{d}_*}$.
And hence $\mathscr{U} = \mathscr{H}^{-1}\mathscr{K}$ where $\mathscr{U} = [\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{D}]^T$;

$$\mathscr{H} = \begin{bmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} & \mathscr{H}_{13} & \mathscr{H}_{14} \\ \mathscr{H}_{21} & \mathscr{H}_{22} & \mathscr{H}_{23} & \mathscr{H}_{24} \\ \mathscr{H}_{31} & \mathscr{H}_{32} & \mathscr{H}_{33} & \mathscr{H}_{34} \\ \mathscr{H}_{41} & \mathscr{H}_{42} & \mathscr{H}_{43} & \mathscr{H}_{44} \end{bmatrix} \quad \text{and } \mathscr{K} = \begin{bmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \\ \mathscr{K}_4 \end{bmatrix}.$$

Thus, we obtain the parameters $\mathfrak{A}_{\not{k}_*}, \mathfrak{B}_{\not{k}_*}, \mathfrak{C}_{\not{k}_*}$, and $\mathfrak{D}_{\not{k}_*}$ in the programming part.
Hence, using an approximate solution to approximate each unknown variables $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, and $\mathfrak{D}$.
Therefore equation (34) satisfies the approximate solution.
This completes the theorem proof.

# 4 Numerical Example

We demonstrated the application based coupled system of fourth-order BVPs of (1) - (2). The numerical results for two cases are presented graphically and compared to solutions found in the literature, Sheikholeslami et al. (2014) consider the following problem

$$(1+N_1)f^{IV} - N_1 g - Re(ff^{'''} - f^{'}f^{''}) = 0, \tag{35}$$
$$N_2 g^{''} + N_1(f^{''} - 2g) - N_3 Re(fg^{'} - f^{'}g) = 0, \tag{36}$$
$$\theta^{''} + Pe_h f^{'}\theta - Pe_h f\theta^{'} = 0, \tag{37}$$
$$\varphi^{''} + Pe_m f^{'}\varphi - Pe_m f\varphi^{'} = 0, \tag{38}$$

with the bc's

$$f^{'} = 0, f = 0, g = 0, \theta = 1, \varphi = 1 \quad at \quad \eta = -1, \tag{39}$$
$$f^{'} = -1, f = 0, g = 1, \theta = 0, \varphi = 0 \quad at \quad \eta = +1. \tag{40}$$

By quasilinearization technique Bellman & Kalaba (1965), the non-linear system of equations (35) - (38) is converted into sequence of the linear systems of differential equations. The computational results of the existing method, as well as the proposed method, are graphically represented. Figures 1 to 4 show the existing results available in the literature. Figures 5 to 12 show the results obtained for two cases. Figure 13 shows the residual plots obtained for two cases. Residual results are given in Tables 1-2.

# 5 Discussion and Results

The coupled system of differential equations (35) - (38) under the boundary conditions (39) - (40) is described in section 4. Figure 1 illustrates the $f$ variation with increasing $Re$ and the


**Figure 2:** Variation in $g$,$f$ with $N_1$,$N_2$ in Sheikholeslami et al. (2014)


**Figure 3:** Variation in $g$ with $N_2$,$N_3$ in Sheikholeslami et al. (2014)


**Figure 4:** Variation in $\theta$,$\phi$ with $Pe_h, Pe_m$ in Sheikholeslami et al. (2014)


**Figure 5:** Variation in $f$ with increasing values of $Re$ in first, second cases


**Figure 6:** Variation in $g$ with increasing values of $Re$ in first, second cases


**Figure 7:** Variation in $g$ with increasing values of $N_1$ in first, second cases


**Figure 8:** Variation in $f$ with increasing values of $N_2$ in first, second cases


**Figure 9:** Variation in $g$ with increasing values of $N_2$ in first, second cases


**Figure 10:** Variation in $g$ with increasing values of $N_3$ in first, second cases


**Figure 11:** Variation in $\theta$ with increasing values of $Pe_h$ in first, second cases

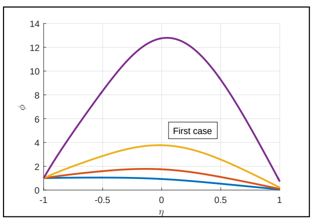
**Table 1:** For the first case, residual error for iterations

| Iterations | $\theta(\eta)$ | $\phi(\eta)$ |
|---|---|---|
| 1 | 2.4231e-06 | 3.8514e-06 |
| 2 | 5.1605e-10 | 7.7091e-10 |
| 3 | 4.7105e-10 | 1.4408e-09 |
| 4 | 4.5901e-10 | 1.4017e-09 |
| 5 | 4.5678e-10 | 1.3941e-09 |
| 6 | 4.5636e-10 | 1.3927e-09 |
| 7 | 4.5627e-10 | 1.3924e-09 |
| 8 | 4.5626e-10 | 1.3924e-09 |

**Table 2:** Residual error for different iterations for second case

| Iterations | $\theta(\eta)$ | $\phi(\eta)$ |
|---|---|---|
| 1 | 2.4734e-06 | 4.7120e-06 |
| 2 | 1.8050e-10 | 6.5094e-10 |
| 3 | 1.4683e-10 | 4.5483e-10 |
| 4 | 1.4925e-10 | 4.6043e-10 |
| 5 | 1.4956e-10 | 4.6114e-10 |
| 6 | 1.4959e-10 | 4.6121e-10 |

$g$ variation with increasing $Re$ in Sheikholeslami et al. (2014). Figure 2 shows the $g$ variation with increasing values of the $N_1$ and also the variation of $f$ with increasing values of $N_2$ in Sheikholeslami et al. (2014). Figure 3 demonstrates the $g$ variation with increasing values of the $N_2$ and also a $g$ variation with increasing values of the $N_3$ in Sheikholeslami et al. (2014). Figure 4 represents variation of $\theta$ with increasing the values of "$Pe_h$", and also the variation of $\phi$ with increasing the values of "$Pe_m$" in Sheikholeslami et al. (2014). In Figure 5, the first graph examined the variation of $f$ as the value of increases the Reynolds number which is first case, and the second graph for the second case. In Figure 6 is plotted for the $g$ variation as the values of Reynolds number in the first case and second graph for second case. In Figure 7, the first graph is sketched for $g$ variation as the values increases of coupling parameter in the first case and second graph for second case. In Figure 8, the first graph we observe that variation of $f$ with increasing values of spin gradient viscosity parameter in the first case and second graph for second case. In Figure 9, the first graph is a graphical representation of the variation of $g$ with increasing values of spin gradient viscosity parameter in the first case and second graph



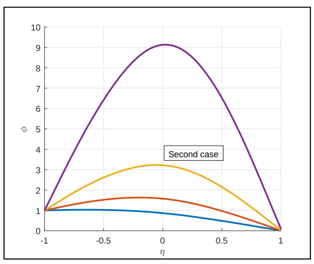




**Figure 12:** Variation in $\phi$ with increasing values of $Pe_m$ in first, second cases


**Figure 13:** Residual errors in first and second cases

for second case. In Figure 10, the first graph exhibits the variation of $g$ with increasing angular velocity values in the first case and second graph for second case. In Figure 11, the first graph deals with the variation of $\theta$ with values increases of Peclet number for diffusion of heat in the first case and second graph for second case. In Figure 12, the first graph labels the variation of $\phi$ with values increases of Peclet number for the diffusion of mass in the first case and second graph for second case. In Figure 13, the first graph indicates residual errors for fluid temperature "$\theta$", and concentration "$\phi$" profiles with residual errors less than $10^{-10}$ for the first case. Figure 13 studies residual errors for profile fluid temperature "$\theta$", and profile concentration "$\phi$" with residual errors less than $10^{-10}$ for the second case.

Figures 5 to 12 show the numerical solutions found using the suggested methodology, while Figures 1 to 4 show the graphical representation of the numerical solutions are given by Sheikholeslami et al. (2014). Also, residual error the first case is represented by Table 1, and the second case is represented by Table 2. And compared between the residual error for the first case and second case, also shown that the accuracy of second case is better than the accuracy of first case. The proposed method calculates each matrix until the tolerance is within $1.0X10^{-5}$ for the solutions Figure 5 to Figure 13. The stability of such a numerical scheme for the proposed methods depends on the stability of the system of BVPs of $(35) - (40)$. If the system of BVPs in $(35) - (40)$ is not stable, and numerical techniques may fail to produce converged solutions.

## 6 Conclusion

In this manuscript, we compared the experimental results, which revealed that the accuracy of the GM with the basis functions of Q5BS is better than that of the GM using the basis functions of Q4BS. For this purpose, we have examined the accuracy of the first and second-case residual errors in Figure 13, Table 1, and Table 2. It is more effective in implementation because it requires only fewer iterations. The numerical results obtained using the proposed technique were compared to those published results previously shown in Figures 1-4, demonstrating that the numerical results of the test problem is more accurate and effective than existing techniques.

## Acknowledgment

We want to express our sincere gratitude to editors and reviewers for their comments and valuable suggestions.

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